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Optimization of Extended Warranties

A Model in a Centralized Supply Chain Environment

Bachelor's thesis within the master of science education at Chalmers

Johan Bertilsson
Gustav Ljungqvist
Amir Vedadi
Annie Westerlund

Department of Mathematical Sciences
Chalmers University of Technology
Gothenburg University
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Gustav Ljungqvist Annie Westerlund

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Johan Bertilsson Amir Vedadi

Supervisor: Mahmoud Shafiee Mathematical Sciences
Examinor: Hjalmar Rosengren

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Abstract

The market of extended warranties is increasing rapidly and as customers' demand better warranty terms, the need to optimize warranty strategies for manufacturers arises accordingly. This project constructs two different types of extended warranty policies and compares their associated optimal profit. Results are obtained by solving a non-linear optimization model in MATLAB and a sensitivity analysis is conducted on some of the input parameters. The findings from the results are that, while one of the policies turns out to be the better one, both are still profitable and that there exists potential in today's market for introducing the evaluated policies on products with certain failure behavior.

Sammandrag

Marknaden för utökade garantier blir allt större. Behovet av att optimera garantistrategier ökar då kunder efterfrågar bättre garantivillkor. Det här projektet utformar två olika typer av utökade garantipolicyer och jämför deras respektive optimala vinst. Genom icke-linjär optimering och känslighetsanalys av vissa parametrar i MATLAB fås ett resultat. En slutsats är att även om den ena policyn är bättre, så är båda vinstgivande och det är på dagens marknad möjligt att introducera dessa policyer för produkter med ett visst felbeteende.

Preface

Under the development of this project, a group journal and individual time logs have been kept in order to describe the progress. The journal mostly shows the group's contribution, but also individual efforts to different tasks and assignments. Problems that occured during the project are also presented, how they were reflected upon and solved occasionally.

In general, the members have not been assigned any specific responsibilities. Instead, a full effort and a total overview from the whole group was deemed important to achieve good results. By setting up scheduled sessions for every week, all members have been present under the project's progression and been able to give their inputs directly. This opened up a more creative atmosphere and encouraged better discussions for proposals and ideas that were brought up continously.

Towards the end of the project, however, the numerical analysis were done by Annie and Gustav, while the finalization of the report were mainly done by Johan and Amir. Worth mentioning is that throughout all stages of the project, all members have been involved and taken responsibility for contribution of work and discussions.

Finally, we wish to extend our sincerest thanks to our supervisor Mahmoud Shafiee for his help which have been very useful in this project.

Contents

1 Introduction

This chapter presents the research's background, aim and purpose. Also problem definition, limitations and method are presented to give the reader an idea of what to expect from this report.

1.1 Background

Most of the products today are sold with a warranty. A product warranty is an agreement between the seller and the customer. A warranty period is a pre-specified time period during which the manufacturer is obligated to repair, replace or compensate the buyer for a failure in the product [1]. The warranty serves different purposes depending on which part that is considered, the manufacturer/seller or the customer/buyer. For the manufacturer, the warranty can act as a protection against misuse of a product or simply promotional. The manufacturer provides the required specifications for care and maintenance of the product and if something happens they will only provide coverage as long as the product has been used accordingly. For the customer, warranties are also protectional. Customers prefer long-lasting products with long warranty periods. The seller's warranty can assure this by repairing faulty products or replacing them for some cost or no cost at all.

Recently, manufacturers have been facing increasing pressure to extend the coverage period of their after sale services by offering extended warranties. In the early 1930s the warranty period for cars was normally three months. In the 1960s it changed to one year and today it could be anywhere between three and five years [2]. In today's market, these extended warranties can be found in many daily used products such as computers, cell phones, home appliances and tools. For example, Apple offers a service plan called AppleCare [3] where customers have the choice to purchase an extended warranty with up to two years of coverage on computers and monitors, and one year on iPhones, iPads and iPods. Samsung offers a service plan called Fast Guard [4] where customers have the choice to purchase an extended warranty with up to four years of coverage on notebooks, printers, monitors, TVs and home appliances.

The market for extended warranties is growing rapidly. The reason why more companies are following in each other's footsteps is not just because of the rising demand for better service from customers. In many cases it has been observed that manufacturer's profit margin on the extended warranties are actually higher than that on the products. It has been stated that the profit margin for post-sale service is roughly 30% as opposed to the 10% profit margin on initial sales [5]. Analysis on US consumer electronic retailers shows that all of Circuit City's operating profit and nearly half of Best Buy's came from warranties [6]. Profit margins on warranties were at that point between 50% and 60%. That is almost 18 times the margin on the products.

Problems arise, however, when offering better warranty terms, since this also increases the cost of service. Even though long warranties are used as advertisement and to reach a higher profit, the manufacturer has to deliver the required service as well. If the company fails, the result will have a negative impact on sales and customer satisfaction. This brings attention to identify where problems lie. A problem could for example be in the logistics chain and if the company's number of stocking points is profitable or not. There is also a problem regarding the channels of distribution, for example whether the manufacturer or the retailer should offer the warranty or not. It could also turn out to be more profitable to let a third party or agency handle the warranty servicing. The problem could also simply be the terms of the offered service plan or policy. It is clear that there is a critical need to find optimal solutions to satisfy all ends.

1.2 Purpose

The purpose of this project is to gain knowledge in how to construct and develop models for new policies of extended warranties. In addition, the project wishes to bring existing research on extended warranty policies closer to the scenarios of a market where full coverage extended warranties are offered.

1.3 Aim

This project should provide a comparison between two models of extended warranty policies in a specific setting. The first policy, often seen in the market of electronics and home appliances and the second policy, being a different type of policy that could be introduced on the market. The aim of the comparison is to present and compare the optimized versions of the two models, that is the versions that maximizes the manufacturer's profit over price and coverage of the extended warranty.

1.4 Problem

The problem is to optimize coverage and price of two different extended warranty policies. The first extended warranty policy offers unlimited repairs and replacements during a specified warranty period. The second is not limited from time but from a combined total amount of repairs and replacements during an unspecified period of time.

Objective functions are constructed seeking to maximize the total profit for the manufacturer where the total profit is the sum of profits from products sold and extended warranties sold. Price and coverage of the extended warranties are varied to find the highest total profit, where the coverage consists of warranty length for the first policy and total amount of repairs and replacements for the second.

Questions that this project will pursue are the following:

- What are the optimal price and coverage for each type of extended warranty?
- Which extended warranty policy is the most profitable of the two?
- Are there any values for the input parameters such that position of being the most profitable extended warranty will shift to the other one? If so, what are these values?

1.5 Limitations

This project uses a model which is simplified by a set of assumptions. This is done in order to analyze the problem of maximizing a manufacturer's profit over price and coverage of the extended warranty. Firstly, it is assumed that the manufacturer has a monopoly on the market of the product, and that no price discrimination is done. This project will only analyze the manufacturer's profit and not engage in the customer's profit.

When constructing extended warranty policies, the considered policies are non-renewing. It means that when a product is replaced, the warranty does not restart, nor will it be prolonged in any way. Furthermore, this project is focused on full-coverage warranties, so that the constructed policies will repair both minor and major failures.

1.6 Method

The methods used in this project can be divided into three steps; research, mathematical formulation and simulation.

1.6.1 Research

Research is done by gathering relevant information needed for the project and examining mathematical models. The information obtained from the research on the topic is then carefully reviewed and analyzed. A deeper understanding comes to existence by discussions through meetings.

1.6.2 Mathematical formulation

Two policies which cover both minor and major failures are constructed. After that, mathematical models of the policies are created and the corresponding optimization problems are formulated.

1.6.3 Simulation and sensitivity analysis

To solve the optimization problem a numerical approach is needed. The expected number of failures is determined by a simulation using Weibull distributed random variables. A sensitivity analysis of some parameters is done after the profit has been maximized over price and coverage of the extended warranty. Product price and length of the base warranty is fixed. The result is analyzed and conclusions are drawn accordingly.

2 Related material and contribution

This chapter focuses on presenting related material and literature in the field. This gives a better insight of which subjects within the field that is yet to be explored and researched and also, in which field our contribution lies more specifically. Key terms that are used throughout subsequent chapters are defined.

2.1 Key terms

Base warranty - A mandatory service plan under which the provider is obligated to repair, replace or maintain a product for free during a specific time period. The base warranty is non negotiable for neither the provider nor the customer.
Extended warranty - A purchasable service plan under which the provider (in this case the manufacturer) agrees to repair, replace or maintain a product either without any additional cost, or at a lower price during the extended warranty period [7]. The extended warranty starts when the manufacturer's base warranty expires.
Minor failure - Failures with the product that can be rectified by minimal repair, restoring the product to its state prior to the occurred failure [8].
Major failure - Catastrophic failures neccessating replacement of the product, restoring the product to its state at the moment of purchase.
Manufacturer/retailer/customer - A manufacturer produces a product and sells it to retailers who in turn sell the product to customers [7].
Supply chain - A supply chain is a system of all instances involved in fulfilling a customer's request for a product or service [9].
Centralized system - Consists of a manufacturer and a consumer. The manufacturer provides both the product and the extended warranty [7].
Monopolistic market - A market where merely one retailer offers a certain product. This creates a monopoly and implies that prices will increase, since there is no competition and the customers cannot buy the product elsewhere [10]. An example of a monopolistic market is "Systembolaget", the only store in Sweden where one can buy alcoholic beverages [11].
Double marginalization - If we consider a supply chain with a manufacturer and a retailer, double marginalization occurs when both parts enjoy a monopolistic market. This will cause the system profit to decrease and the price of the product to increase [12].
Convex function - A function is called convex if we can draw a line between two arbitrary points and the function lies below this line [13].
Objective function - In a minimization/maximization problem an objective function is the function that is to be minimized/maximized subject to certain given constraints.
Non-linear optimization problems - Optimization problems where the objective function or the constraints are non-linear.

2.2 Literature review

Li et al. [7] described three outcomes for different extended warranty providers. The paper considered three models, one centralized system with simply a manufacturer and a consumer where the manufacturer provides both the product and the extended warranty, one system with a manufacturer, a retailer and a consumer where the retailer provides the extended warranty and one system similar to the latter except that the manufacturer provides the extended warranty instead. Li et al. concluded that the centralized system generated the largest profit due to the double marginalization occurring when a retailer is present in a system. They merely considered a monopolistic setting.

Chen et al. [14] brought in competition by discussing the setting of one manufacturer and two retailers where the retailers provide the extended warranty. The paper considered three options for manufacturers when choosing a pricing strategy for setting the wholesale price to maximize the profit. In neither of these models different warranty policies were reflected upon.

Rangan and Khiabani [8] proposed three different policies for extended warranties where, in every case, the extended warranty period terminates when a major failure occurs. The possibility of an extended warranty that provides a full replacement after a major failure was not considered. Furthermore Rangan and Khiabani made an assumption that the product would be completely replaced for not only major but also minor failures during the base warranty period.

Yeh and Peggo [15] discussed the possibility of a customer being able to renew the extended warranty an amount of i times after each time it had expired. They suggested two types of policies. The first policy promised the manufacturer would minimally repair every failure during a specific time period, while the other policy stated that the manufacturer would minimally repair failures a limited amount of k times. Yeh and Peggo assumed every failure to be a minor failure. They never considered the case when a product malfunctions completely and needs replacement.

