CHALMERS
UNIVERSITY OF TECHNOLOGY

Bielliptic Surfaces and their Geometry

Master's thesis in Engineering mathematics and computational science

XUDONG LIU

DEPARTMENT OF MATHEMATICAL SCIENCES
CHALMERS UNIVERSITY OF TECHNOLOGY
Gothenburg, Sweden 2024
www.chalmers.se

Master's thesis 2024

Bielliptic Surfaces and their Geometry

XUDONG LIU

CHALMERS
UNIVERSITY OF TECHNOLOGY

Department of Mathematical Sciences
Division of algebraic geometry and number theory
Chalmers University of Technology
Gothenburg, Sweden 2024

Bielliptic Surfaces and their Geometry
XUDONG LIU

Supervisor: Dennis Eriksson, Department of Mathematical Sciences
Examiner: Dennis Eriksson, Department of Mathematical Sciences

Master's Thesis 2024
Department of Mathematical Sciences
Division of algebraic geometry and number theory
Chalmers University of Technology
SE-412 96 Gothenburg
Telephone +46 31 772 1000

Typeset in LaTeX
Printed by Chalmers Reproservice
Gothenburg, Sweden 2024

Bielliptic Surfaces and their Geometry
XUDONG LIU
Department of Mathematical Sciences
Chalmers University of Technology

Abstract

Geometry is a high concern in modern mathematics. One way to begin the study is by handling a nice example. The bielliptic surfaces can play such a role. It is constructed using elliptic curves, some nice curves in some way equivalent to a torus.

The prerequisites of bielliptic surfaces involve algebraic geometry and elliptic curves. The final result is about the intersection of bielliptic surfaces, so the intersection theories of surfaces will also be introduced. Works of classification and works of Néron-Severi lattices are crucial for the study of bielliptic surfaces in the last section.

Algebraic geometry focuses on the method of solving geometry problems in algebraic ways. The fundamental of the study is abstract algebra. It studies curves, surfaces, and some other higher-dimension objects like hyperspaces. The key point is describing geometry structures by zeros of polynomials. Many results are derived over the complex field, where many nice properties can be found.

The elliptic curve is a kind of algebraic curve of genus one. Weierstrass equations are the algebraic forms of elliptic curves. The composition law defines an operation on the elliptic curves. Another important property is that the lattices over the complex field determine the elliptic curves, which can be derived from the construction of the Weierstrass $\wp$-function. Isogenies are introduced as the maps between elliptic curves.

The topic of intersection theory on surfaces concerns the intersection number of two curves on the given surface, which is the number of intersection points counted with algebraic multiplicity. The definition of intersection number can be generalized to n varieties in high dimensions. In the article, the situation of two curves on a surface is enough. One important result is Bézout's theorem, a theorem of the intersection number of plane curves.

The definition of bielliptic surfaces is based on the elliptic curves. With all the knowledge before, the final result about the intersection number of bielliptic surfaces can be given.

Keywords: elliptic curve, bielliptic surfaces, intersection theory, intersection number.

Acknowledgments

In the period I was in Goteborg, my family gave me lots of financial support for living and studying. My teacher Dennis Eriksson, who was introduced by Minchen Xia, helped me many times when I had difficulties in understanding mathematics and writing the article. As both the administrator of the master's program and my functional analysis teacher, Håkan Andreasson also benefited me.

Xudong Liu, Gothenburg, March 2024

Nomenclature

Below is the nomenclatures that have been used throughout this thesis.

K	Algebraic closed field
C	Algebraic curve
S	Algebraic surface
E, F	Elliptic curve
Λ	Lattice in $\mathbb{C}$

Contents

List of Tables

1

Introduction

This section is about the outline of the article. There are some interpretations, but only a few details are given.

1.1 Algebraic Geometry

As a part of modern mathematics, algebraic geometry is the method of studying geometry by abstract algebra. It is the foundation of studying bielliptic surfaces. Although this branch of mathematics is wide and complex, the knowledge needed in the thesis is just some simple facts.

The simplest situation is probably a line in space $\mathbb{C}^2$ given by the linear equation

$$aX + bY + c = 0.$$

A slightly different example is the line in space $\mathbb{C}^3$

$$\begin{cases} a_1X + b_1Y + c_1Z + d_1 = 0 \\ a_2X + b_2Y + c_2Z + d_2 = 0. \end{cases}$$

The equations are not unique, but the ideal generated by the two equations is. Similarly, a geometry structure V denoted by some equations $f_1 = 0, f_2 = 0, ..., f_n = 0$ can be also denoted by the zeros of the unique ideal $I = (f_1, f_2, ..., f_n)$. Conversely, for any ideal I in $\mathbb{C}[X_1, X_2, ..., X_n]$, the relation

$$f(P) = 0, P \in \mathbb{C}^n, \forall f \in I$$

also denotes a unique geometry structure V. Thus some geometry structures could be described by algebraic expressions, which makes commutative algebra tools useful in geometry study. In this way, the zero set V of such an ideal I is called variety.

An important space used in the thesis is projective space $\mathbb{P}^n$. It is based on the equivalence relation of $K^n \setminus \{(0, 0, ..., 0)\}$

$$(x_1, x_2, ..., x_n) \sim (kx_1, kx_2, ..., kx_n), k \in K^*.$$

In the projective space, the geometry structure can be described by homogeneous polynomials. Homogeneous polynomials are f with all nonzero terms having the same degree, like $X^3 + X^2Y + Y^3$. Some problems could be simplified once it is embedded in the projective space.

With these foundations, the geometry concepts can be better defined by algebraic expressions, like curves C, D, surface S, intersection multiplicity $m_P(C, D)$ of C, D at point P, Néron–Severi group $NS(S)$ of some surface S. The details of definitions are omitted in the introduction, but the concepts are used to give some basic explanations.

Following, the elliptic curves and intersection theory are introduced as the necessary knowledge of bielliptic surfaces.

1.2 Elliptic Curves

Any elliptic curve over $\mathbb{C}$ could be described by a Weierstrass equation in $\mathbb{C}[X, Y, Z]$, or a variety E in the projective space $\mathbb{P}^3$. Let $x = X/Z, y = Y/Z$, the equation is of form

$$E : y^2 = 4x^3 - 27c_4x - 54c_6.$$

Composition law, or the additional group law of E, is defined by points P, Q, R of a line L intersecting E. That is,

$$P, Q, R \in L \cap E.$$

Take $O = [0 : 1 : 0]$ as the zero element given by the Weierstrass equation, by the composition law there is

$$P \oplus Q \oplus R = O.$$

A lattice Λ with bases $\omega_1, \omega_2 \in \mathbb{C}$ is defined as

$$\Lambda := \{k_1\omega_1 + k_2\omega_2 : k_1, k_2 \in \mathbb{Z}\}.$$

Over $\mathbb{C}/\Lambda$, an interesting result is constructing E with Weierstrass $\wp$-Function. It is a periodic function of the form

$$\wp(z; \Lambda) := \frac{1}{z^2} + \sum_{0 \neq w \in \Lambda} \left(\frac{1}{(z - w)^2} - \frac{1}{w^2}\right).$$

With the help of $\wp(z; \Lambda)$, there exists an isomorphism ϕ keeping the group structure to the elliptic curve $E(\mathbb{C})$

$$\phi : \mathbb{C}/\Lambda \to E(\mathbb{C}) \subset \mathbb{P}^2(\mathbb{C}),\ z \mapsto [\wp(z), \wp'(z), 1].$$

One important result, which gives more details about the relationships between $\mathbb{C}/\Lambda$ and elliptic curve E, is that these categories are equivalent:

1. Objects: Elliptic curves E over $\mathbb{C}$.
   Maps: Isogenies.

2. Objects: Elliptic curves E over $\mathbb{C}$.
   Maps: Complex analytic maps taking O to O.
3. Objects: Lattices $\Lambda \subset \mathbb{C}$, up to homothety.
   Maps: $Map(\Lambda_1, \Lambda_2) = \{\alpha \in \mathbb{C} : \alpha\Lambda_1 \subset \Lambda_2\}$.

1.3 Intersection Theory

To illustrate the intersection number, the concept of intersection multiplicity $m_P(C, D)$ of curve C, D at point P is necessary. There is an example. If two curves C, D are given by $Y = X^2, X = 0$, then

$$m_{(0,0)}(Y - X^2, X) = \dim O_{\mathbb{A}^2,(0,0)}/(Y - X^2, X) = 2$$

with the fact of $\deg X^2 = 2$. This case shows how to calculate the multiplicity algebraically.

For two curves C, D, the number of intersection points $\#(C \cap D)$ (counted with multiplicity) is an intuitive problem, denoted as $C \cdot D$. However, the intersection theory can be generalized to the intersection of some high-dimension geometry objects $V_1 \cdot V_2 \cdot \ ... \ \cdot V_n$. The situation of bielliptic surfaces is simple: two curves C, D on a surface S.