Chien et al. [16] proposed a generalized replacement policy where a system is replaced whenever the system has had minor failures, k, amount of times, a major failure occurs or its age has reached a threshold time, T.

Wang and Zhang [17] do not consider a threshold time for the system's age in their model. They introduce two replacement models. The first is based on the limiting availability and the second on the long-run average cost rate of the system. The optimal replacement policy is found by maximizing the limiting availability and minimizing the long-run average cost rate respectively for each model.

The policies discussed in the papers of Chien et al. [16] and Wang and Zhang [17] are from the perspective of customers and not manufacturers. Their policies are neither constrained by a period of time nor an amount of repairs. Instead they focus primarily on keeping the operating cost to a minimum during a long period of time. Their optimization problem considers for this reason how many times it is viable to minimally repair a minor failure instead of replacing the product/system.

Since Rangan and Khiabani [8] modeled extended warranties they modified the Glickman-Berger demand function (eq. 9) and made it depend not only on the price of the product and the length of the base warranty but also on the length of the extended warranty. However, they did not consider the price of the extended warranty to be an affecting factor, and also assumed that everyone who buys a product also buys the extended warranty.

2.3 Contributions to the model

In this project, the first constructed policy terminates after a given period of time T and the other terminates after k number of repairs and replacements, that is, a k-limited amount policy. In this project the warranties considered are free renewal warranties (FRW) and non-renewing.

In the new model, not only minor but also major failures are considered and only the products with major failures will be replaced. Instead of replacing those with minor failures, they will be minimally repaired. This will not only reduce the impact on the environment and make the profit increase (under the assumption that it is cheaper to minimally repair than to replace a product), but it will also make more sense in a realistic point of view, for example you do not receive a new laptop every time a key malfunctions.

The policies investigated in this project are chosen to better represent the situation on a market where full coverage warranties are offered. Frequently occurring on such markets is the limited time policy while the k-limited amount policy has been introduced to possibly bring attention to whether this type of policy can pose a serious alternative to the limited time policy in these scenarios.

In this project the demand function is formulated as two separate parts, one standard demand function for the product and one modified for the extended warranty since all consumers who buy the products usually do not buy the extended warranty as well.

The demand function that Rangan and Khiabani [8] used for extended warranties lacks some credibility, since they only consider the length of the extended warranty as an affecting role of the demand. It is resonable to believe that the price of the extended warranty should also affect the demand. When the length of an extended warranty is enhanced the price increases as well, which in turn affects the demand for the extended warranty negatively. To compensate for the flaw in the previously used demand function, the inverse of the price of the extended warranty is inserted into the demand function. Moreover, a demand function for k-limited amount policies is constructed. It is dependent on the number of repairs and replacements, k, instead of the expected time of the extended warranty. To make the comparison reasonable the new demand function is behaving in the same fashion as the one dependent on expected time.

To sum up, this model differs from previous models in the sense that:

1. The system observed in this project is a centralized system. Also, a monopolistic setting is considered.
2. Two new policies that cover both minor and major failures are constructed.
3. The product will not be replaced for all types of failures, merely the products with major failures and instead minimally repair those with minor.
4. In the k-limited amount policy, no distinction is made between minor and major failures when counting failures.
5. The demand function is slightly modified by separating the demand for the product and the extended warranty, and multiplying the demand for extended warranties with a factor consisting of the inverse of the extended warranty price.
6. A new demand function, dependent on the value k instead of the length of the extended warranty, is designed.

3 Theory

This chapter mainly focuses on clarifying the theory that has been used in the project. Also, the theory behind the calculations is described to easier understand the mathematical formulation.

3.1 Classification of Warranty Policies

There is no commonly recognized classification of warranties, although attempts to categorize different types have been made [5]. The first touchstone is whether the warranty involves development after point of sale or not. Post sale warranty development is principally used in large, industrial and complex items e.g. aircrafts and power plants, which is of little interest in this project.

For the warranties that do not involve development there are two main groups, A and B. Group A consists of warranties that are applicable to the sale of single items while group B are warranties that are applicable to the sale of batches or groups of items. This project primarily focuses on group A.

To demonstrate the classification for products belonging to group A or B we start from the manufacturer's viewpoint as seen in figure 1. The next stage is to determine whether the warranty contract is of one-dimensional or two-dimensional character. Two-dimensional means that the terms of warranty are declared in two different variables, often age and usage. For example, a car has a 3 year/30 000 mileage warranty (see figure 2), whichever limit is reached first puts an end to the warranty. This project, however, is based on one-dimensional warranties whose terms are declared in only one variable.

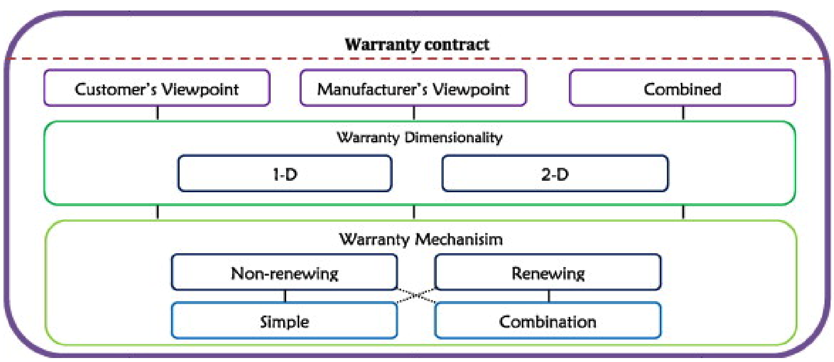

Figure 1: A classification of different warranty contracts [18].

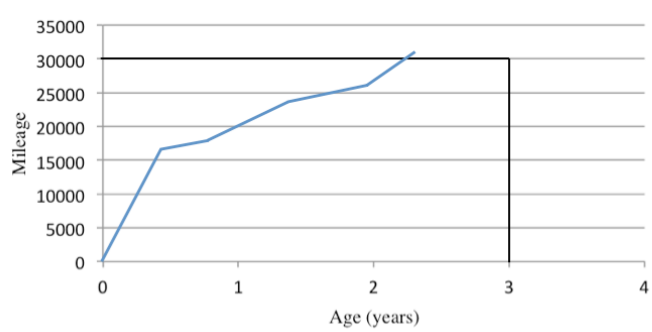

Figure 2: 3 year/30 000 mileage two-dimensional warranty.

The mechanism of the warranty consists of two categories, renewing policy or non-renewing policy. Renewing policy meaning that the warranty is renewed whenever a failure occurs, only allowing the warranty to expire when the product has functioned without failure for the time interval of the warranty. Non-renewing means that the warranty period is fixed and independent of failures.

Finally, there are two main principles when it comes to the point of repair or replacement of a product during the warranty period, free renewal (FRW) and pro-rata (PRW) warranties. As illustrated in figure 3, FRW charges nothing from the customer for the repair/replacement while the PRW has a fee that increases proportionally with the elapsed time from purchase.

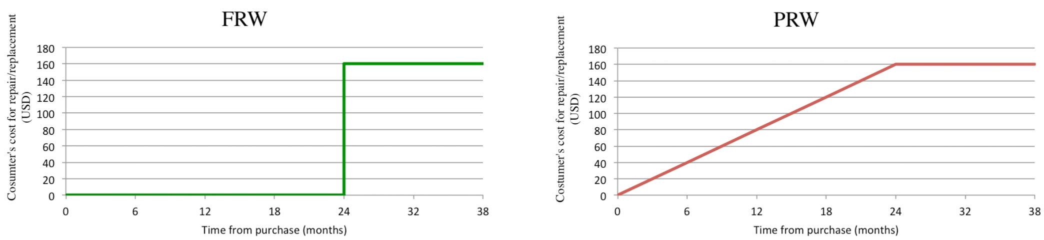

Figure 3: Cost for customers under FRW and PRW, where the base warranty expires after 24 months.

3.2 Distribution of failures

A probability distribution is needed in order to estimate the amount of product failures at certain life lengths. In this project, the Weibull distribution is chosen to be the distribution function of failure times. The reason why is simple. Survival or failure times are very commonly described using the Weibull distribution [19]. Also, the Weibull distribution is very flexible and by varying its parameters, it can be adapted to many different products. This provides the possibility of applying this research on many different cases.

There are two types of the Weibull distribution, one with two parameters and one with three. The three parameter Weibull distribution features a parameter γ that decides the minimum life length of a product, that is, no failures will happen when $t < \gamma$ [19]. However, for most products it is not impossible that a failure occurs even at a very early stage of the product's life, and therefore this project will not use a third parameter.

The probability density function, pdf, of the two-parameter Weibull distribution is the following

$$f(t) = \tfrac{\kappa}{\theta}(\tfrac{t}{\theta})^{\kappa-1}e^{-(t/\theta)^{\kappa}} \tag{1}$$

Here, κ is called the *shape parameter* and θ is called the *scale parameter*. Just as the names suggest, κ modifies the shape of the curve, and θ modifies the amplitude of it. Figure 4 displays the Weibull distribution for different shape parameters. The failure rate function of a product with the distribution of failure times described by a probability density function, pdf, $f(t)$ and a cumulative distribution function, cdf, $F(t)$ is defined as follows

$$r(t) = \frac{f(t)}{1 - F(t)} \tag{2}$$

For the Weibull distribution the failure rate is given by

$$r(t) = \frac{\kappa}{\theta}\left(\frac{t}{\theta}\right)^{\kappa-1} \tag{3}$$

Note that the Weibull distribution with $\kappa = 1$ is equivalent to the exponential distribution, which has a constant failure rate.

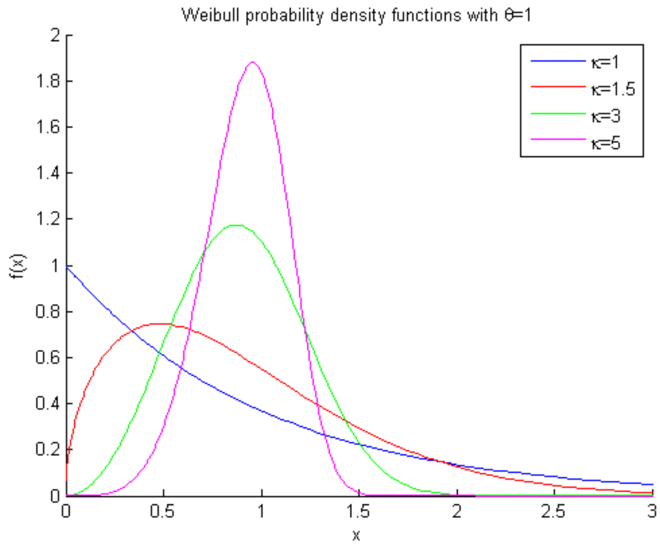

Figure 4: The Weibull distribution.