The intersection number can be defined as

$$C \cdot D = \sum_{P \in C \cap D} m_P(C, D).$$

One important and interesting result is Bézout's theorem, a theorem about how the degrees define the global intersection number. For distinct plane curve $C, D \subset \mathbb{P}^2$, it is of form

$$C \cdot D = \deg C \cdot \deg D.$$

An example is the fundamental theorem of algebra: any nontrivial polynomial $f(x)$ has $d = \deg f$ roots counting with multiplicity. The roots of $f(x)$ correspond to the intersection point of $f(X, Y) = Y^d f(x) = 0, x = X/Y$ and $Y = 0$. The degrees of two polynomials are

$$\deg(f(X, Y)) = d, \deg Y = 1.$$

The result is also true for homogeneous polynomials. In this way, the fundamental theorem of algebra is a special case of Bézout's theorem.

There is also a general version of Bézout's theorem. For n distinct hypersurfaces $H_1, H_2, ..., H_n$ defined as zeros of homogeneous polynomials $f_1, f_2, ..., f_n$, if the intersection number is finite, there is

$$H_1 \cdot H_2 \cdot \ ... \ \cdot H_n = \deg f_1 \cdot \deg f_2 \cdot \ ... \ \cdot \deg f_n.$$

1.4 Bielliptic Surfaces

Let E, F be two elliptic curves, G acts on E, F as automorphism, the bielliptic surface S has form

$$S \cong E \times F/G.$$

G, a fixed point free action, acts on E, F in the following way: for some injective homomorphism $a : G \to Aut(F)$, $g \in G$ acting on $(x, y) \in (E, F)$ is

$$g(x, y) = (x + g, a(g)y).$$

Another concept here is Néron–Severi group $NS(S)$ for the bielliptic surfaces S, an equivalence relation of intersection theory. Denote $[C]$ for the class of curve C, then for any $C' \in [C]$ and any other curve D,

$$C \cdot D = C' \cdot D.$$

There is an important fact: any curve C in S can be factored into $a[E]+b[F]$, where a, b are coefficients, and $[E], [F]$ are the equivalent classes of E, F.

The result of the intersection theory in the last section is

$$[E]^2 = [F]^2 = 0, [E] \cdot [F] = \#G.$$

Factor the curves C_1, C_2, then there is

$$[C_1] \cdot [C_2] = (a_1[E] + b_1[F]) \cdot (a_2[E] + b_2[F]) = (a_1b_2 + a_2b_1)\#G.$$

The coefficients a, b are similar to the degrees, so the result works similarly with Bézout's theorem.

2

Algebraic Geometry Foundations

In this section, some backgrounds of algebraic geometry will be established. It concentrates on describing geometry structure by algebraic terms, especially polynomials. An example is $y = x^2$ related to a parabola in the $\mathbb{R}^2$ coordinate. These foundations are the bases of elliptic curves and bielliptic surfaces.

An algebraic variety (**Definition** 2.1.1,2.1.4) describes geometry objects in the form of polynomial rings and ideals. Curve and surface (**Definition** 2.1.8) are defined in this way as a 1-dimensional and 2-dimensional projective variety.

Another important topic is the degree of map (**Definition** 2.3.1). Some properties of degrees are introduced, as the necessary tools used in the last section.

For a curve C, a divisor (**Definition** 2.2.1) $D = \sum_{P \in C} n_P(P)$, where all but finite n_P are zeros, is the formal additional group of points P with coefficients n_P, in the thesis are mainly zeros and poles with multiplicities for a rational function.

The genus of a curve, a parameter that can be used to classify, can be calculated by the Riemann-Roch theorem (**Theorem** 2.4.4) in the form of divisors.

2.1 Algebraic Variety

In affine space, point sets are associated with ideals. It is similar in projective space, the difference being that polynomials mentioned in projective space are homogeneous.

Definition 2.1.1. *The affine n-space over K is the set*

$$\mathbb{A}^n = \mathbb{A}^n(K) := \{P = (x_1, x_2, ..., x_n) : x_k \in K\}.$$

For the ideal I in the polynomial ring $K[X] = K[X_1, X_2, ..., X_n]$, the associated set is

$$V_I := \{P \in \mathbb{A}^n : f(P) = 0, \forall f \in I\}.$$

V is an algebraic set if it has the form V_I. The ideal of V is

$$I(V) := \{f \in K[X] : f(P) = 0, \forall P \in V\}.$$

Definition 2.1.2. *The projective n-space over* K*, denoted by* $\mathbb{P}^n$ *or* $\mathbb{P}^n(K)$*, is* $(\mathbb{A}^{n+1} \setminus \{(0, 0, ..., 0)\})/\sim$*, where the equivalence relation*

$$\sim: (x_0, x_1, ..., x_n) \sim (kx_0, kx_1, ..., kx_n), k \in K.$$

Each equivalent class x *is denoted by* $[x_0 : x_1 : \ ... \ : x_n]$*.*
Definition 2.1.3. $f \in K[X]$ *is a homogeneous polynomial if*

$$f(kx_0, kx_1, ..., kx_n) = k^{n+1} f(x_0, x_1, ..., x_n), \forall k \in K.$$

An ideal in $K[X]$ *is homogeneous if it can be generated by homogeneous polynomials.*
Definition 2.1.4. *For the ideal homogeneous* I *in the polynomial ring* $K[X] = K[X_1, X_2, ..., X_n]$*, the associated set is*

$$V_I := \{P \in \mathbb{P}^n : f(P) = 0, \forall f(X) \in I, f(X) \ is \ homogeneous\}$$

V *is a projective algebraic set if it has form* V_I*. The homogeneous ideal of* V *is*

$$I(V) := \{f(X) \in K[X] : f(P) = 0, \forall P \in V, f(X) \ is \ homogeneous\}.$$

The ideal $I \subset K[X]$ gives a natural equivalence relation $f \sim g : f - g \in I$ for the polynomial ring, where comes the coordinate ring.
Definition 2.1.5. *A (projective) algebraic set* V *is called a (projective) variety if its (homogeneous) ideal* $I(V)$ *is a prime ideal. The coordinate ring* $K[V]$ *is defined by*

$$K[V] := K[X]/I(V).$$

The rational ring $K(V)$ *is generated by polynomials* f *and* g^{-1}*, where*

$$f \in K[X], g \in K[X] \setminus K[V], h_1 \sim h_2 \in K(V) \ if \ and \ only \ if \ h_1 - h_2 \in I(V).$$

A morphism between varieties is a regular map, as the definition below.
Definition 2.1.6. *Between two projective varieties* $V_1 \to V_2$*, a map* $\phi = [f_1 : f_2 : ..., f_n], f_i \in K(V_1)$ *is regular at* $P \in V_1$ *if there exists* $g \in K(V_1)$*, s.t.*

1. gf_i *is well defined for all* i*.*
2. $(gf_i)(P) \neq 0$ *for some* i*.*

ϕ *is a morphism if it is regular everywhere, i.e. regular at any* $P \in V_1$*.*
The algebraic curve and surface are defined after the dimension, defined in the view of field extension, of variety. And the nonsingularity is also defined.
Definition 2.1.7. *The dimension of an affine variety* V*, denoted by* $\dim(V)$*, is the transcendence degree of* $K(V)/K$*.*

For a projective variety $V \subset \mathbb{P}^n$*, there exists* $\mathbb{A}^n \subset \mathbb{P}^n$*, s.t.* $V \cap \mathbb{A}^n \neq \phi$*. The dimension of* V *is defined in* $\mathbb{A}^n$ *as*

$$\dim(V) := \dim(V \cap \mathbb{A}^n).$$

Definition 2.1.8. *A curve* C *(or surface* S*) is a projective variety of dimension* 1 *(or* 2*).*

Definition 2.1.9. *For a variety $V \ni P$, where the ideal $I(V)$ generated by $f_1, f_2, ..., f_m \in K[X]$, V is nonsingular (or smooth) at P if the rank is $n - \dim V$ for the $m \times n$ matrix*

$$(\frac{f_i}{X_j}(P)).$$

V is nonsingular (or smooth) if it is smooth at every point P.

2.2 Divisors

In this section, C is always a smooth curve over K.

Divisor is the formal sum of points on the curve. The points here are often the zeros and poles with coefficients as the multiplicities.

Definition 2.2.1. *The divisor group of a curve C is defined as*

$$Div(C) := \{D = \sum_{P \in C} n_P(P)\}.$$

It is an abelian group generated by points $P \in C$, where $n_P \in \mathbb{Z}$ and $n_P = 0$ for all but finitely many P. The degree of D is

$$\deg D := \sum_{P \in C} n_P.$$

$Div^0(C)$ is the degree-0 subgroup of $Div(C)$

$$Div^0(C) := \{D \in Div(C) : \deg D = 0\}.$$

Let $ord_P(f)$ be the order of zeros of f at P. For $0 \neq f \in K(C)^*$, where C is a smooth curve, define

$$div(f) := \sum_{P \in C} ord_P(f)(P).$$

Definition 2.2.2. *A divisor D is principal if it has form $D = div(f)$ for some $f \in K(C)$. The divisors D_1, D_2 are linearly equivalent when $D_1 - D_2$ is principal with the notation $D_1 \sim D_2$. The Picard group $Pic(C)$ of a curve C is the quotient of $Div(C)$ by its subgroup of principal divisors.*

Proposition 2.2.3. *Let $f \in K(C)^*$.*

1. *$div(f) = 0$ if and only if $f \in K$.*
2. $\deg div(f) = 0$.