3.3 The renewal reward theorem

In order to calculate the long run average cost, the Ross renewal reward theorem [20] is used. It is stated as follows:

$$LAC = \frac{Expected\ cost\ of\ a\ cycle}{Expected\ length\ of\ a\ cycle} \tag{4}$$

To understand the long run average cost equation we need to clarify the concept of a renewal process. A Poisson process could be seen as a counting process where the times between events are independent and identically distributed exponential random variables. The definition of a renewal process states that if the sequence of non-negative random variables $\{X_1, X_2,...\}$ is independent and identically distributed, then the counting process $\{N(t), t \geq 0\}$ is said to be a renewal process. This is a generalization of the Poisson process but with an arbitrary distribution instead of an exponential [21].

In many cases of renewal reward processes, the notation of cycles is usually the way to describe stochastic events and probability models. Consider a renewal process $\{N(t), t \geq 0\}$ with interarrival times $X_n, n \geq 1$, and suppose that a reward is given for every time a renewal occurs. This reward will here be denoted by R_n, and is earned at the time of the nth renewal. Assume that $R_n, n \geq 1$, are independent and identically distributed but allow R_n to depend on the length of the nth renewal interval X_n. If we let

$$R(t) = \sum_{n=1}^{N(t)} R_n \tag{5}$$

then $R(t)$ represents the total reward earned at time t. Now let $E[R] = E[R_n]$ and $E[X] = E[X_n]$. If $E[R] < \infty$ and $E[X] < \infty$, then

$$a) \text{ with probability 1, } \lim_{t\to\infty} \frac{R(t)}{t} = \frac{E[R]}{E[X]} \tag{6}$$

$$b) \lim_{t\to\infty} \frac{E[R(t)]}{t} = \frac{E[R]}{E[X]} \tag{7}$$

When the cycle is completed which is every time a renewal occurs, the long-run average reward per unit time is equal to the expected reward during a cycle divided by the expected length of a cycle.

3.4 The Glickman-Berger demand function

Cobb and Douglas [22] constructed a log-linear production function to describe the total production, Y, as a function of labor, L, and capital input, K.

$$Y = AL^a K^b \tag{8}$$

where A is an amplifying productivity factor and a and b are the labor and capital elasticity.

Glickman and Berger [23] used this idea in order to calculate the demand for a product. The demand function is determined by price and warranty length and has been used perpetually in economic modeling since then, for example by Ladany and Shore [24] and Manna [25]. The Glickman-Berger function has the following appearance:

$$q(C_c, W) = z_1 C_c^{-\alpha}(z_2 + W)^{\beta} \tag{9}$$

where $\alpha > 0$ and $0 < \beta < 1$ are interpreted as the price and warranty time elasticity respectively. $z_1 \geq 0$ is a product demand amplitude factor and $z_2 \geq 0$ is a constant time displacement to ensure the demand of the product to be non-zero even if the length of the warranty is equal to zero. Figure 5 shows two plots of the Glickman-Berger demand function with $z_1 = z_2 = 1$, and $\alpha = 1.5$, $\beta = 0.5$ to the left and $\alpha = 0.4$, $\beta = 0.5$ to the right.

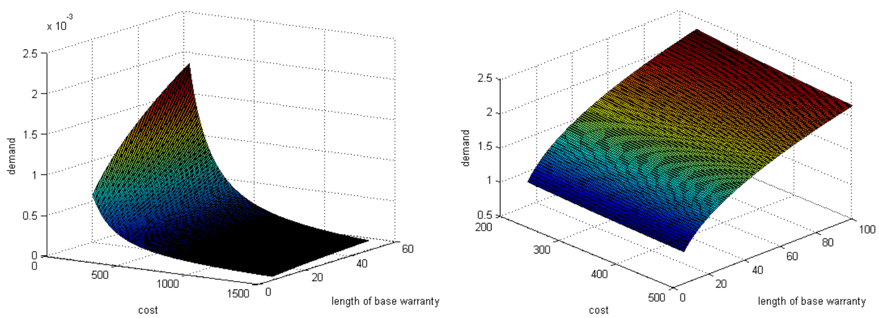

Figure 5: Plot of the Glickman-Berger demand function for different α.

4 Mathematical formulation

This chapter provides an analytical overview of the equations and formulations that have been developed. Presenting a step-by-step walkthrough will also clarify how the results were obtained. To begin with, the definitions of the two compared policies are presented.

4.1 Policy 1: Limited Time Policy

Starting from the expiration of the base warranty W, the manufacturer will minimally repair all minor failures and replace all products with major failures an infinite amount of times during a given extended warranty time T.

4.2 Policy 2: Limited Amount Policy

Starting from expiration of the base warranty W, the manufacturer will minimally repair all minor failures and replace all products with major failures a given number of times, k. When the combined amount of repairs and replacements have reached the value k, the extended warranty expires.

The cost for the manufacturer during the extended warranty is k times the average repair cost, hence the price of the extended warranty will be bigger than this number. Therefore the price of the extended warranty could be very high if k is large.

4.3 The notation used

p, q: probabilities that a failure is classified as minor or major, where $p, q \in [0, 1]$ and $p+q = 1$.

C_c, C_m: purchasing cost of the product for the customer and production cost for the manufacturer, respectively.

C_{EWi}: cost of extended warranty under policy i.

c_1, c_2: repair and replacement costs for the manufacturer of minor and major failures, respectively, where $c_1 < c_2 < C_m$.

W: length of base warranty.

T_i: time period of extended warranty policy i.

$f(t), F(t)$: probability density function and cumulative distribution function of failure time of a new product.

$r(t)$: failure rate function of a product. $r(t) = \frac{f(t)}{1-F(t)}$

$N(t), M(t)$: number and expected number of failures of any type in $(0, t)$.

$LAC_{m_i,p}, LAC_{c_i,p}$: long run average cost under policy i for the manufacturer and the customer when only the product is purchased, respectively.

$LAC_{m_i,pEW}, LAC_{c_i,pEW}$: long run average cost under policy i for the manufacturer and the customer when both the product and the extended warranty is purchased, respectively.

$LAP_{m_i,p}$: long run average profit under policy i for the manufacturer when only the product is purchased.

$LAP_{m_i,pEW}$: long run average profit under policy i for the manufacturer when both the product and the extended warranty is purchased.

Π_i: total profit under policy i for the manufacturer.

4.4 Cost analysis

The model is separated into two parts. The first part describes the cost for every customer who only buys the product, and the second part the cost for every customer that buys both the product and the extended warranty.

4.4.1 Policy 1: Limited Time Policy

The simplified expressions considering the case where the customer only buys the product are the following where $M(t)$ is solved numerically. Using the Ross renewal reward theorem [20] the long run average cost for the customer is simply the price of the product over the base warranty period

$$LAC_{c_1,p} = \frac{C_c}{W} \tag{10}$$

If a customer only buys the product, the long run average cost for the manufacturer is the cost of producing the product and the repair costs of minor and major failures over the base warranty period.

$$LAC_{m_1,p} = \frac{C_m + M(W)(pc_1 + qc_2)}{W} \tag{11}$$

In the case where the customer buys both the product and the extended warranty, we conclude that $E[T_1] = T_1$ since T_1 is fixed. Moreover, we also conclude that the long run total cost for the customer is the price of the product and the price of the extended warranty. Thus the expressions considering the case where the customer buys both the product and the extended warranty are the following

$$LAC_{c_1,pEW} = \frac{C_c + C_{EW}}{W + T_1} \tag{12}$$

and

$$LAC_{m_1,pEW} = \frac{C_m + M(W + T_1)(pc_1 + qc_2)}{W + T_1} \tag{13}$$

Now recall that a minor failure does not significantly change the failure rate and that a product is replaced after each major failure, which resets the failure rate. This is illustrated in figure 6, where minor failures occur at times t_1, t_2 and t_4, and a major failure takes place at time t_3. Assume that there will be m_i minor failures during the time interval $[0, t_i]$, where t_i is the time it takes until next major failure occurs for all $i \in \{1, 2, \dots, n\}$. Furthermore the time of the jth minor failure in the interval $[0, t_i]$ is denoted by $\hat{t}_j$ for $j \in \{1, 2, \dots, m_i\}$. Let $r_1(t) = p \cdot r(t)$ be the failure rate for minor failures and $r_2(t) = q \cdot r(t)$ be the failure rate for major failures. Then the case when a customer merely buys the product is described according to the following expressions. The long run average cost for the manufacturer is:

$$LAC_{m_1,p} = \frac{C_m + \sum\limits_{i=1}^{n} [c_1 \sum\limits_{j=1}^{m_i} \int\limits_{\hat{t}_{j-1}}^{\hat{t}_j} r_1(t)\mathrm{d}t + c_2 \int\limits_{0}^{t_i} r_2(t)\mathrm{d}t]}{W} \tag{14}$$

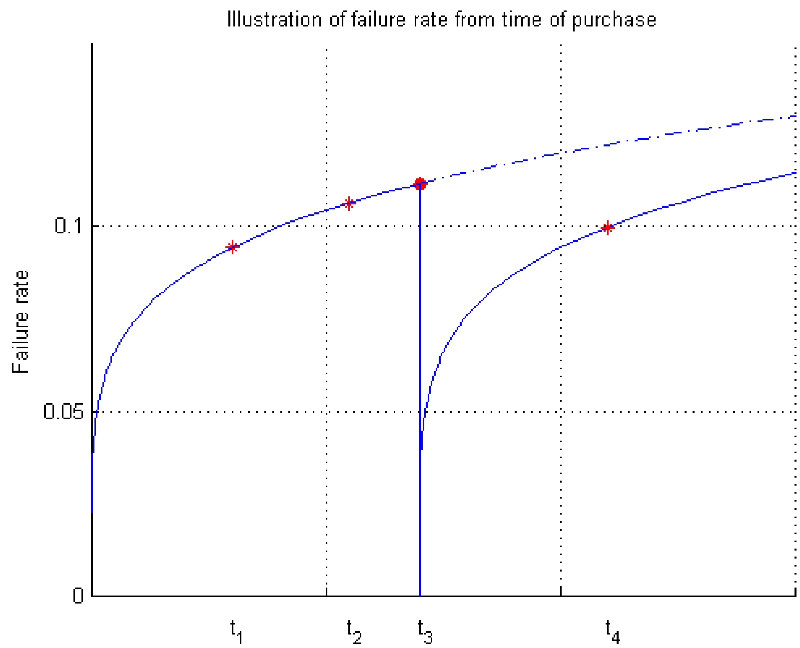

Figure 6: Plot of the failure rate.