Proof. See [1, p.28, Proposition 3.1].

□

The canonical divisor (**Definition** 2.2.4), used in the Riemann-Roch theorem, is defined by differential form. The proposition following the definition ensures the definition is proper.

Definition 2.2.4. *The space of (meromorphic) differential forms of a curve C, denoted by Ω_C, is the $K(C)$-vector space spanned by form $dx, x \in K(C)$ satisfying*

1. *$d(x + y) = dx + dy, \forall x, y \in K(C)$.*

2. $d(xy) = ydx + xdy, \forall x, y \in K(C)$.
3. $d\alpha = 0, \forall \alpha \in K$.
The canonical divisor class

$$K_C := div(\omega)$$

is the class of $\omega \in \Omega_C$ *in* $Pic(C)$*, where*

$$div(\omega) := \sum_{P \in C} ord_P(\omega)(P) \in Div(C)$$

which is well defined by ***Proposition*** *2.2.5.*
Proposition 2.2.5. *For all* $\omega_1, \omega_2 \in \Omega_C$*, there exists an* $f \in K(C)$*, s.t.*

$$\omega_1 = f\omega_2, div(\omega_1) = div(f) + div(\omega_2).$$

Proof. See [1, p.32, Remark 4.4].

□

2.3 Maps between curves

$\phi : C_1 \to C_2$ is always a map of curves C_1, C_2 over K in this part.

For a nonconstant rational map ϕ, its composition induces an injection

$$\phi^* : K(C_2) \to K(C_1), \phi^* f = f \circ \phi.$$

The following contents are about the degree of ϕ, as important facts used in the last section.
Definition 2.3.1. *For map* $\phi : C_1 \to C_2$*, the degree* $\deg \phi = 0$ *when* ϕ *is constant, otherwise*

$$\deg(\phi) := \dim[K(C_1) : \phi^* K(C_2)].$$

ϕ *is separable, inseparable, or purely inseparable if the extension* $K(C_1) : \phi^* K(C_2)$ *has the corresponding property. The notation is* $\deg_s(\phi)$ *for the separable degree. Particularly,* $\deg_s(\phi) = \deg(\phi)$ *when* $K = \mathbb{C}$*.*
Proposition 2.3.2. *Let* $\phi : C_1 \to C_2$ *be a nonconstant map of smooth curves, for all but finite* $Q \in C_2$*,*

$$\#\phi^{-1}(Q) = \deg_s(\phi).$$

Particularly, $\#\phi^{-1}(Q) = \deg(\phi)$ *when* $K = \mathbb{C}$*, since* $\deg_s(\phi) = \deg(\phi)$*.*

Proof. See [1, p.23, Proposition 2.6].

□

Proposition 2.3.3. *For nonconstant map* $\phi : C_1 \to C_2$ *of smooth curves,*

$$(\deg \phi^* D) = (\deg \phi)(\deg D), \text{ where } D \in Div(C_2).$$

Proof. See [1, p.29, Proposition 3.6].

□

2.4 Genus

The Riemann-Roch theorem gives the genus in the form of $l(D) = \dim L(D)$, where $L(D)$ is a linear function space, where D is a divisor.

Definition 2.4.1. *A divisor $D = n_P(P)$ is positive if $n_P \geq 0, P \in C$, denoted by $D \geq 0$. The notation $D_1 \geq D_2$ means that $D_1 - D_2$ is positive.*

Definition 2.4.2. *For $D \in Div(C)$,*

$$L(D) := \{f \in K(C) : div(f) + D \geq 0\} \cup \{0\}.$$

$L(D)$ is a finite-dimensional K-vector space as [2, p.122, Theorem 5.19], its dimension

$$l(D) := \dim_K L(D).$$

Proposition 2.4.3. *If $\deg D < 0, D \in Div(C)$, then $L(D) = \{0\}, l(D) = 0$.*

Proof. Let $0 \neq f \in L(D)$, then there is a contradiction by **Proposition** 2.2.3

$$\deg D = \deg(D + div(f)) - \deg div(f) \geq -\deg div(f) = 0.$$

□

Theorem 2.4.4. *(Riemann–Roch theorem) For the canonical divisor K_C on a smooth curve C, the genus $g \in \mathbb{N}$ of C satisfies*

$$l(D) - l(K_C - D) = \deg D - g + 1, \forall D \in Div(C).$$

Proof. See [2, p.295, Theorem 1.3].

□

Corollary 2.4.5. *By the Riemann–Roch theorem,*

1. $l(K_C) = g$.
2. $\deg K_C = 2g - 2$.
3. *If* $\deg D > 2g - 2$, *then* $l(D) = \deg D - g + 1$.

Proof.
1. Let $D = 0$, then $L(0) = K, l(0) = 1$.
2. Let $D = K_C$.
3. $\deg(K_C - D) = \deg K_C - \deg D < 0$, thus by **Proposition** 2.4.3, $l(K_C - D) = 0$.

□

3

Elliptic Curves

This section will focus on elliptic curves, which are denoted E. A fact is any E over $\mathbb{C}$ is related to a torus, by the isomorphism between E and a lattice $\mathbb{C}/\Lambda$. The knowledge of elliptic curves is an important basis for the last section since two elliptic curves E, F can construct a bielliptic surface S.

Elliptic curves (**Definition** 3.1.1) are non-singular curves of genus 1 with a point O. Another description of E is the Weierstrass equation. There is also composition law (**Definition** 3.1.3), or "geometry group law", on E defined by intersections of E and some lines.

This section will focus on E over a complex field $\mathbb{C}$. One important result, transforming geometry structures into algebraic structures, is that each E is related to a lattice $\mathbb{C}/\Lambda$, where the Weierstrass $\wp$-function (**Definition** 3.2.6) connects the two forms of expressions. Therefore an elliptic curve E over $\mathbb{C}$ is isomorphic to a torus since the lattice $\mathbb{C}/\Lambda$ has the same topology structure.

Isogeny (**Definition** 3.3.1), or isomorphism up to a finite kernel between two elliptic curves E_1, E_2, is another important subject. Each isogeny ϕ has its dual $\hat{\phi}$ (**Definition** 3.3.4). For E over $\mathbb{C}$, the isogeny is also a homomorphism of lattices $\mathbb{C}/\Lambda_1, \mathbb{C}/\Lambda_2$.

3.1 General Knowledge

3.1.1 Weierstrass Equations

The general definition of elliptic curves is based on the genus, where O in the definition is "0" in the group law shown after. For calculation, each E can be associated to a Weierstrass equation.

Definition 3.1.1. *An elliptic curve is a pair (E, O), where E is a non-singular curve of genus 1, $O \in E$. Write E/K if E is defined over K as a curve and $O \in E(K)$.*

For any elliptic curve E, there exists $a_1, ..., a_6 \in K$ and base point $O = [0, 1, 0]$, that E has an equation of the form

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3.$$

Let $x = X/Z, y = Y/Z$

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

There are simple forms

$$E : y^2 = 4x^3 + b_2x^2 + 2b_4x + b_6, char(K) \neq 2$$

$$E : y^2 = 4x^3 - 27c_4x - 54c_6, char(K) \neq 2, 3$$

where

$$b_2 = a_1^2 + 4a_2, b_4 = 2a_4 + a_1a_3, b_6 = a_3^2 + 4a_6$$
$$c_4 = b_2^2 - 24b_4, c_6 = -b_2^3 + 36b_2b_4 - 216b_6.$$

Proposition 3.1.2. *Let E be an elliptic curve defined over K.*

1. *For a given (E, O), there is an isomorphism*

$$\phi : E \to \mathbb{P}^2, \phi = [x, y, 1]$$

map E/K onto a curve given by the Weierstrass equation

$$C : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

where $x, y \in K(E), a_1, ..., a_6 \in K, \phi(O) = [0 : 1 : 0]$.

2. *For any two Weierstrass equations of E, there exists a transformation*

$$x = u^2x' + r, y = u^3y' + su^2x' + t$$

where $r, s, t \in K, u \in K$.

3. *Each Weierstrass equation gives E/K with the base point $O = [0 : 1 : 0]$.*

Proof. See [1, p.59, Proposition 3.1].

□

3.1.2 Group Law

E under the composition law, or "geometry group law", forms an Abelian group. The multiplying-by-m map $[m]$ of any point $P \in E$ is defined based on this group structure.

Definition 3.1.3. *Composition Law.* *The following process gives the group law of $P, Q \in E$:*
For the line $L_1 \ni P, Q$ ($P = Q$ then L is tangent at P), there exists a third intersection point $R \in L_1$. The line $L_2 \ni R, O$ intersects E at a third point $S \in L_2$ as the sum of P, Q, with the notation

$$P \oplus Q := S$$

Proposition 3.1.4. *For the composition law of E:*

1. *$P \oplus O = P$.*
2. *$P \oplus Q = Q \oplus P$.*
3. *An inverse $\ominus P$ exists for any P, satisfying $P \oplus (\ominus P) = O$.*

4. $(P \oplus Q) \oplus R = P \oplus (Q \oplus R)$.