This means that the expected cost of a cycle for the manufacturer is the cost of the expected amount of minor and major failures during the period and the manufacturing cost. Since it is assumed that a minor failure does not significantly change the failure rate, and we know that the failure rate, $r(t)$ is continuous, the expression can be rewritten as

$$LAC_{m_1,p} = \frac{C_m + \sum_{i=1}^{n}[c_1 \int_0^{t_i} r_1(t)\mathrm{d}t + c_2 \int_0^{t_i} r_2(t)\mathrm{d}t]}{W} \tag{15}$$

If we now consider the case when the customer buys both the product and the extended warranty and that the number of major failures during the time $[0, W + T_1]$ is l analogously, then the long run average cost for the manufacturer is

$$LAC_{m_1,pEW} = \frac{C_m + \sum_{i=1}^{l}[c_1 \int_0^{t_i} r_1(t)\mathrm{d}t + c_2 \int_0^{t_i} r_2(t)\mathrm{d}t]}{W + T_1} \tag{16}$$

4.4.2 Policy 2: Limited Amount Policy

When calculating the long run average costs for this policy it is already known how many repairs that will be carried out, namely k repairs. We have that

$$M(W + T_2) = M(W) + k \tag{17}$$

$M(W)$ is calculated in the same way as for the first policy. The expressions for the manufacturer's and customers' long run average cost when an extended warranty is not bought is naturally the same as previously stated. The problem is now to determine the length of a cycle when an extended warranty is bought. Simplified, we have that

$$LAC_{m_2,pEW} = \frac{C_m + (M(W) + k)(pc_1 + qc_2)}{W + E[T_2]} \tag{18}$$

Here $M(W)$ is calculated in the same way as for the first policy. We recall from eq. 15 that if there are n major failures during the base warranty W then

$$(pc_1 + pc_2)M(W) = \sum_{i=1}^{n}[c_1 \int_0^{t_i} r_1(t)\mathrm{d}t + c_2 \int_0^{t_i} r_2(t)\mathrm{d}t] \tag{19}$$

Using this, the expressions are

$$LAC_{m_2,pEW} = \frac{C_m + \sum_{i=1}^{n}[c_1 \int_0^{t_i} r_1(t)\mathrm{d}t + c_2 \int_0^{t_i} r_2(t)\mathrm{d}t] + k(pc_1 + qc_2)}{W + E[T_2]} \tag{20}$$

$$LAC_{c_2,pEW} = \frac{C_c + C_{EW2}}{W + E[T_2]} \tag{21}$$

The expressions for both policy 1 and 2 can be derived analytically if the failures are exponentially distributed since the failure rate, $r(t)$ then is constant. In this project, however, the failures are assumed to be Weibull distributed. In this case $\kappa \neq 1$ and hence the calculations are carried out numerically.

4.5 Demand functions

To calculate the demand for the product the Glickman-Berger demand function (eq. 9) is used. Recall that

$$q_p = z_1 C_c^{-\alpha}(W + z_2)^{\beta} \tag{22}$$

As noted in the literature review, the demand function considered for extended warranties is simply an extension of the Glickman-Berger demand function. The demand will then be dependent on the expected time and price of the extended warranty. The demand for a warranty with policy i is

$$q_{EWi}(E[T_i], C_{EWi}) = \tilde{z}_1 C_c^{-\alpha}(W + z_2)^{\beta} E[T_i]^{\gamma} C_{EWi}^{-\delta} \tag{23}$$

Where $\tilde{z}_1 > 0$ is the extended warranty demand amplitude factor and $z_2 \geq 0$ is the warranty period displacement. Furthermore $\alpha > 0$ is the product price elasticity, $0 < \beta < 1$ and $\gamma > 0$ are the warranty and extended warranty length elasticities respectively, and $\delta > 0$ is the extended warranty price elasticity.

For the first policy, the expected length of the extended warranty, $E[T_1]$ is known to be equal to T_1. Hence, the demand function for an extended warranty with policy 1 is

$$q_{EW1}(T_1, C_{EW1}) = \tilde{z}_1 C_c^{-\alpha}(W + z_2)^{\beta} T_1^{\gamma_1} C_{EW1}^{-\delta} \tag{24}$$

When studying the second policy, we see that $E[T_2]$ is not known, as the policy is limited by an amount of failures, not by time. The expected duration of the extended warranty could be calculated and inserted into the formula, given the Weibull parameters (κ, θ). This would, however, be equivalent to assuming that customers know the Weibull parameters, or that customers guess the length and guess exactly the expected value of the extended warranty every time. Neither of these assumptions are realistic. A more logical approach is to let the demand function depend on what the customer knows, in this case, the combined amount of repairs and replacements, k.

$$q_{EW2}(k, C_{EW2}) = \hat{z}_1 C_c^{-\alpha}(W + z_2)^{\beta} k^{\gamma_2} C_{EW2}^{-\delta} \tag{25}$$

To get comparable results, the demand function needs to behave in a similar way as if it had been dependent on the time $E[T_2]$. Therefore numerical analysis will be used to fit this demand function to the corresponding demand function using the time, to find the k-elasticity γ_2 and the extended warranty demand amplitude factor $\hat{z}_1$.

4.6 Profit analysis

The general expression for the long run average profit, LAP_{m_i}, per product cycle for the manufacturer is of course the difference between the cost for the customer, LAC_{c_i}, and the cost for the manufacturer, LAC_{m_i}.

$$LAP_{m_i} = LAC_{c_i} - LAC_{m_i} \tag{26}$$

Let $LAP_{m_i,p}$ denote the long run average profit when a product is sold without an extended waranty under policy i. Then, we obtain the following expression for the long run average cost when a customer merely buys the product:

$$LAP_{m_i,p} = LAC_{c_i,p} - LAC_{m_i,p} \tag{27}$$

where $LAC_{c_i,p}$ is the long run average cost for the customer when only buying a product, and $LAC_{m_i,p}$ is the corresponding long run average cost for the manufacturer. In the same fashion, we obtain the long run average profit, $LAC_{m_i,pEW}$, under policy i for a product sold with an extended warranty:

$$LAP_{m_i,pEW} = LAC_{c_i,pEW} - LAC_{m_i,pEW} \tag{28}$$

To calculate the total profit, Π_i, the long run average profit is multiplied by the total sales volume of products with and without extended warranties under each policy i. Thus, the total profits for policy 1 and 2 respectively are the following:

$$\Pi_1(C_{EW1}, T_1) = (q_p - q_{EW1})LAP_{m_1,p} + q_{EW1}LAP_{m_1,pEW} \tag{29}$$

$$\Pi_2(C_{EW2}, k) = (q_p - q_{EW2})LAP_{m_2,p} + q_{EW2}LAP_{m_2,pEW} \tag{30}$$

where q_p, q_{EW1}, q_{EW2}, $LAP_{m_i,p}$, and $LAP_{m_i,pEW}$ are given by equations 22, 24, 25, 27 and 28 respectively.

4.7 Maximization problem

Here the maximization problems for each policy are presented under given constraints.

4.7.1 Policy 1: Limited Time Policy

The objective function is $\Pi_1(C_{EW1}, T_1)$ which gives the maximization problem for policy 1:

$$\left.\begin{array}{ll} \text{maximize} & \Pi_1(C_{EW1}, T_1) \\ & \\ \text{subject to} & C_{EW1} - C_c \leq 0 \\ & T_1 - \hat{T}_1 \leq 0 \\ & C_{EW1}, T_1 \geq 0 \end{array}\right\} \tag{31}$$

where the first constraint declares that the price of the extended warranty is less than or equal to the price of the product, the second that the length of the extended warranty is controlled by interval $[0, \hat{T}_1]$ and the last constraint that the price and length of the warranty can not be less than zero, that is the time period can not be negative and manufacturer cannot pay the customer to take the extended warranty.

4.7.2 Policy 2: Limited Amount Policy

There is no specific time period of the extended warranty in policy 2 and the expected time of the policy is unknown to the customer, instead the amount of repairs and replacements, k, is known. The objective function is therefore dependent on k, $\Pi_2(C_{EW2}, k)$. This gives us the following problem:

$$\left.\begin{array}{ll} \text{maximize} & \Pi_2(C_{EW2}, k) \\ & \\ \text{subject to} & C_{EW2} - C_c \leq 0 \\ & k - \hat{k} \leq 0 \\ & C_{EW2}, k \geq 0 \end{array}\right\} \tag{32}$$

The constraints are constructed as in maximization problem 1 (eq. 31).

5 Non-linear optimization

This project deals with non-linear optimization, since the objective function is non-linear. Therefore this chapter will be devoted to the methods used to solve these types of optimization problems. In this section the problem is the following:

$$\left.\begin{array}{l} \min f(x) \\ \text{subject to} \quad x \in S \end{array}\right\} \tag{33}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is in C^1, $S \subseteq \mathbb{R}^n$ is a non-empty, closed set and

$$S := \{x \in \mathbb{R}^n \quad | \quad g_i(x) \leq 0, \quad i = 1, 2, \dots, m\} \tag{34}$$

5.1 Some cones of interest

Consider the set S as described in the minimization problem (eq. 33) and take $x \in \mathbb{R}^n$, then

$$T_s(x) := \{p \in \mathbb{R}^n | \exists \{x_k\}_{k=1}^{\infty} \subset S, \{\lambda_k\}_{k=1}^{\infty} \subset (0, \infty) : \lim_{k\to\infty} x_k = x, \lim_{k\to\infty} \lambda_k(x_k - x) = p\} \tag{35}$$

is called the tangent cone for S at x [13], that is, $T_s(x)$ is the set of all tangential directions, p, at x for S.

We now consider the active constraints and define a cone, $G(x)$, related to them at the given point $x \in \mathbb{R}^n$.

$$G(x) := \{p \in \mathbb{R}^n \quad | \quad \nabla g_i(x)^T p \leq 0, i \in \mathcal{I}(x)\} \tag{36}$$

where $\nabla g_i(x)$ is the gradient of all active constraints describing the feasible set, S. In other words, the defined cone (eq. 36) is the set of all directions, $p \in \mathbb{R}^n$, that are descent directions with respect to the active constraints $g_i(x)$, at x.

5.2 Constraint qualifications (CQ)

There are a number of different CQ. Beneath follows two examples of the most common constraint qualifications [13]:

Abadie's CQ - Take a point $x \in S$, if we have that $T_s(x) = G(x)$ then Abadie's CQ is satisfied. Moreover, if all inequality constraints are affine, that is if the set S is a polyhedron, then Abadie's CQ holds.
Linear Independence CQ (LICQ) - If the functions $g_i(x), i \in \mathcal{I}(x)$ defining the active inequality constraints have linearly independent gradients, $\nabla g_i(x)$, at a point $x \in S$, then the LICQ holds. If LICQ is satisfied at a point $x \in S$, it is also implied that Abadie's CQ holds.

Since all inequality constraints in the optimization problems (eq. 31, 32) are affine, we can conclude that Abadie's CQ is satisfied for all $[C_{EW2}, k]$ (or in (eq. 31) $[C_{EW1}, T_1]$) in the feasible set. Notice also that LICQ will be satisfied as well.

5.3 The Karush-Kuhn-Tucker optimality conditions

A first suggestion when solving a non-linear optimization problem could be to use the Karush-Kuhn-Tucker (KKT) conditions for convex problems. If we assume that the KKT conditions hold at $x^* \in S$ and that the minimizing problem (eq. 33) is convex, that is, $f(x)$ is convex and all constraints $g_i(x)$ are convex. Then, this is enough for the point $x^* \in S$ to be a global minimum [13]. However, it is important to remember that the KKT conditions do not imply local optimality if the problem is not convex. Consider the first maximization problem (eq.

31). The gradients of the constraints are the following:

$$\left.\begin{aligned}
\nabla g_1(C_{EW1}, T_1) &= \begin{bmatrix}1\\0\end{bmatrix}, \nabla g_2(C_{EW1}, T_1) = \begin{bmatrix}0\\1\end{bmatrix}\\
\nabla g_3(C_{EW1}, T_1) &= \begin{bmatrix}-1\\0\end{bmatrix}, \nabla g_4(C_{EW1}, T_1) = \begin{bmatrix}0\\-1\end{bmatrix}
\end{aligned}\right\} \tag{37}$$

Since it is established that Abadie's CQ is satisfied at the maximization problem (eq. 31), the KKT conditions are necessary (or if the problem is convex, sufficient) for a point to be optimal. Then, there exist $\mu_1, \mu_2, \mu_3, \mu_4$, such that:

$$\left.\begin{aligned}
&\nabla \Pi_1(C_{EW1}, T_1) = \mu_1 \begin{bmatrix}1\\0\end{bmatrix} + \mu_2 \begin{bmatrix}0\\1\end{bmatrix} + \mu_3 \begin{bmatrix}-1\\0\end{bmatrix} + \mu_4 \begin{bmatrix}0\\-1\end{bmatrix}\\
&\mu_1(C_{EW1} - C_c) = 0\\
&\mu_2(T_1 - \hat{T}_1) = 0\\
&\mu_3(-C_{EW1}) = 0\\
&\mu_4(-T_1) = 0\\
&\mu_1 \geq 0\\
&\mu_2 \geq 0\\
&\mu_3 \geq 0\\
&\mu_4 \geq 0
\end{aligned}\right\} \tag{38}$$

There is, however, a problem in solving the obtained system of equations. The problem lies in the fact that the objective function (eq. 29), is stochastic and not well known, hence the gradient cannot be calculated explicitly and the system of equations will be too difficult and complicated to solve. Another problem is the convexity of the problem. If one would like to add another dimension to the problem, it would, for the same reasons, be difficult establish whether the objective function is concave or not. You would not be able to decide convexity of the objective function from a plot anymore. The same reasoning applies to the second maximization problem (eq. 32) as well. Therefore, we will not use the KKT conditions.

5.4 Penalty methods

When dealing with penalty algorithms one tries to alter the problem (eq. 33) to become an equivalent but unconstrained problem instead [13]. For $x \in \mathbb{R}^n$ the problem obtained is on the form:

$$\min \quad f(x) + \chi_s(x) \tag{39}$$

where $\chi_s(x) = 0$ when $x \in S$ and $\chi_s(x) = \infty$ for all other x. $\chi_s(x)$ is called the "indicator function" of the set S. It is non-differentiable and discontinuous. Its main use is to make sure that we do not worry about optimizing the objective function, $f(x)$, until feasibility is achieved.

There are two types of penalty methods, the exterior and the interior (also called the barrier method). The exterior penalty methods start at optimal but infeasible points. The sequence of optimal points converge to a stationary (or if the problem is convex, globally optimal) point in the original problem. In this project however, an interior penalty method is used.

Suppose that there is a point, $\bar{x}$, such that $g_i(\bar{x}) < 0$, $i \in \{1, \ldots, m\}$. Instead of approximating $\chi_s(x)$ on $\mathbb{R}^n$, the interior penalty method sets a barrier against leaving the feasible set, and thus only creates approximations inside it. The method produces a sequence of interior points, converging to a (if the problem is convex) globally optimal solution of the original problem.

Let us now choose a function $\phi : \mathbb{R}_- \to \mathbb{R}_+$ so that ϕ is a continuous function and it holds that $\phi(s) \to \infty$ as the negative sequences $\{s_k\}$ tend to zero. An example of such a function is $\phi(s) = \frac{-1}{s}$. $\chi_s(x)$ can then be approximated accordingly:

$$\chi_s(x) \approx \nu \check{\chi}_s(x) := \begin{cases} \nu \sum_{i=1}^{m} \phi\big(g_i(x)\big) & \text{if } g_i(x) < 0 \\ +\infty & else \end{cases} \tag{40}$$

where the penalty parameter, $\nu > 0$ and $\nu \in \mathbb{R}$. We obtain the approximated problem

$$\min\ f(x) + \nu \check{\chi}_s(x) \tag{41}$$

Suppose that $\phi'(s) \geq 0$ for all $s < 0$ and study the sequence of stationary points $\{x_k\}$. Assume further that $x_k \to \hat{x}$ for some $\nu_k \to 0$ as $k \to \infty$. If LICQ is satified at $\hat{x}$, then $\hat{x}$ verifies the KKT conditions.

6 Numerical analysis

In this section parameters and constraints used in calculations are set and argued for. After that, the main results and a sensitivity analysis are presented.

6.1 Defining the experiment

The setting for the experiment is limited to product types with a failure behavior that is equivalent to short lifetime. The reason for this is that extended warranties in general are offered on products with relatively long lifetimes (for example laptops and home appliances) and not on shorter lasting products such as earphones and chargers. Whether extended warranties could be successfully offered on products with short lifetime or not, is therefore investigated. The length of the base warranty period W is set to 1 year. Although many companies are legally bound to offer a base warranty period for half a year, the majority usually extends this period to one year. The parameters α, β and z_2 are taken from previous research [26] while the others have been assumed for use in the numerical experiment:

$W = 1$, $C_c = 350$, $C_m = 200$, $c_1 = 80$, $c_2 = 170$

$\alpha = 4.2$, $\beta = 0.85$, $\gamma_1 = 1.4$, $\delta = 1.4$

$z_1 = 10^{15}$, $z_2 = 0.12$, $\tilde{z}_1 = 10^{17}$

$\kappa = 3$, $\theta = 1.5$

Now fitting the demand functions (eq. 24 and 25) to each other, we get

$\hat{z}_1 = 5.157 \cdot 10^{16}$, $\gamma_2 = 1.1808$

Using these assigned parameters, the demand, or in other words the number of sold products, is 22738. The time in the demand is measured in years. The Weibull parameters (κ, θ) gives a probability density function and a cumulative distribution function which can be seen in figure 7. With these parameters, the mean time to the first failure of any kind is 1.34 years, and the mean time to the first major failure is 2.29 years. The expected time to the first failure after the base warranty has run out is 0.58 years.

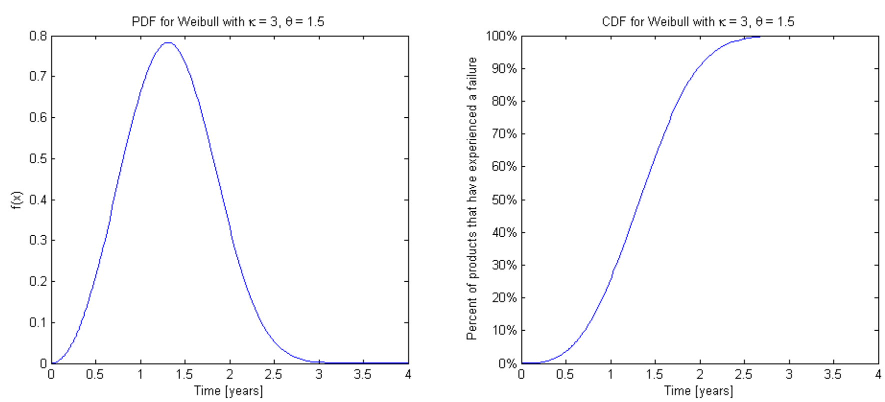

Figure 7: Weibull graphs with our chosen parameters.

In the numerical experiment we further limit the decision variables based on a number of reasons. The limitations are:

$$\left.\begin{array}{ll} 1 \leq T_1 \leq 4, & T_1 \in (\mathbb{N} \cup (\mathbb{N} + \frac{1}{2})) \\ 2 \leq k \leq 6, & k \in \mathbb{N} \\ 10 \leq C_{EWi} \leq C_c = 350, & C_{EWi} \in \mathbb{R} \end{array}\right\} \quad (42)$$

The extended warranty period T_1 can only be a positive integer or half-integer, as it is not quite realistic that a manufacturer would provide a warranty covering for example 1.816 years. The amount of failures covered by policy 2, k, is obviously a positive integer. As an upper limit to the price of extended warranty, the cost of the product is chosen. This is, since it is not realistic to offer an extended warranty that is more expensive than the product itself, as mentioned in the mathematical formulation. The upper bounds of T_1 and k are explained by the mean time to the first failure and the mean time to the first major failure, mentioned earlier. There is seldom any reason to offer an extended warranty covering a longer period of time than the mean time to a major failure, as this would be unprofitable for the manufacturer.

The choices of lower bounds, however, are more complicated and trace back to the used log-linear demand function.

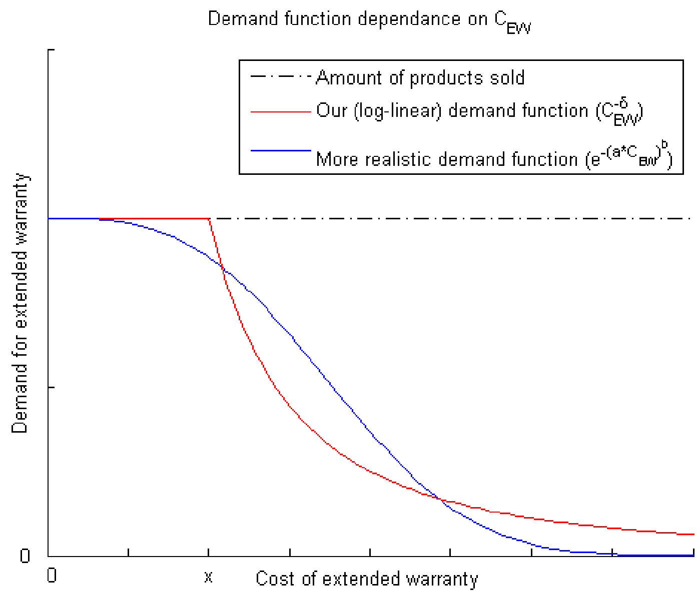

Figure 8: Behavior of demand functions.

As our extended warranty demand functions (eq. 24 and 25) depend on the inverse of C_{EWi}, they tend to infinity as the price of the extended warranty approaches zero. It is obviously impossible to sell more extended warranties than products, so therefore the demand function gets the shape which can be seen in figure 8, where also a possibly more realistic demand function is displayed. Now, if we have a price at or around x in the graph, we are in a situation where our model yields an unnaturally high demand for extended warranties. Hence, for some very low extended warranty prices the profit will be very high. However, this will only be valid for short extended warranty periods. Otherwise, the manufacturer will not receive a profit for the extended warranty sold, and it is irrelevant how high the demand is. This is why we do not consider extended warranties of policy 1 with half a year, or warranties of policy 2 with $k = 1$.

6.2 Results

The optimal solutions from numerical calculations are found in tables 1 and 2, where C^*_{EWi}, T^*_1, k^*, Π^*_i and q^*_{EWi} denote the optimal values for the variables C_{EWi}, T_1, k, Π_i and q_{EWi} respectively. As seen in the tables, the optimal price for policy 1 is far less than for policy 2, yielding that 42.10 % of the customers buying the product also buys the extended warranty under policy 1, while only 17.75 % buys it under policy 2. How the profits of the extended warranties vary when the decision variables change is shown in figure 9.