Proof. See [1, p.51, Proposition 2.2].

□

Notation. $\oplus$ and $\ominus$ will be written $+$ and $-$. For $m \in \mathbb{Z}$ and $P \in E$, let

$$[0]P = O, [m]P - [m-1]P = P$$

That is, $[m]$ is the map of multiplying by m.

For given E, the composition law relates to divisors. The relation connects the geometry and algebraic structure of E.

Lemma 3.1.5. *For $P, Q \in C$ a curve of genus 1, the divisors $(P) \sim (Q)$ if and only if $P = Q$.*

Proof. $\Leftarrow$: Obvious.
$\Rightarrow$: Since C has genus 1, $l(Q) = \deg(Q) = 1$ by the Riemann–Roch theorem. If $(P) \sim (Q)$, then for any f satisfies

$$div(f) = (P) - (Q)$$

that is $f \in L(Q)$. $f \in K$ since $H^0(0) \subset H^0(Q)$ and both of dimension 1, thus $P = Q$.

□

Proposition 3.1.6. *For a given (E, O) with genus 1,*

1. *There exists σ mapping $D \in Div^0(E)$ to $\sigma(D) = P \in E$*

$$\sigma : Div^0(E) \mapsto E, s.t. D \sim (P) - (O).$$

2. *There exists a bijection*

$$\kappa : E \xrightarrow{\sim} Pic^0(E), P \mapsto divisor\ class\ of\ (P) - (O)$$

.

3. *For the Weierstrass equation of E, σ induces a homomorphism from the composition law to $Pic^0(E)$.*

Proof.

1. $l(D + (O)) = 1$ by The Riemann–Roch theorem. Let $0 \neq f \in L(D + (O))$, since $div(f) \geq -D - (O)$ and $\deg(div(f)) = 0$, there exists a $P \in E$,

$$div(f) = -D - (O) + (P).$$

Thus there is a point P satisfies $D \sim (P) - (O)$.

2. The map is injective by **Lemma** 3.1.5 since

$$\kappa(P) = \kappa(Q) \Leftrightarrow (P) - (O) \sim (Q) - (O) \Leftrightarrow (P) \sim (Q) \Leftrightarrow P = Q.$$

The map is surjective since σ gives κ^{-1} by letting $D = (P) - (O) \in Div^0(C)$.

3. It suffices to show that

$$\kappa(P+Q) = \kappa(P) + \kappa(Q).$$

For (E, O) in $\mathbb{P}^2$, let

$$f(X,Y,Z) = \alpha X + \beta Y + \gamma Z = 0.$$

give the line L intersects E at P, Q, R, and let

$$f'(X,Y,Z) = \alpha' X + \beta' Y + \gamma' Z = 0$$

be the line L' intersects E at $P+Q, R, O$.

By the fact that the line $Z = 0$ intersects E at O with multiplicity 3, we have

$$div(f/Z) = (P) + (Q) + (R) - 3(O)$$

$$div(f'/Z) = (R) + (P+Q) - 2(O).$$

Then $(P+Q)-(P)-(Q)+(O) = div(f'/f) \sim 0$, thus $\kappa(P+Q)-\kappa(P)-\kappa(Q) = 0$.

□

3.2 Elliptic Curves over $\mathbb{C}$

3.2.1 Elliptic Functions

From now on, Λ will always be a lattice, a discrete subgroup of $\mathbb{C}$, in this section. The fundamental parallelogram is a single "unit" of a given lattice Λ. Elliptic functions are periodic functions on a lattice Λ.

Definition 3.2.1. *A lattice Λ with bases $\omega_1, \omega_2 \in \mathbb{C}$ is defined as*

$$\Lambda := \{k_1\omega_1 + k_2\omega_2 : k_1, k_2 \in \mathbb{Z}\}.$$

A fundamental parallelogram for Λ is a set

$$D := \{a + t_1\omega_1 + t_2\omega_2 : 0 \le t_1, t_2 < 1\}$$

where $a \in \mathbb{C}$ and $\{\omega_1, \omega_2\}$ is a basis for Λ. Note that the natural map $D \to \mathbb{C}/\Lambda$ is bijective.

Definition 3.2.2. *An elliptic function $f(z)$ relative to a lattice Λ is a meromorphic function on $\mathbb{C}$ that satisfies*

$$f(z+\omega) = f(z), \forall \omega \in \Lambda, z \in \mathbb{C}.$$

The set of all such $f(z)$ is denoted by $\mathbb{C}(\Lambda)$, which is a field.

The following proposition is similar to Liouville's theorem [4, p.122] in complex analysis.

Proposition 3.2.3. *A holomorphic elliptic function, or an elliptic function with no poles, is constant. Similarly, an elliptic function with no zeros is constant.*

Proof. For any holomorphic elliptic function $f(z)$ on the fundamental parallelogram $\bar{D}$, there is

$$\sup_{\mathbb{C}} f(z) = \sup_{\bar{D}} f(z).$$

$f(z)$ is continuous on the compact set $\bar{D}$, so it is bounded on $\bar{D}$, even on $\mathbb{C}$. Thus $f(z)$ is a constant by Liouville's theorem.

□

Notation. The notation $\sum_{w \in \mathbb{C}/\Lambda}$ denotes a sum over a fundamental parallelogram D for Λ. By implication, w is independent of the choice of D, and only finitely many terms of the sum are nonzero.

Divisors for Λ over $\mathbb{C}$ are more concrete than the general situation. The order of a function in $\mathbb{C}(\Lambda)$ can be defined after the theorem below showing the number of poles and number of zeros are equal.

Theorem 3.2.4. *For $f \in \mathbb{C}(\Lambda)$,*

1. $\sum_{w \in \mathbb{C}/\Lambda} res_w(f) = 0.$
2. $\sum_{w \in \mathbb{C}/\Lambda} ord_w(f) = 0.$
3. $\sum_{w \in \mathbb{C}/\Lambda} ord_w(f) w \in \Lambda.$

Proof. See [1, p.162, Theorem 2.2].

□

Definition 3.2.5. *For an elliptic function $f(z)$, the order $\sum_{w \in D} ord_w(f)$ is the order of zeros, which equals the order of poles, of f at w in a fundamental parallelogram D.*

The divisor group on $\mathbb{C}/\Lambda$, denoted as $Div(\mathbb{C}/\Lambda)$, consists of the formal sums

$$\sum_{w \in \mathbb{C}/\Lambda} n_w(w), n_w \in \mathbb{Z}$$

where $n_w = 0$ for all but finitely many w.

For $D = n_w(w) \in Div(\mathbb{C}/\Lambda)$, define degree of D

$$\deg D = \sum n_w$$

and 0-degree divisor group

$$Div^0(\mathbb{C}/\Lambda) = \{D \in Div(\mathbb{C}/\Lambda) : \deg D = 0\}.$$

The principal divisor of $f \in \mathbb{C}(\Lambda)$ is

$$div(f) = \sum_{w \in \mathbb{C}/\Lambda} ord_w(f)(w) \in Div^0(\mathbb{C}/\Lambda).$$

Which gives a homomorphism

$$div : \mathbb{C}(\Lambda) \to Div^0(\mathbb{C}/\Lambda).$$

3.2.2 Weierstrass $\wp$-Function

One important fact of the elliptic curves over $\mathbb{C}$ is that each of them is associated to an elliptic function. Weierstrass $\wp$-function connects the two forms. In this part about the Weierstrass $\wp$-function, the lattice Λ is fixed.

Definition 3.2.6. *Let $\Lambda \subset \mathbb{C}$ be a lattice. The Weierstrass $\wp$-function for Λ is*

$$\wp(z;\Lambda) := \frac{1}{z^2} + \sum_{0 \neq w \in \Lambda} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2}\right).$$

The following theorem seems obvious. However, the convergence needs to be checked as proven in the reference.

Theorem 3.2.7. *The Weierstrass $\wp$-function is an even elliptic function.*

Proof. See [1, p.165, Theorem 3.1].

□

The theorem below shows the relation between Weierstrass $\wp$-function and elliptic functions. That is, elliptic functions can be represented by $\wp$. The main idea is to construct a function $g(z)$ in the form of $\wp$ having the same zeros and poles as the elliptic function $f(z)$, then by **Proposition** 3.2.3 there exists a constant c s.t. $g = cf$.

Theorem 3.2.8. *Let $\Lambda \subset \mathbb{C}$ be a lattice, every elliptic function is a rational combination of $\wp, \wp'$, i.e.*

$$\mathbb{C}(\Lambda) = \mathbb{C}(\wp(z), \wp'(z)).$$

Proof. For an elliptic function $f(z) \in \mathbb{C}(\Lambda)$

$$f(z) = \frac{1}{2}(f(z) + f(-z)) + \frac{1}{2}(f(z) - f(-z)).$$

It suffices to prove the theorem for functions that are either odd or even. Further, if $f(z)$ is odd, then $f(z)\wp'(z)$ is even by **Theorem** 3.2.7, so we are reduced to the case that f is an even elliptic function.