Limited time policy	*Variable*	*Value*
Extended Warranty Price	C^*_{EW1}	133.9691
Time (in months) Of Extended Warranty	T^*_1	12
Maximum Total Profit	Π^*_1	$2.7797 \cdot 10^6$
Share Of Extended Warranties	$\frac{q^*_{EW1}}{q_p}$	42.10 %
Number Of Extended Warranties	q^*_{EW1}	$9.5720 \cdot 10^3$

Table 1: Optimal values for policy 1.

Limited amount policy	*Variable*	*Value*
Extended Warranty Price	C^*_{EW2}	253.9983
Number Of Repairs/Replacements	k^*	2
Maximum Total Profit	Π^*_2	$2.6167 \cdot 10^6$
Share Of Extended Warranties	$\frac{q^*_{EW2}}{q_p}$	17.75%
Number Of Extended Warranties Sold	q^*_{EW2}	$4.0357 \cdot 10^3$

Table 2: Optimal values for policy 2.

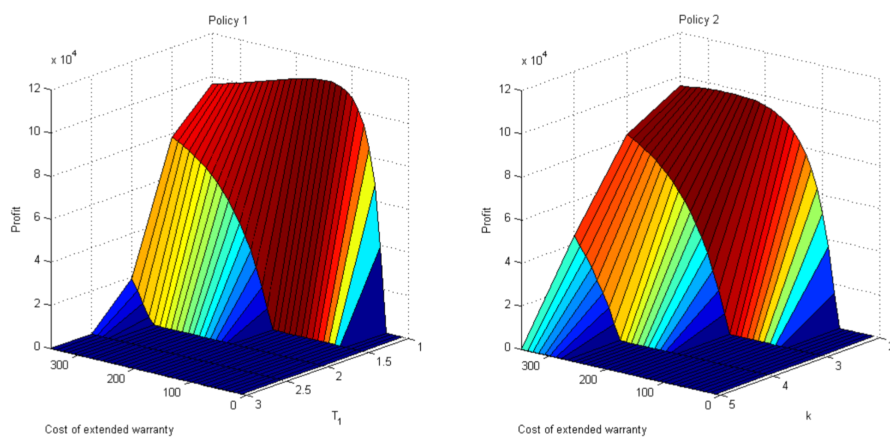

Figure 9: Profits from extended warranties.

6.3 Sensitivity analysis

To verify and explore the results a sensitivity analysis is carried out.

6.3.1 Varying elasticities

First, the elasticities for price and length of the extended warranty are varied, as they are difficult to estimate without extensively investigating the customer perception of the policies and their variables.

		δ		
		1.2	1.4	1.6
γ_1	1.2	(249, 1)	(137, 1)	(110, 1)
	1.4	(250, 1)	**(134, 1)**	(107, 1)
	1.6	(245, 1)	(149, 1)	(114, 1)

Table 3: Sensitivity analysis of elasticities for policy 1, with values (C^*_{EW1}, T^*_1), where the original values are in bold.

As can be seen in in tables 3 and 4, the extended warranty price sensitivity parameter δ naturally has some effect on the optimal price of the extended warranty. Larger δ implies price awareness of the customer which in turn yields a lower optimal price. The elasticity of

		δ		
		1.2	1.4	1.6
γ_1	1.2	(350, 2)	(243, 2)	(209, 2)
	1.4	(350, 2)	**(254, 2)**	(194, 2)
	1.6	(350, 2)	(259, 2)	(199, 2)

Table 4: Sensitivity analysis of elasticities for policy 2, with values (C^*_{EW2}, k^*), where the original values are in bold.

the length of the extended warranty, γ_1, does not affect the results that much in this scenario. This is due to the fact that the product malfunctions quite often, so it is not profitable to offer for example a 1.5 or 2 year extended warranty even if the elasticity for it is a bit higher than our standard values. The demand for a 2 year extended warranty for example does increase with γ_1, but with failures coming as often as they do here, the manufacturer would not make a profit for such a long extended warranty.

6.3.2 Varying the base warranty length

As stated earlier, policy 1 is the most profitable for our standard scenario. However, there are several variations that can be done to the parameters to make policy 2 yield more profit. In the standard case the expected time to the second failure after the base warranty has run out is 1.12 years, that is, a policy 2 extended warranty with $k = 2$ has $E[T_2] = 1.12$. Since the failure rate is increasing, this number will decrease if W is increased. The difference between $E[T_2]$ for different k-values will also be smaller, which actually increases the probability that policy 2 is better. This is because of the fact that there is always an optimal time $E[T_i]$, the problem is just to come as close to it as possible. If T_1 or k would have been continous then it would not be a problem, but in reality and in this model they are discrete. Now, when the difference between $E[T_2]$ for different k-values decreases the discrete grid of time points is refined, increasing the probability that $E[T_2]$ will be closer to the optimal time than T_1.

To give an example of this, we use the standard parameter values and variable restrictions described in chapter 6.1, but increase the base warranty length W. Changing it to 1.5 or 2 years both leads to policy 2 becoming more profitable than policy 1.

6.3.3 Varying the Weibull parameters

When the product fails more often, policy 2 will be benefited according to the same reasoning as above, because the difference between $E[T_2]$ for different k-values decreases. However one can not say that it will always lead to better results, as T_1 still can get closer to the optimal time than $E[T_2]$, it is just that the chance is smaller, but for example decreasing θ to 1.3 will make policy 2 yield the higher profit. The effects of some other variations of θ can be seen in figure 10. The same can be said about increasing κ, which can be seen in figure 11, as $\kappa = 3.5$ leads to a higher profit for policy 2. It is however hard to say anything in general for when one of the policies will be better than the other.

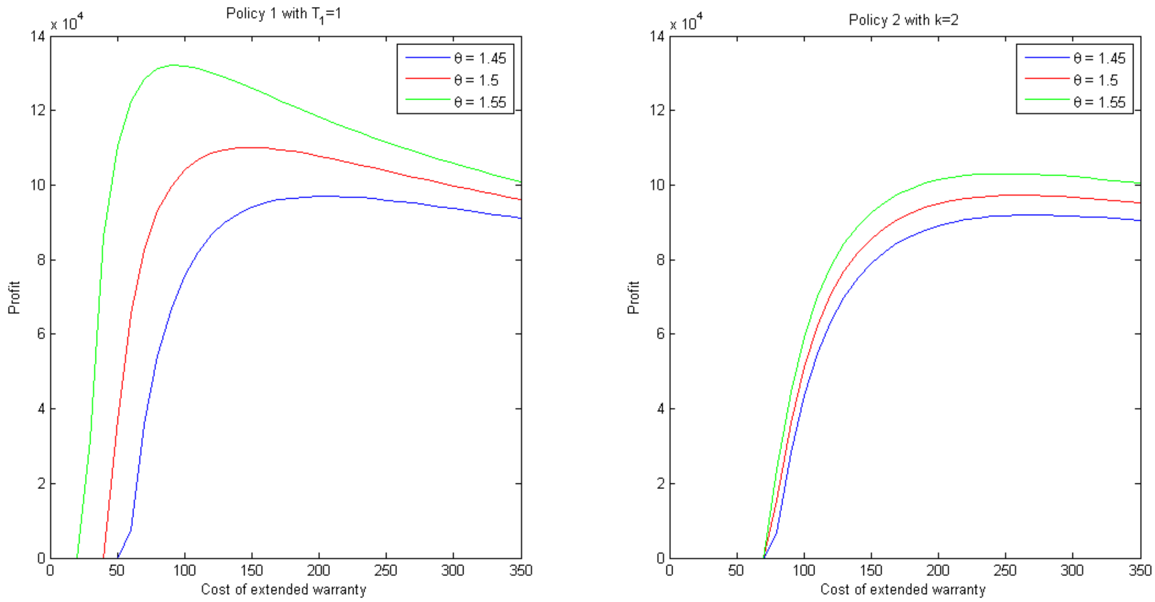

Figure 10: The Weibull scale parameter θ varied.

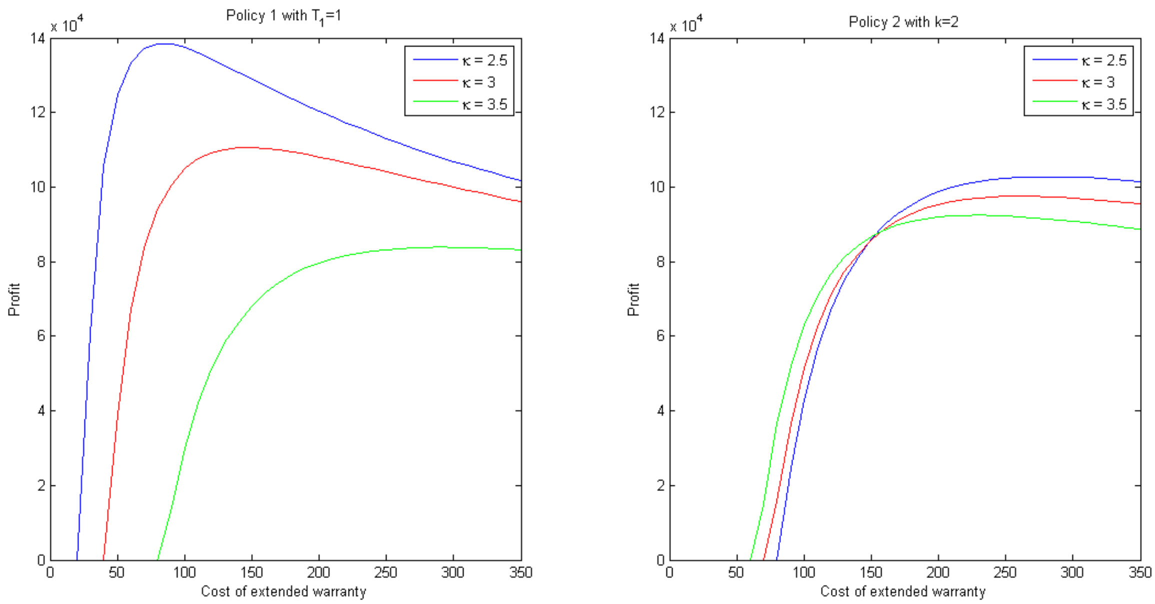

Figure 11: The Weibull shape parameter κ varied.

7 Discussion

This chapter is divided into two sections. Conclusions are drawn based on results from the numerical analysis in the first section, followed by where future research should be aimed to extend the project further in the second section.

7.1 Conclusions

As presented in tables 1 and 2, the time limited policy possesses the higher total profit for the considered case. Sensitivity analysis showed that the optimal solutions to the optimization problems are highly dependent on the price elasticity, δ, but close to independent of the extended warranty time elasticity, γ. It should be noted though, that this applies only to our case and it is far from certain that this will hold for the general case.

Recall that the calculations were carried out numerically, and hence do not show the completely true optimal values. Both k and T_1 were discrete in our program. Since T_1 is a variable in time, it would be beneficial to have it stepping every month instead of every half year. The reason why policy 1 is more beneficial than policy 2 has much to do with the fact that k and T_1 are discrete. Policy 1 and 2 would actually be equivalent as far as the results go if k and T_1 had not been discrete. If they both were continuous, the maximum profit would be the same and the value on k would give an expected time equal to T_1. Should the failures occur more frequently, the limited amount policy easier adapts as the gaps between different values for k on the time axis decrease. Ultimately, the optimal case for the second policy would be if the manufacturer had a product that does not malfunction at all during the base warranty, but have failures occuring rather frequently right after that. This would reduce the expected time for the extended warranty cycle, and hence making it more profitable, yet it would appear quite appealing to the customers.

As both policies in our case turned out to be profitable, there seems to be potential for successfully offering extended warranties for products with shorter lifetime than those being sold together with extended warranties in the market today. There is a possibility that extended warranties should not be reserved for expensive long life products only. Here, one has to consider the customer perception. Perhaps the real-life demand for extended warranties on shorter life products is too small. Perhaps nobody has tried to introduce this kind of warranty on the market. However, under the assumption that our presumed costs for the product are somewhat correct, the insignificant presence of extended warranties offered on short lifetime products on the market is hard to explain.

7.2 Future research

Though the models developed in this project have been progressed in a direction somewhat closer to the reality than models proposed in previous articles, there are still some issues that need to be recognized for future research.

First and foremost, the centralized model in which the manufacturer benefits a monopoly situation. The monopolistic setting seems to be the biggest flaw in the pursuit for a more realistic model. Monopolistic settings are rarely seen in markets world wide today. The centralized model is thus restraining the utility of this project's proposed model. To be able to apply the model to scenarios where competition exists, one must consider extensive game theory and, more specifically, the n-player game theory. Moreover, as distribution and sales often take place in supply chains one could also include different supply chain models, since our project only considers the centralized model.

This project does not at all consider customer satisfaction. To extend our model even more towards reality one should also this. Customer satisfaction depends on many factors that directly affect the manufacturer's profit in the long-run. For example, customers often tell others about their purchase. Studies have shown that satisfied customers tell 3 - 5 people while dissatisfied customers tell 8 - 20 people. Based on these data, models show that a company with 95 % customer satisfaction controls more than twice the market share over other companies with 90 % customer satisfaction [27].

In the second proposed policy, the limited amount policy, it is assumed that the customer immediately returns the product for repairing/replacement when a failure occurs. However, as the customers are only allowed to submit the failed product for repairing/replacement a limited amount of times, one could make a reasonable assumption that the customers will wait with the submission of a failed product until the next failure occurs if the occurred minor failure is sufficiently small, so that the main function of the product is still maintained. For example, suppose that a key on your laptop's keyboard starts to function poorly but you still would be able to use the laptop without greater problems. You would probably wait with returning it since you only will be allowed a limited amount of submissions. With current definitions of minor and major failures, the customer is assumed to return this marginally faulty product even if it is somewhat functional. This assumption would however necessitate the need of a new category of failures, that only takes these failures into account, or simply a change in the existing definitions. New definitions of failures would also better adapt to more complex products and make the model even more compatible for realistic scenarios, but on the other hand make the modelling far more complicated.

More importantly, the customer's perception of the limited amount of repairs and replacements has in this project been assumed to have the same impact on the demand as the length of the extended warranty has on the demand of policy 1. Whether these demand functions are equivalent or not is unclear and there is certainly a need to further develop a demand function that is dependent on limited amount of repairs in order to legitimate the assumption. That is, investigate the value of the parameter γ_2.

As the use of the modified Glickman-Berger function (eq. 23) for the demand of the extended warranty showed some limitations, a modified version, earmarked for extended warranties, should be considered. In its present state the demand tends to infinity as the price of the extended warranty reaches zero when reasonably, the demand maximum should be limited to the demand of the product (see figure 8). This leads to the non ideal action to cut off the demand when it has reached the same value as the product demand. A more suitable behavior for this demand function would be for it to tend to the product demand as the price reaches zero thus removing the need for the cut-off action. A possible solution to this problem could perhaps be to simply add a third displacement, z_3, to the extended warranty price.

Finally, in the calculations of this project, the price of the product as well as the length of the base warranty have been treated as fixed values. Should the situation allow it, interesting results could be drawn from calculations where these values are treated as variables instead of fixed values.

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A MATLAB code

A.1 Failure function for limited time policy

```
function [total, minorBase, majorBase, minor, major, qest] = failures(kappa, theta, T, W, q)
% [total, minor, major, ratio] = failures(kappa, theta, T, q)
% Returns:
%   total = minor + major
%   minorBase = Amount of minor failures during the baase warranty
%   majorBase = Amount of major failures during the baase warranty
%   minor = Amount of minor failures during time T
%   major = Amount of major failures during time T
%   qest  = Percentage of failures that are major, should be very close to q
% Parameters:
%   kappa  = Weibull shape parameter
%   theta = Weibull scale parameter
%   T     = Time EW + W
%   W     = Time base warranty
%   q     = Chance that a failure is major (type 2)

tot1=0; tot2=0;
minorBase = 0;
majorBase = 0;
TIMES=10000;
for i=1:TIMES
    x = 0; % time passed
    timeSinceRenewal = 0;

    base1 = 0; % amount of minor failures base warranty
    base2 = 0; % amount of major failures base warranty
    M=0; % amount of minor failures extended warranty
    N=0; % amount of major failures extended warranty

    while x < T
        x = x + wblrnd(theta, kappa); % The product is new
        timeSinceRenewal = timeSinceRenewal + x;
        z = randi(1/q,1); % An integer on [1,5] if p=0.2
        while z~=1 % minor failures

            if x < W
                base1 = base1 + 1;
            elseif x < T
                M=M+1;
            end

            %What is a random life length given that it is alive right now?
            u = rand(1);
            temp = (timeSinceRenewal^kappa - (theta^kappa)*log(1-u))^(1/kappa);

            x = temp; %next failure
            timeSinceRenewal = temp;
            z = randi(1/q,1); %is it minor or major?

            if x > T
                break;
            end
        end
```

```
        if x < W
            base2 = base2 + 1; %major failure during base warranty
            timeSinceRenewal = 0;
        elseif x < T
            N=N+1; % major failure during extended warranty
            timeSinceRenewal = 0; % Renewing product.

        end
    end
    tot1 = tot1 + M + base1;
    tot2 = tot2 + N + base2;
    minorBase = minorBase + base1;
    majorBase = majorBase + base2;

end

minorBase = minorBase/TIMES;
majorBase = majorBase/TIMES;
minor = tot1/TIMES;
major = tot2/TIMES;
total = (tot1+tot2)/TIMES;
qest = tot2/(tot1+tot2);


end
```

A.2 Failure function for limited amount policy

```
function [expectedTime, minorBase, majorBase] = failuresAmount(kappa, theta, amount, w, q)
% expectedTime = failuresAmount(beta, theta, amount, q)
% Returns the time given an amount of failures.
% Parameters:
%   minorBase = Amount of minor failures during the base warranty
%   majorBase = Amount of major failures during the base warranty
%   kappa   = Weibull shape parameter
%   theta  = Weibull scale parameter
%   amount = Amount of failures the warranty covers
%   q      = Chance that a failure is major and needs a replacement

tot = 0;
mintot = 0;
majtot = 0;
TIMES=3000; % Amount of times to repeat the experiment. A higher number
            % gives more reliable results but takes longer to compute.

for i=1:TIMES
    %fprintf('New product.\n');
    x = 0; % time passed
    timeSinceRenewal = 0;
    M = 0; % amount of type 1 failures
    N = 0; % amount of type 2 failures
    k = 0;

    while k < amount
        % Here, the product is new.
        x = x + wblrnd(theta, kappa); % We get the time of the next failure.
```

```
        timeSinceRenewal = timeSinceRenewal + x;
        z = randi(1/q,1);   % An integer on [1,5] if p=0.2. Decides what
                            % type of failure that just happened.
        while z~=1 % It was a minor failure

            if x < w
                M = M + 1;
            else
                k = k + 1 ;
                if k == amount
                     break;
                end
            end
            %What is a random life length given that it is alive right now?
            u = rand(1);
            temp = (timeSinceRenewal^kappa - (theta^kappa)*log(1-u))^(1/kappa);
            % According to the truncated distribution

            x = temp; % Next failure
            timeSinceRenewal = temp;
            z = randi(1/q,1); %Again, is it type 1 or 2?
        end

        if x < w
            N = N + 1; %major failure during base warranty
            timeSinceRenewal = 0;
        else
            k = k + 1; % major failure during extended warranty
            timeSinceRenewal = 0;
             % Now we renew the product.
            if k == amount
                break;
            end
        end
    end

    tot = tot + x;
    mintot = mintot + M;
    majtot = majtot + N;
end

expectedTime = tot / TIMES - w;
minorBase = mintot / TIMES;
majorBase = majtot / TIMES;

end
```

A.3 Demand function

```
function q=demand(cost_c, w)
% q = demand(cost_c, w)
% Returns the demand of the product.
% Parameters:
% cost_c = Cost of the product
% w      = Length of the base warranty

alpha=4.2;
```

```
beta=0.85;
z_1 =10^15;
z_2=0.12;

q=z_1.*cost_c.^(-alpha).*(w+z_2).^beta;
end
```

A.4 Demand function for extended warranty, policy 1

```
function q_EW = demandEW(cost_c, cost_e, w, t_e)
% q_EW = demandEW(cost_c, cost_e, w, t_e)
% Returns the demand of the product together with its extended warranty.
% Parameters:
%   cost_c = Cost of the product
%   cost_e = Cost of the extended warranty
%   w      = Length of the base warranty
%   t_e    = (Expected) length of the extended warranty
alpha=4.2;
beta=0.85;
gamma=1.4;
delta=1.4;
z_1Base = 10^15;
z_1 =10^17;
z_2=0.12;
d_EW = z_1.*cost_c.^(-alpha).*(w+z_2).^beta.*t_e.^(gamma).*(cost_e).^(-delta);
d = z_1Base.*cost_c.^(-alpha).*(w+z_2).^beta;

if d_EW <= d
    q_EW=z_1.*cost_c.^(-alpha).*(w+z_2).^beta.*t_e.^(gamma).*(cost_e).^(-delta);
else
    q_EW = d;
end

end
```

A.5 Demand function for extended warranty, policy 2

```
function q_EW = demandEWk(cost_c, cost_e, w, k)
% q_EW = demandEW(cost_c, cost_e, w, k)
% Returns the demand of the product together with its extended warranty.
% Parameters:
%   cost_c = Cost of the product
%   cost_e = Cost of the extended warranty
%   w      = Length of the base warranty
%   k      = Amount of failures covered

alpha=4.2;
beta=0.85;
gamma2=1.1808;
delta=1.4;
z_base=10^15;
z_1=10^17*0.5157;
z_2=0.12;
z_3=0;

y=z_base.*cost_c.^(-alpha).*(w+z_2).^beta;
x=z_1.*(cost_c.^(-alpha)).*((w+z_2).^beta).*(k.^(gamma2)).*((cost_e).^(-delta));
```

```
q_EW = x;
if y < x % Can't sell more EWs than products
    q_EW = y;
else
    q_EW = x;
end
end
```