Let D be the fundamental parallelogram $\{t_1\omega_1 + t_2\omega_2 : t_1, t_2 \in [-\frac{1}{2}, \frac{1}{2}]\}$ for Λ, and let H be the positive "half" $\{t_1\omega_1 + t_2\omega_2 : t_1 \in [0, \frac{1}{2}], t_2 \in [-\frac{1}{2}, \frac{1}{2}]\}$ of D.
Suppose the divisor of f has the form

$$\sum_{w \in H} n_w((w) + (-w)) \tag{3.1}$$

for certain integers $n_w \in \mathbb{Z}$. Consider the function

$$g(z) = \prod_{w \in H \setminus \{0\}} (\wp(z) - \wp(w))^{n_w}.$$

The divisor of $\wp(z) - \wp(w)$ is $(w) + (-w) - 2(0)$, so we see that f and g have the same zeros and poles except possibly at $w = 0$, and by **Theorem** 3.2.4 they have the same order at 0. Then $f(z)/g(z)$ is a holomorphic elliptic function, thus a constant by **Proposition** 3.2.3. Therefore there exists a constant c

$$f(z) = cg(z) \in \mathbb{C}(\wp(z), \wp'(z)).$$

Thus it suffices to show (3.1).

The assumption that f is even implies for every $w \in \mathbb{C}$

$$ord_w f = ord_{-w} f.$$

So 3.1 holds for $2w \in D \setminus \Lambda$. The rest is to show that $ord_w f$ is even for $2w \in \Lambda$, or $w \equiv -w \ (mod\ \Lambda)$. To see this, we differentiate $f(z) = f(-z)$ repeatedly to obtain

$$f^{(i)}(z) = (-1)^i f^{(i)}(-z).$$

If $2w \in \Lambda$, then $f^{(i)}(z)$ has the same value at w and $-w$, so

$$f^{(i)}(w) = f^{(i)}(w - 2w) = f^{(i)}(-w) = (-1)^i f^{(i)}(w).$$

$f^{(i)}(w) = 0$ for odd values of i, so $ord_w f$ is even. Thus 3.1 holds.

□

The rest of this part is the relation between $\wp$ and $E/\mathbb{C}$. The theorem below shows $\wp, \wp'$ satisfies the Weierstrass equation. If the defined curve of $\wp, \wp'$ is non-singular, by **Theorem** 3.1.2 the relation gives an elliptic curve. This is done at the end of this part. Therefore we have $\wp$, a bridge connecting $\mathbb{C}(\Lambda)$ and $E/\mathbb{C}$.
Theorem 3.2.9. *For all $z \in \mathbb{C}/\Lambda$, $\wp(z)$ and its derivative $\wp'(z)$ satisfy the relation*

$$\wp'(z)^2 = 4\wp(z)^3 - 60G_4\wp(z) - 140G_6$$

where for $k \in \mathbb{Z}^+$,

$$G_{2k} = \sum_{w \in \Lambda/\{0\}} w^{-2k}.$$

Proof. See [1, p.169, Theorem 3.5].

□

Notation. Standard notations are

$$g_2 = g_2(\Lambda) = 60G_4(\Lambda), g_3 = g_3(\Lambda) = 140G_6(\Lambda).$$

Then the relation of $\wp(z), \wp'(z)$ is

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3.$$

Proposition 3.2.10. 1. *Roots for the polynomial*

$$f(x) = 4x^3 - g_2x - g_3$$

are distinct. Thus the discriminant $\Delta(\Lambda) = g_2^3 - 27g_3^2$ is nonzero.

2. *Let $E/\mathbb{C}$ be the elliptic curve*

$$E : y^2 = 4x^3 - g_2x - g_3.$$

Then there exists a complex analytic isomorphism

$$\phi : \mathbb{C}/\Lambda \to E(\mathbb{C}) \subset \mathbb{P}^2(\mathbb{C}),\ z \mapsto [\wp(z), \wp'(z), 1].$$

Proof. See [1, p.170, Proposition 3.6].

□

3.3 Maps of Elliptic Curves

3.3.1 Isogeny

A special class of morphisms between E_1, E_2 is an isogeny ϕ, as the homomorphism of group structures on E_1, E_2.
Definition 3.3.1. *An isogeny is a morphism of elliptic curves* $\phi : E_1 \to E_2, \phi(O) = O$*, and* E_1, E_2 *are isogenous if* $\phi(E_1) \neq \{O\}$.
Theorem 3.3.2. *For an isogeny* $\phi : E_1 \to E_2$*,* $\phi(P+Q) = \phi(P) + \phi(Q)$ *for all* $P, Q \in E_1$.

Proof. Obvious when $\phi(E_1) = \{O\}$. Otherwise, ϕ induces a homomorphism of the divisor group

$$\phi_* : Pic^0(E_1) \to Pic^0(E_2), class\ of\ \sum n_i(P_i) \mapsto\ class\ of\ \sum n_i(\phi P_i).$$

On the other hand, we have group isomorphisms by **Theorem** 3.1.6

$$\kappa_i : E_i \to Pic^0(E_i), P \mapsto class\ of\ (P) - (O).$$

Then there is the commutative diagram:

$$\begin{array}{ccc} E_1 & \xrightarrow[\kappa_1]{\cong} & Pic^0(E_1) \\ \Big\downarrow \phi & & \Big\downarrow \phi_* \\ E_1 & \xrightarrow[\kappa_2]{\cong} & Pic^0(E_2) \end{array}$$

κ_1, κ_2, and ϕ_* are all group homomorphisms. Thus ϕ is also a homomorphism. $\square$

By convention, set $\deg[0] = 0$. Thus for all chains of isogenies $E_1 \xrightarrow{\phi} E_2 \xrightarrow{\psi} E_3$,

$$\deg(\psi \circ \phi) = \deg(\psi)\deg(\phi).$$

3.3.2 The Dual Isogeny

The pullback of ϕ, a map of degree 0 Picard groups,

$$\phi^* : Pic^0(E_2) \to Pic^0(E_1)$$

induces a special isogeny $\hat{\phi}$, the dual isogeny, as follows.
Theorem 3.3.3. *Let* $\phi : E_1 \to E_2$ *be a nonconstant isogeny of degree* m.
1. *There exists a unique isogeny* $\hat{\phi} : E_2 \to E_1, \hat{\phi} \circ \phi = [m]$.
2. *As a group homomorphism,* $\hat{\phi}$ *equals*

$$\begin{array}{ccc} E_2 \to Div^0(E_2) & \xrightarrow{\phi^*} & Div^0(E_1) \overset{sum}{\to} E_1 \\ Q \mapsto (Q) - (O) & & \sum n_P(P) \mapsto \sum [n_P]P \end{array}$$

.

Proof. See [1, p.81, Theorem 6.1].

□

Definition 3.3.4. *For an isogeny $[0] \neq \phi : E_1 \to E_2$ be, the dual isogeny is $\hat{\phi} : E_2 \to E_1$ given by* ***Theorem*** *3.3.3.*
The theorem below is about the operational laws of dual.
Theorem 3.3.5. *Let $\phi : E_1 \to E_2$ be an isogeny of degree m, $\lambda : E_2 \to E_3, \psi : E_1 \to E_2$ are other isogenies*

1. $\hat{\phi} \circ \phi = [m]$ *on* $E_1, \phi \circ \hat{\phi} = [m]$ *on* E_2.
2. $\widehat{\lambda \circ \phi} = \hat{\phi} \circ \hat{\lambda}$.
3. $\widehat{\phi + \psi} = \hat{\phi} + \hat{\psi}$.
4. $\widehat{[d]} = [d], \deg[d] = d^2, \forall d \in \mathbb{Z}$.
5. $\deg \hat{\phi} = \deg \phi$.
6. $\hat{\hat{\phi}} = \phi$.

Proof. See [1, p.83, Theorem 6.2].

□

3.3.3 Maps on $\mathbb{C}$

Let Λ_1 and Λ_2 be lattices in $\mathbb{C}$, and suppose that $\alpha\Lambda_1 \subset \Lambda_2$ for some $\alpha \in \mathbb{C}$. Then scalar multiplication by α induces a holomorphic homomorphism

$$\phi_\alpha : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2, \phi_\alpha(z) = \alpha z \ (mod\ \Lambda_2).$$

ϕ_α gives all the possible isogenies as the theorem below.
Theorem 3.3.6. *For holomorphic maps ϕ,*

1. *There is a bijection*

$$\begin{array}{ccc} \{\alpha \in \mathbb{C} : \alpha\Lambda_1 \subset \Lambda_2\} & \to & \{\mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2 \ with\ \phi(0) = 0\} \\ \alpha & \to & \phi_\alpha \end{array}$$

.

2. *For E_1 and E_2 corresponding to lattices Λ_1 and Λ_2, the natural inclusion induces a bijection*

$$\{isogenies\ \phi :\ E_1 \to E_2\} \to \{holomorphic\ maps\ \phi : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2, \phi(0) = 0\}.$$

Proof. 1. To show it is bijective, sufficient to show there is an inverse. That is, any $\phi_0 : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2, \phi_0(0) = 0$ associates to a unique α, or ϕ_α. There exists a unique $\alpha \in \mathbb{C}$ such that $\phi_0(1) = \alpha = \phi_\alpha(1)$. Let $\phi := \phi_0 - \phi_\alpha$, since $\mathbb{C}$ is simply connected, we can lift ϕ to a holomorphic map $f : \mathbb{C} \to \mathbb{C}, f(0) = 0$ so that the following diagram commutes:

$$\begin{tikzcd} \mathbb{C} \arrow[r, "f"] \arrow[d] & \mathbb{C} \arrow[d] \\ \mathbb{C}/\Lambda_1 \arrow[r, "\phi"] & \mathbb{C}/\Lambda_2 \end{tikzcd} .$$

Thus
$$f(z+\omega) \equiv f(z) \ (mod\ \Lambda_2)\ for\ all\ \omega \in \Lambda_1\ and\ all\ z \in \mathbb{C}.$$
The difference $f(z+\omega) - f(z)$ must be independent of z. Differentiating, we find that
$$f'(z+\omega) = f'(z)\ for\ all\ \omega \in \Lambda_1\ and\ all\ z \in \mathbb{C}$$
as a holomorphic elliptic function, $f'(z)$ is constant by **Proposition** 3.2.3, so $f(z) = \beta z + \gamma$ for some $\beta, \gamma \in \mathbb{C}$. The assumption that $f(0) = 0$ implies that $\gamma = 0$, since $\phi(z) = \phi_0(z) - \phi_\alpha(z)$,
$$f(1) = \phi(1) = \phi_0(1) - \phi_\alpha(1) = \alpha - \alpha = 0$$
tells us that $\beta = 0$, so $\phi = f = 0$. Thus $\phi_0 = \phi_\alpha$.

2. See [1, p.171 , Theorem 4.1].

□

The corollary below results from the association of $E/\mathbb{C}$ and $\mathbb{C}(\Lambda)$ given by $\wp(z)$.
Corollary 3.3.7. *Let $E_1/\mathbb{C}$ and $E_2/\mathbb{C}$ be elliptic curves corresponding to lattices Λ_1 and Λ_2. Then E_1 and E_2 are isomorphic over $\mathbb{C}$ if and only if Λ_1 and Λ_2 are homothetic, i.e., there exists some $\alpha \in \mathbb{C}^*$ such that $\Lambda_1 = \alpha\Lambda_2$.*

Proof. Omitted.

□

The relation between $E/\mathbb{C}$ and $E(\Lambda)$ is summarized as follows theorems.
Theorem 3.3.8. *(Uniformization Theorem) For $A, B \in \mathbb{C}, A^3 - 27B^2 = 0$, there exists a unique lattice $\Lambda \subset \mathbb{C}$ satisfying $g_2(\Lambda) = A$ and $g_3(\Lambda) = B$.*

Proof. See [1, p.173, Theorem 5.1].

□

Theorem 3.3.9. *The following categories are equivalent:*

1. *Objects: Elliptic curves over $\mathbb{C}$.*
   Maps: Isogenies.
2. *Objects: Elliptic curves over $\mathbb{C}$.*
   Maps: Complex analytic maps taking O to O.
3. *Objects: Lattices $\Lambda \subset \mathbb{C}$, up to homothety.*
   Maps: $Map(\Lambda_1, \Lambda_2) = \{\alpha \in \mathbb{C} : \alpha\Lambda_1 \subset \Lambda_2\}$.

Proof. See [1, p.175, Theorem 5.3].

□

4

Intersection Theory of Surfaces

A topic for algebraic surfaces S is the properties of intersection points P of two curves $C, D \subset S$. The situation of the intersection theory of a surface is studied in this section, for the last result of bielliptic surfaces.

There are two results of the intersection theory in this section. The first is self-intersection. It is derived from a generalization form of intersection in the Picard group. For surfaces in $\mathbb{P}^2$, the second result is Bézout's theorem giving the intersection number depends on the degree of curves.

A curve is irreducible if its divisor has only one component, and it is reduced if the coefficients of all the irreducible components are 1. Two distinct reduced curves are curves that have no common component and no nontrivial factorization of components. In this section, C, D are distinct reduced irreducible curves on a smooth projective surface S over K. C, D are defined by equations $f, g \in K(S)$ locally, that is,

$$\forall P \in C(or\ D), \exists U \subset S, s.t. P \in U$$

where $C|_U(or\ D|_U)$ is defined by $f|_U(or\ g|_U)$, U is open. Define the function in the coordinate ring

$$\bar{g} := g\ mod\ I(C).$$

4.1 Intersection Number

The multiplicity needs to be defined to introduce the intersection number. The properties of the intersection number will be shown afterward to illustrate the pairing structure.

Definition 4.1.1. *For distinct irreducible curves C, D of equations f, g, the multiplicity of a single point P is*

$$m_P(C, D) := \dim_K O_{X,P}/(f, g)$$

where $O_{X,P}$ is the localization of regular functions in $k[X]$ at P.

Remark 4.1.2. *Notice that m_P is well-defined. At P, C, D might be locally defined by different $f_1 \neq f_2, g_1 \neq g_2$, where $f_1, f_2, g_1, g_2 \in K(X)$. But the ideals*

$$(f_1) = O_{C,P} = (f_2), (g_1) = O_{D,P} = (g_2).$$

Therefore

$$(f_1, g_1) = ((f_1), (g_1)) = ((f_2), (g_2)) = (f_2, g_2).$$

Thus

$$m_P(C, D) = \dim_K O_{X,P}/(f_1, g_1) = \dim_K O_{X,P}/(f_2, g_2).$$

Thus $m_P(C, D)$ is well-defined.

Example 4.1.3. *One simple situation of the multiplicity is $\{y = x^2\} \cap \{y = 0\}$ at $(0, 0)$ is 2, as the part of the intersection theory in the first section (introduction). There is a more complex but similar example. Consider the multiplicity at a point P for two curves C, D defined by functions f, g. Let $X = K[x, y], f = x, P = (0, 0), g = \sum_{j,k} a_{j,k} x^j y^k, a_{jk} \in K$, then*

$$O_{X,P}/(f, g) = O_{\mathbb{A}^2,(0,0)}/(x, \sum_{j,k} a_{j,k} x^j y^k) = O_{\mathbb{A},(0)}/(\sum_k b_k y^k)$$

where $b_k = a_{0,k}$. Thus

$$m_P(C, D) = \dim_K O_{X,P}/(f, g) = \min\{k : b_k \neq 0\}.$$

Definition 4.1.4. *For distinct irreducible curves C, D, the intersection number is*

$$C \cdot D := \sum_{P \in C \cap D} m_P(C, D).$$

The intersection number defines a symmetric and bilinear pairing of the Picard group

$$(Pic(S), Pic(S)) \to \mathbb{Z}.$$

There is also an equivalence relation based on such a pairing, as the theorem below.

Theorem 4.1.5. *For any $g \in K(X)$ and curve C in X, if $div(g)$ is distinct from C, then*

$$C \cdot div(g) = 0.$$

Proof. Since C is locally defined by f, and $\bar{g} = g \ mod \ I(C)$ is well-defined for C being distinct from $div(g)$,

$$\begin{aligned} m_P(C, div(g)) &= \dim O_{X,P}/(f, g) = \dim(O_{X,P}/(f))/(g) = \dim O_{C,P}/(\bar{g}) \\ &= ord_P(\bar{g}) \end{aligned}$$

.

Therefore

$$C \cdot div(g) = \sum_{C \cap div(g)} m_P(C, div(g)) = \sum_P ord_P \bar{g} = \deg \bar{g}.$$

Thus by **Proposition** 2.2.3

$$C \cdot div(g) = \deg \bar{g} = 0.$$

□

Remark 4.1.6. *The intersection number $(C, D) \to C \cdot D$ is defined for distinct C, D, but it is not enough for more general situations. Factoring $C = \sum_j a_j R_j, D = \sum_k b_k R_k$ into irreducible components, then*

$$C \cdot D = (\sum_j a_j R_j) \cdot (\sum_k b_k R_k) = \sum_{j,k} (a_j b_k) R_j \cdot R_k.$$

Now $R_j \cdot R_k$ is well defined for $R_j \neq R_k$, but the self-intersection is needed for the general $C \cdot D$. That is, there should be a definition for the form

$$R_j^2 = R_j \cdot R_j.$$

4.2 Self-Intersection

To give a well-defined self-intersection structure, the following is necessary.