A.6 Long run average cost for customer when buying an extended warranty

```
function L_cEW=LAC_cEW(cost_c,cost_e,w,t_e)
% Parameters:
% cost_c = Cost to buy the product on the market for the customer
% cost_e = Cost to buy the extended warranty for the customer
% w      = Length of base warranty (typically 12 months)
% t_e    = (Expected) length of the extended warranty

L_cEW = (cost_c + cost_e)/(w + t_e);

end
```

A.7 Long run average cost for manufacturer when customer buys an extended warranty, policy 1

```
function L_mEW=LAC_mEW(cost_m,c_1,c_2,minor,major,w,t_e)
% Long run average cost for the manufacturer when a customer buys an
% extended warranty with policy 1.
% Parameters:
% cost_m = Manufacturing cost of the product
% c_1    = Cost for the manufacturer to repair a minor failure
% c_2    = Cost for the manufacturer to repair a major failure
% minor  = Amount of minor failures during the time of the warranties (base
%          and extended)
% major  = Amount of major failures during the time of the warranties (base
%          and extended)
% w      = Length of the base warranty (typically 12 months)
% t_e    = Length of the extended warranty

L_mEW = (cost_m + c_1*minor + c_2*major)/(w + t_e);

end
```

A.8 Long run average cost for manufacturer when customer buys an extended warranty, policy 2

```
function L_mEW2=LAC_mEW2(cost_m,c_1,c_2,q,k,minorBase,majorBase,w,t_e)
% Long run average cost for the manufacturer when a customer buys an
% extended warranty with policy 2.
% Parameters:
% cost_m    = Manufacturing cost of the product
% c_1       = Cost for the manufacturer to repair a minor failure
% c_2       = Cost for the manufacturer to repair a major failure
% q         = Probability of a major failure
% k         = Amount of failures covered by the extended warranty
% minorBase = Amount of minor failures during the base warranty
% majorBase = Amount of major failures during the base warranty
```

```
% w         = Length of the base warranty (typically 12 months)
% t_e       = Expected length of the extended warranty

L_mEW2=(cost_m + c_1*((1-q)*k + minorBase) + c_2*(q*k + majorBase))/(w+t_e);

end
```

A.9 Long run average cost for the customer when only buying a product

```
function L_cw=LAC_cW(cost_c,w)
% Parameters:
% cost_c = Cost to buy the product on the market for the customer
% w      = Length of base warranty (typically 12 months)

L_cw=cost_c/w;

end
```

A.10 Long run average cost for the manufacturer when customers only buys a product

```
function L_mW=LAC_mW(cost_m,c_1,c_2,minorBase,majorBase,w)
% Long run average cost for the manufacturer when a customer does not buy
% an extended warranty.
% Parameters:
% cost_m    = Manufacturing cost of the product
% c_1       = Cost for the manufacturer to repair a minor failure
% c_2       = Cost for the manufacturer to repair a major failure
% minorBase = Amount of minor failures during the base warranty
% majorBase = Amount of major failures during the base warranty
% w         = Length of the base warranty (typically 12 months)

L_mW=(cost_m + c_1*minorBase + c_2*majorBase)/w;

end
```

A.11 Profit function, policy 1

```
function profit_mEW1 = profit_mEW1(x)
% Profit of the manufacturer when a customer buys an extended warranty of
% type 1.
% Parameters:
% cost_m = Manufacturing cost of the product
% cost_c = Cost to buy the product on the market for the customer
% cost_e = Cost to buy the extended warranty for the customer
% c_1    = Cost for the manufacturer to repair a minor failure
% c_2    = Cost for the manufacturer to repair a major failure
% minor  = Amount of minor failures during the time of the warranties (base
%          and extended)
% major  = Amount of major failures during the time of the warranties (base
%          and extended)
% w      = Length of the base warranty (typically 12 months)
% t_e    = Length of the extended warranty


cost_e = x(1);
```

```
t_e = x(2);

% assign values to parameters
cost_c = 350;
cost_m = 200;
c_1 = 80;
c_2 = 170;
q = 0.2;
w = 1;

% failure distribution parameters for drills
kappa = 3;
theta = 1.5;

%calculate expected number of minor and major failures using t_e and w
[tot  minorBase majorBase minor major ratio] = failures(kappa,theta,(t_e + w),w,q);

% total long run profit
profit_mEW1 = -((demand(cost_c, w)- demandEW(cost_c,cost_e,w,t_e))* (LAC_cW(cost_c,w)-LAC_mW(cost

end
```

A.12 Profit function, policy 2

```
function profit_mEW2 = profit_mEW2(x)
% Profit of the manufacturer when a customer buys an extended warranty of
% type 2.
% Parameters:
% cost_m    = Manufacturing cost of the product
% cost_c    = Cost to buy the product on the market for the customer
% cost_e    = Cost to buy the extended warranty for the customer
% c_1       = Cost for the manufacturer to repair a minor failure
% c_2       = Cost for the manufacturer to repair a major failure
% q         = Probability of a major failure
% k         = Amount of failures covered by the extended warranty
% minorBase = Amount of minor failures during the base warranty
% majorBase = Amount of major failures during the base warranty
% w         = Length of the base warranty (typically 12 months)
% t_e       = Expected length of the extended warranty

cost_e = x(1);
k = x(2);

% assign values to parameters
cost_c = 350;
cost_m = 200;
c_1 = 80;
c_2 = 170;
q = 0.2;
w = 1;

% failure distribution parameters for drills
kappa =3;
theta =1.5;


[t_e, minorBase, majorBase] = failuresAmount(kappa, theta, k, w, q);
```

```
profit_mEW2 = -((demand(cost_c,w) - demandEWk(cost_c, cost_e, w, k))...
*(LAC_cW(cost_c,w)-LAC_mW(cost_m,c_1,c_2,minorBase,majorBase,w))...
+ demandEWk(cost_c, cost_e, w, k)*(LAC_cEW(cost_c,cost_e,w,t_e)...
 - LAC_mEW2(cost_m,c_1,c_2,q,k,minorBase,majorBase,w,t_e)));

end
```

A.13 Main program, policy 1

```
%
% Main program 1,
%
% maximizing profit of policy 1
% w.r.t. length and price of extended warranty
%
% % % % % % % % % % % % % % % % % % % % % % % %

clc,
clear all

cost_c = 350;       % price of product
TPrim = 4;          % maximum length of extended warranty
minPrice = 10;      % minimum price of extended warranty
maxPrice = cost_c;     % maximum price of extended warranty
times = 0;

for T = 1:0.5:TPrim

x0 = [160 ; T]; % starting point

% Constraints, matrix & vector
A = [1 0 ;
    0 1 ;
    -1 0 ;
    0 -1 ];

b = [maxPrice ; T; -minPrice ; -T];


% minimizing
options=optimset('Algorithm','interior-point','AlwaysHonorConstraints',...
    'bounds', 'InitBarrierParam', 0.15,'TolX',1e-10, 'Tolfun', 1e-13);

times = times + 1;
[xStar(:,times) fval(times)] = fmincon(@profit_mEW1,x0,A,b,[],[],[],[],[],options);

end

%find optimal
minf = min(fval);
index = find(fval==minf);

% optimal solutions
extendedWarrantyPrice = xStar(1,index)
timeOfExtendedWarranty = xStar(2,index)*12
```

```
maxProfit = -minf


shareOfExtendedWarranties = demandEW(cost_c, extendedWarrantyPrice, 1, ...
    timeOfExtendedWarranty/12)/demand(cost_c, 1)

numberOfExtendedWarranties = demandEW(cost_c, extendedWarrantyPrice, 1,...
    timeOfExtendedWarranty/12)

numberOfSoldProducts = demand(cost_c, 1)
```

A.14 Main program, policy 2

```
%
% Main program 2,
%
% maximizing profit of policy 2
% w.r.t. the number k, and price of extended warranty
%
% % % % % % % % % % % % % % % % % % % % % % % % % % %


clc,
clear all

khat = 6;           % maximum length of extended warranty
minPrice = 50;      % minimum price of extended warranty
cost_c = 350;       % price of product
maxPrice = cost_c;     % maximum price of extended warranty


for k =2:khat

    x0 = [160 ; k];  % starting point

    % Constraints, matrix & vector
    A = [1 0 ;
        0 1 ;
        -1 0 ;
        0 -1 ];

    b = [maxPrice ; k ; -minPrice ; -k];

    % minimizing function
    options=optimset('Algorithm','interior-point','AlwaysHonorConstraints', ...
        'bounds', 'InitBarrierParam', 0.15,'TolX',1e-11,'Tolfun', 1e-13);

    [xStar(:,k-1) fval(k-1)] = fmincon(@profit_mEW2,x0,A,b,[],[],[],[],[],options);

end

%find optimal
minf = min(fval);
index = find(fval==minf);

% optimal solutions
extendedWarrantyPrice = xStar(1,index)
```

```
numberOfRepairs = xStar(2,index)
maxProfit = -minf


shareOfExtendedWarranties = demandEWk(cost_c, extendedWarrantyPrice, 1, ...
    numberOfRepairs)/demand(cost_c, 1)

numberOfExtendedWarranties = demandEWk(cost_c, extendedWarrantyPrice, 1, ...
    numberOfRepairs)

numberOfSoldProducts = demand(cost_c, 1)
```

A.15 To plot profit functions

```
clear all
cost_e = 10:10:350; upperT = 3; kmin=2; kmax=5; kvec=kmin:kmax;
kappa = 3; theta = 1.5; Tvec = 1*(1:0.5:upperT); w=1; q=0.2;
prof1 = zeros(length(Tvec),length(cost_e));
prof2 = zeros(kmax-kmin+1,length(cost_e));
dem = zeros(length(Tvec),length(cost_e));
[totBase minorBase majorBase qestBase] = failures(beta,theta,w,q);
tot=zeros(1,length(Tvec)); minor=zeros(1,length(Tvec));
major=zeros(1,length(Tvec)); qest=zeros(1,length(Tvec));
eTimeYears=zeros(1,length(kvec));

%%
for j = 1:length(cost_e)
    for i = 1:length(Tvec)
        prof1(i,j) = profit_mEW1(cost_e(j),Tvec(i));
    end
    for ks = kmin:kmax
        prof2(ks-1,j) = profit_mEW2(cost_e(j),ks);
    end
end

prof1(prof1<0) = 0; % Removes negative entries
prof2(prof2<0) = 0; % to give a better scale

figure(1), clf
subplot(1,2,1)
surf(cost_e, Tvec, prof1)
xlabel('Cost of extended warranty'), ylabel('T_1'), zlabel('Profit'), title('Policy 1')
axis([0 350 1 upperT 0 120000])
subplot(1,2,2)
surf(cost_e, kvec, prof2)
xlabel('Cost of extended warranty'), ylabel('k'), zlabel('Profit'), title('Policy 2')
axis([0 350 kmin kmax 0 120000])

% To find highest points of the surfaces
[r,c] = find(prof1==max(prof1(:)));
Opt = [Tvec(r) cost_e(c)]
OptProfit = max(prof1(:))

[s,d] = find(prof2==max(prof2(:)));
OptK = [kvec(s) cost_e(d)]
OptProfitK = max(prof2(:))
```