Proposition 4.2.1. *For any curve C, the intersection number*

$$C \cdot (C - div(g)), g \in K(X)$$

is well-defined. That is,

1. *Existence: there exists some functions g, that $C, C - div(g)$ are distinct.*
2. *Consistency: for any g_j, g_k,*

$$C \cdot (C - div(g_j)) = C \cdot (C - div(g_k)).$$

Proof.
1. Existence:
   There is a uniformizer g by [1, p.18, Remark 1.1.1], s.t.

$$ord_C(g) = 1, g \in K(C).$$

   So $C \subset div(g)$, and

$$(div(g) - C), C \text{ not distinct} \Rightarrow ord_C(g) > 1.$$

   Thus $C, C - div(g)$ are distinct.
2. Consistency:
   By **Corollary** 4.1.5

$$\begin{aligned} &C \cdot (C - div(g_j)) - C \cdot (C - div(g_k)) \\ &= -C \cdot (div(g_j) - div(g_k)) = -C \cdot (div(g_j - g_k)) = 0 \end{aligned}$$

.

□

Definition 4.2.2. *The self-intersection of C is*

$$C^2 := C \cdot (C - div(g)), g \in K(X).$$

Remark 4.2.3. *The structure of pairing can be illustrated, the intersection number* $(C, D) \to C \cdot D$ *is a pairing*

$$Pic(S) \times Pic(S) \to \mathbb{Z}.$$

*The equivalence relation, which is well-defined by **Theorem** 4.1.5, is*

$$0 \sim div(f), f = g/h \in K(S).$$

It is easy to check that such an equivalence relation is well-defined for projective space.

4.3 Bézout's theorem

A lemma is introduced to give a simplified version of Bézout's theorem.
Lemma 4.3.1. *Any plane curve* $C \subset \mathbb{P}^2$ *is defined by a homogeneous polynomial* F.

Proof. Firstly, there are some polynomials F, s.t. $C \subset div(F_j)$. To illustrate the statement, let

$$C' := C|_{Z=1} = C \cap \mathbb{A}^2.$$

By [2, p.7, Proposition 1.13], C' is defined by an irreducible polynomial

$$f \in K[x, y], x = X/Z, y = Y/Z.$$

There exist some homogeneous polynomials $F_j \in K[X, Y, Z], j \in \mathbb{Z}$, where $f = F_j|_{Z=1}$. There is $C \subset div(F_j)$, since $C'|_{Z=1} = C|_{Z=1}$,

$$(X, Y, Z) \in C \Rightarrow (x, y, 1) = (X/Z, Y/Z, 1) \in C', \ for\ Z \neq 0$$

where C' is defined by f. Notice that $div(F_j)$ is closed, by taking its boundary, all the points of $\{Z = 0\} \cap C$ are also taken.

The rest is to show, there exists a F_j, s.t. $C = div(F_j)$. For $j, k \in \mathbb{Z}, F_j \neq F_k$,

$$(F_j/F_k)|_{Z=1} = (F_j|_{Z=1})/(F_k|_{Z=1}) = f_j/f_k = 1.$$

Since only the constant function in $K[x, y]$ can always be 1, F_j/F_k has only factors without X, Y but Z, so

$$F_j = Z^a F_k.$$

Therefore C is defined by the irreducible F without any factor Z, which means $\deg F = \deg f$. Otherwise, there is another component other than the curve.

□

Theorem 4.3.2. *(Bézout's theorem) For distinct plane curves* $C, D \in \mathbb{P}^2$*, the intersection number is*

$$C \cdot D = ab, a = \deg C, b = \deg D.$$

Proof. C, D are given by nonzero functions F, G by the lemma, where $\deg F = a, \deg G = b$. Since $C \in \mathbb{P}^2$, for line $L : X = 0$, s.t. F/L^a is a homogeneous rational function, then by **Remark** 4.2.3

$$[C] = [F/L^a \cdot L^a] = [F/L^a] + [L^a] = a[L].$$

Similarly, for D there is a distinct line $M : Y = 0$, s.t.

$$[D] = b[M].$$

Thus

$$C \cdot D = aL \cdot bM = (ab)L \cdot M.$$

If $L : X = 0, M : Y = 0$, then $L \cdot M = 1$ by 4.1.3. Thus

$$C \cdot D = ab.$$

□

5

Biellipitic Surfaces

Bielliptic surfaces are constructed by elliptic curves. The classification of bielliptic surfaces, which won't be proved here, is based on such structure. These results can conclude the intersection theory of bielliptic surfaces.

Only some simple introductions of bielliptic surfaces are illustrated to avoid too many other topics. The first thing is the structure: bielliptic surfaces have the form $E \times F/G$, the product of two elliptic curve quotients in a group (**Definition** 5.1.1). The next part focuses on the Néron–Severi group (**Definition 5.2.1**), an important concept in intersection theory. The structure of the Néron–Severi group is also introduced (**Theorem** 5.2.3). Based on these facts, there is the result of the intersection theory of S (**Proposition** 5.3.2).

In this section, S is always a bielliptic surface, E, F are elliptic curves, G is the group acting on S, and $a : G \to Aut(F)$ is always an injective homomorphism.

5.1 Introduction

The structure of bielliptic surfaces is $S = (E \times F)/G$, where E, F are elliptic curves and G is the group acting on S. Under the equivalence relation of G, bielliptic surfaces can be classified into finite classes.

Definition 5.1.1. *For any bielliptic surface S, there exist elliptic curves E, F and the group G acting on S, s.t.*

$$S = (E \times F)/G$$

where for some injective homomorphism $a : G \to Aut(F)$, $g \in G$ acting on $(x, y) \in (E, F)$ is

$$g(x, y) = (x + g, a(g)y).$$

There are two maps of S

$$f : S \to E/G, g : S \to F/a(G) \cong \mathbb{P}^1.$$

Proposition 5.1.2. *There exists a finite subgroup G_0 of translations of F, and an isomorphism*

$$a(G) \cong G_0 \times \mathbb{Z}/n\mathbb{Z}, n \in \{2, 3, 4, 6\}$$

for some injective homomorphism $a : G \to Aut(F)$. For $char(K) = 0$, the classification of bielliptic curves is listed in the table 5.1.

Type	G	G_0	n
(a_1)	$\mathbb{Z}/2\mathbb{Z}$	$\{0\}$	2
(a_2)	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/2\mathbb{Z}$	2
(b_1)	$\mathbb{Z}/3\mathbb{Z}$	$\{0\}$	3
(b_2)	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathbb{Z}/3\mathbb{Z}$	3
(c_1)	$\mathbb{Z}/4\mathbb{Z}$	$\{0\}$	4
(c_2)	$(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z})$	$\mathbb{Z}/2\mathbb{Z}$	4
(d)	$\mathbb{Z}/6\mathbb{Z}$	$\{0\}$	6

Table 5.1: Classification of bielliptic curves

Proof. See [3, p.20, List 1.17].

□

Remark 5.1.3. *The actions of group G on S are listed below, where the additions are defined as the composition law (**Definition** 3.1.3).*
(a_1) *There exists a nontrivial 2-torsion point $a \in E$, s.t.*

$$G \cong (\mathbb{Z}/2\mathbb{Z}) \cong \langle a \rangle, a.(x, y) = (x + a, -y).$$

(a_2) *There exists nontrivial 2-torsion points $a, b, c \in E$, s.t.*

$$G \cong (\mathbb{Z}/2\mathbb{Z})^2 \cong \langle b \rangle \times \langle a \rangle, a.(x, y) = (x + a, -y), b.(x, y) = (x + b, y + c).$$

(b_1) *There exists a nontrivial 3-torsion point $a \in E$, where $\alpha(a) = \omega$ is an automorphism of order 3, s.t.*

$$G \cong (\mathbb{Z}/3\mathbb{Z}) \cong \langle a \rangle, a.(x, y) = (x + a, \omega(y)).$$

(b_2) *There exists nontrivial 2-torsion points $a, b \in E$, where $\alpha(a) = \omega$ is an automorphism of order 3, s.t.*

$$G \cong (\mathbb{Z}/3\mathbb{Z})^2 \cong \langle b \rangle \times \langle a \rangle, a.(x, y) = (x + a, \omega(y)), b.(x, y) = (x + b, y + c).$$

(c_1) *There exists a nontrivial 4-torsion point $a \in E$, s.t.*

$$G \cong (\mathbb{Z}/4\mathbb{Z}) \cong \langle a \rangle, a.(x, y) = (x + a, i(y)).$$

(c_2) *There exists nontrivial 4-torsion points $a, b \in E$, s.t.*

$$G \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z}) \cong \langle b \rangle \times \langle a \rangle, a.(x, y) = (x + a, i(y)), b.(x, y) = (x + b, y + c).$$

(d) *There exists a nontrivial 6-torsion point $a \in E$, where $\alpha(a) = \omega$ is an automorphism of order 3, s.t.*

$$G \cong (\mathbb{Z}/6\mathbb{Z}) \cong \langle a \rangle, a.(x, y) = (x + a, -\omega(y)).$$

*The composition law of E is isomorphic to the addition of $\mathbb{C}/\Lambda$ by **Proposition** 3.2.10, so one way to understand how G acts on S is to consider the addition and multiplication of $\mathbb{C}/\Lambda$. There are some examples for case (d). Consider*

$$E = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau_1), F = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau_2)$$

then

$$a.(x, y) = (x + \frac{1}{6}, e^{\frac{1}{3}\pi i}y), x \in E, y \in F.$$

F can also be the elliptic curve given by the Weierstrass equation

$$w^2 = z^3 + 1$$

G acts on F as

$$a.(z, w) = (e^{\frac{2}{3}\pi i}z, -w), (z, w) \in F$$

where there is the natural result

$$a^6.(z, w) = (z, w).$$

5.2 Néron–Severi Group

Recall that a connected set is a topology subspace that is not the union of multiple disjoint non-empty open sets. Then there is the definition of the Néron–Severi group as below.

Definition 5.2.1. *Consider a relationship for D, D' if there is a curve C,*

$$\exists \phi : S \to C, s.t. P, P' \in C, \text{ where } D = \phi^{-1}(P), D' = \phi^{-1}(P').$$

This generates the equivalence relationship $D \sim D'$, and $NS(S)$ is defined as the quotient of $Div(S)$ by this relationship.

A particular fact is that if $D \sim D'$ in the same $NS(S)$ class, then they are in the same intersection class, i.e. for any other curve R,

$$[D] = [D'], D \cdot R = D' \cdot R.$$

For the intersection theory of surfaces, an element of $NS(S)$ is just an equivalence class $[C]$, which could be better illustrated in [2, p.140, Remark 6.10.3]. The classification of $NS(S)$ by [3, p.27, Table 2.2] is listed in the table 5.2. The proposition following gives the relation between $NS(S)$ and maps f, g of bielliptic surfaces S.

Type	$NS(X)$
(a_1)	$\mathbb{Z}^2 \oplus (\mathbb{Z}/2\mathbb{Z})^2$
(a_2)	$\mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$
(b_1)	$\mathbb{Z}^2 \oplus \mathbb{Z}/3\mathbb{Z}$
(b_2)	$\mathbb{Z}^2$
(c_1)	$\mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$
(c_2)	$\mathbb{Z}^2$
(d)	$\mathbb{Z}^2$

Table 5.2: Classification of Néron–Severi group

Definition 5.2.2. *The Néron-Severi lattice is the quotient of $NS(S)$ by its torsion subgroup:*

$$Num(S) := NS(S)/NS(S)_{tors}.$$

The structure of $Num(S)$ for elliptic surfaces S is described in the next theorem.

Theorem 5.2.3. *For $S \cong (E \times F)/G$, if the characteristic of the field K is different from 2 and 3, then the classification of the Néron-Severi lattice is listed in the table 5.3*

Type	Basis of $Num(S)$
(a_1)	$\{\frac{1}{2}[E], [F]\}$
(a_2)	$\{\frac{1}{2}[E], \frac{1}{2}[F]\}$
(b_1)	$\{\frac{1}{3}[E], [F]\}$
(b_2)	$\{\frac{1}{3}[E], \frac{1}{3}[F]\}$
(c_1)	$\{\frac{1}{4}[E], [F]\}$
(c_2)	$\{\frac{1}{4}[E], \frac{1}{2}[F]\}$
(d)	$\{\frac{1}{6}[E], [F]\}$

Table 5.3: Classification of Néron-Severi lattice

In this list, the classes of E and F refer to the general fibers of g and f, which are in fact isomorphic to E and F.

Proof. See [3, p.39, Theorem 3.3].

□

5.3 Intersection Theory of Bielliptic Surfaces

In this part, $K = \mathbb{C}$.

Intersection theory could be applied to bielliptic surfaces, based on the structure of the Néron-Severi group (**Theorem** 5.2.3). The essential point is to factor $[C] \cdot [D]$ into linear combinations of $[E], [F]$ (**Remark 4.1.6**), where $[C], [D], [E], [F] \in NS(S)$. That is

$$[C] = a[E] + b[F], [D] = c[E] + d[F].$$

The intersection number is the bilinear form

$$[C] \cdot [D] = ac[E]^2 + (ad + bc)[E] \cdot [F] + bd[F]^2.$$

The next proposition is about the intersection numbers $[E]^2, [F]^2, [E] \cdot [F]$, by which there is

$$[C] \cdot [D] = (ad + bc)\#G.$$

For any divisor of curve $R = \sum_P n_P(P)$, the degree $\deg R = \sum_P n_P$. The above relation can be considered the bielliptic surfaces version of Bézout's theorem since it tells how the coefficients a, b, c, d define the intersection number.

Lemma 5.3.1. *In the Néron–Severi group $NS(S)$, all the fibers of f construct the class $[F]$, and all smooth fibers of g consist of the class $[E]$. That is, for any $P \in E/G, Q \in F/\alpha(G)$,*

$$f^{-1}(P) = [F], g^{-1}(Q) = [E]$$

where f, g are the projections

$$f : S \to E/G, g : S \to F/a(G) \cong \mathbb{P}^1.$$

Proof. For $f^{-1}(P)$, all the preimages form a class by **Remark** 5.2.1. Let 0_E be the class of 0 in E/G, then

$$F \cong \{(0_E, y) : y \in F\} \in f^{-1}(0_E).$$

Therefore, $f^{-1}(P) = [F]$. The proof of $g^{-1}(Q) = [E]$ is similar.

□

Proposition 5.3.2. *For bielliptic surface $S \cong E \times F/G$ and $[E], [F] \in NS(S)$,*

1. $$[E]^2 = [F]^2 = 0.$$

2. $$[E] \cdot [F] = \#G.$$

Proof.

1. By **Proposition** 5.3.1, $[E], [F] \in NS(S)$ consist of elements as $f^{-1}(P), g^{-1}(Q)$, where the classes are independent of choices of P, Q. Since for $P_1 \neq P_2, Q_1 \neq Q_2$, by 5.2.1

   $$f^{-1}(P_1) \cap f^{-1}(P_2) = g^{-1}(Q_1) \cap g^{-1}(Q_2) = \phi.$$

   Thus

   $$[E]^2 = [F]^2 = 0.$$

2. Denote by $R = E \times F$ and $S = (E \times F)/G$ and consider the natural projection $\sigma : R \to S$,

   $$\begin{tikzcd} & & E/G \\ R \arrow[r, "\sigma"] & S \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & & F/a(G) \end{tikzcd}.$$

   Taking inverse images of points $(P, Q) \in S$ under σ has the effect of multiplying intersection multiplicities by $\#G$. This means that

   $$\begin{aligned} \#\{(f\sigma)^{-1}(P) \times (g\sigma)^{-1}(Q)\} &= \#\{\sigma^{-1}\left(f^{-1}(P) \times g^{-1}(Q)\right)\} \\ &= \#G\#\{f^{-1}(P) \times g^{-1}(Q)\}. \end{aligned} \tag{5.1}$$

Then it is enough to understand $\#\{(f\sigma)^{-1}(P) \times (g\sigma)^{-1}(Q)\}$, for any given points P, Q in E/G and $F/a(G)$. To this effect, we describe the inverse images of $E \times F \to E/G$ and $E \times F \to F/a(G)$.
First of all, the points in the inverse image of $P \in E/G$ under the map $E \times F \to E/G$ are the points of the form $\bigcup_{g\in G} g(P_0) \times F$, where P_0 is a point in E mapping to P. Similarly, the points in the inverse image of $Q \in F/a(G)$ under the map $E \times F \to F/a(G)$ are of the form $\bigcup_{g\in G} E \times g(Q_0)$ where Q_0 is a point in F mapping to Q.
All the intersections, in total $(\#G)^2$ of them, to be computed to find the left-hand side of (5.1) are of the form $P_0 \times F$ and $E \times Q_0$.
Now, the intersection of two such objects is independent of the choice of P_0 and Q_0 since they determine the same classes in the Néron-Severi group of $E \times F$. Note that both $P_0 \times F$ and $E \times Q_0$ are locally given by equations, namely $\ell = 0$ and $\ell' = 0$, where $\ell = 0$ and $\ell' = 0$ denote local equations for P_0 and Q_0 in E and F, considered over $E \times F$.
Now the multiplicity is defined as

$$m_{(P_0,Q_0)}((P_0 \times F), (E \times Q_0)) = \dim \mathcal{O}_{E\times F, P_0, Q_0}/(\ell, \ell') = 1$$

since the two curves intersect transversally. This means that

$$\#\{(f\sigma)^{-1}(P) \cdot (g\sigma)^{-1}(Q)\} = (\#G)^2.$$

By (5.1), we find that

$$\#\{f^{-1}(P) \cdot g^{-1}(Q)\} = \#G$$

which was to be proven.

□

Bibliography

[1] Silverman, J. H. (2009). The arithmetic of elliptic curves (Vol. 106). Springer.

[2] Hartshorne, R. (2013). Algebraic geometry (Vol. 52). Springer Science & Business Media.

[3] De Narvaez, D. B. (2021). Moduli of bielliptic surfaces [Phdthesis]. Technische Universität München.

[4] Martin, D., & Ahlfors, L. (1966). Complex analysis. New York: McGraw-Hill.

DEPARTMENT OF SOME SUBJECT OR TECHNOLOGY
CHALMERS UNIVERSITY OF TECHNOLOGY
Gothenburg, Sweden
www.chalmers.se

CHALMERS
UNIVERSITY OF TECHNOLOGY