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Fourier-Whittaker Coefficients of Eisenstein Series on $SL(3, \mathbb{A})$

With Applications in String Theory

Master's Thesis in Physics and Astronomy

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Abstract

We explore means of evaluating Fourier-Whittaker coefficients on p-adic Lie groups by evaluating in explicit detail these coefficients with respect to a generic unitary character on the upper unipotent subgroup of $\mathrm{SL}(3, \mathbb{Q}_p)$. The method is by expanding the integrand into an explicit complex-valued function on $\mathbb{Q}_p^3$ and evaluating the resulting integral. This has applications in evaluating similar integrals that appear in compactified type IIB string theory.

Keywords: Lie groups, Lie algebras, Automorphic forms, Eisenstein series, p-adic numbers, string theory, mathematical physics.

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Introduction

The notion of an automorphic form is an important one in several branches of mathematics, as well as string theory and some other parts of physics such as statistical physics. The automorphic forms provide a deep connection between number theory, group theory, and representation theory.

An automorphic form is in essence a function from a topological group to the complex plane which transforms nicely under some discrete subgroup and satisfies some condition on its derivatives and growth. One of the simplest and most well known cases, although they are seldom called automorphic, is probably the complex exponentials $e^{2\pi inx}$. These are functions from the group of real numbers under addition, and they transform trivially under the subgroup of integers under addition, as well as satisfying $\frac{\partial^2 f}{\partial x^2} = -(2\pi n)^2 f$.

The most well known automorphic forms which are actually referred to as automorphic forms are the modular forms. Here the group itself is $\mathrm{SL}(2, \mathbb{R})$ and the discrete subgroup is the modular group $\mathrm{SL}(2, \mathbb{Z})$, and we demand that the function is holomorphic. These and others can be formed by considering Eisenstein series; sums over the image of some nice function over the discrete subgroup in question.

These appear in string theory when considering corrections in maximally super-symmetric string theories. Then the moduli-spaces are symmetric spaces, and it is known that the corrections must be automorphic forms on these symmetric spaces. We will work out one example in some detail: the first order corrections to gravity in type IIB string theory in 10 dimensions. The Fourier-expansions of these corrections give us physical information.

In this thesis we will work out in explicit detail how one might evaluate the Fourier-transforms of the Eisenstein series of the group $\mathrm{SL}(3, \mathbb{R})$. The results have been known for a long time and can be found by exploiting the symmetries inherent in the group. This is not always possible, so we shall do it without these symmetries.

The reason for doing this is to find methods of evaluating the corresponding Fourier-transforms of more complicated Lie groups such as the exceptional groups E_7 or E_8. The automorphic forms of these groups appear as coefficients in higher-order quantum corrections to Einstein gravity. Their Fourier-transforms contain information about the physics involved, but evaluating their Fourier-transforms is difficult. So instead we will look at $\mathrm{SL}(3, \mathbb{R})$ which is in some sense the easiest non-trivial example.

The theory of automorphic forms over real groups is best approached using the framework of the rational Adeles. These are objects which contain the real numbers as well as other number-fields known as the p-adics, which contain the number-theoretic information that is lost when going from the rational numbers to the reals. As we have mentioned, the automorphic forms find applications in number theory, so the number theoretical content has to enter the theory somehow. By going through the Adeles the number theory becomes much more natural and easier to manage.

The first part of this thesis, and a quite large one at that, is an overview of Lie theory and the basics of complex, real and Adelic Lie groups, and the p-adic numbers and the Adeles. A secondary goal here might be that this thesis can provide a good overview for someone who knows very little about any or all of these subjects. We do however not go into any great detail and almost all proofs are omitted except when they are particularly illuminating. We do try to provide at least handwavy motivations for all concepts introduced.

The rest of the thesis is mostly dedicated to deriving the so-called Fourier-Whittaker coefficient of the Eisenstein series of $\mathrm{SL}(3, \mathbb{Q}_p)$, where $\mathbb{Q}_p$ is the field of p-adic numbers. We go from the basic definition of the Eisenstein series as the sum over an abstract coset of an Adelic group all the way to the evaluation of an explicit Fourier-transform over $\mathbb{Q}_p^3$.

We will also cover some ways one might attempt to evaluate these Fourier transforms that fail, and point out why they fail, so that others need not try the same approach unless they suspect their integral is some special case where this works nicely.

There will also be two detours through two big theorems that make up the theory of Fourier-Whittaker coefficients of Eisenstein series: Langlands constant term formula and the Casselman-Shalika formula. The proofs we depict for these two theorems serve as inspiration for the methods we use in our own calculations.

2

Preliminaries

In this chapter we will introduce some of the necessary theory that we will be using, but which might not be familiar to your average fifth year physics student.

To start we will cover the basics of group theory and Lie theory before moving on to the construction of semi-simple Lie algebras using the Cartan matrix. This perspective writes each semi-simple Lie algebra as a combination of multiple $\mathfrak{sl}(2, \mathbb{C})$ Lie algebras with simple commutation relations.

We will also introduce the p-adic numbers and the Adeles which will feature heavily in the theory of automorphic forms. They provide an alternative notion of continuity than what we are used to from the real numbers, and encapsulate quite naturally the number theoretic information that exists even in real automorphic forms.

Anyone already familiar with these topics can skip most of this, since we will not be doing anything new or out of the ordinary. The exceptions might be the Fourier transforms we derive in Section 2.2.6, which are probably not integrals that everyone knows by heart. But in any case they are not that difficult to evaluate.

This material is entirely non-controversial so we will not be sourcing all statements individually. The primary source of this thesis is the (as of yet unprinted) book by D. Persson et al [1], who is also the supervisor of this thesis. That book contains (in principle) everything needed. For primary sources and a more complete description we refer to; C.C. Pinter [2] for discrete groups and the notions from abstract algebra; the books by D. Bump [3] or V.G. Kac [4] for complex Lie groups; Deitmar's book [5] for the p-adics, and L. Brekke and P.G.O Freund's text on p-adics in physics [6] for a softer treatment; and for the Adelic Lie groups we refer to A. Weil [7], who first introduced them.

2.1 Lie Groups and Lie Algebras

This thesis is about the evaluation of Fourier-Whittaker coefficients of the semi-simple Lie group $\mathrm{SL}(3,\mathbb{R})$, so it makes sense that we need to understand some Lie theory. Therefore we are going to walk through the basics of group theory and Lie algebras, before we tackle the Cartan-matrix perspective of semi-simple Lie algebras. This way of looking at a Lie group is the one we will be using in the entirety of the thesis.

2.1.1 Basic Group Theory

A *group* G is a set G_S (typically written just G, making no distinction between the set and the group) together with an associative multiplication between two objects of the set such that the multiplication and inverses are closed in G. That is, we demand that for any x,y,$z \in G$

$$xy \in G, \tag{2.1}$$

$$(xy)z = x(yz), \tag{2.2}$$

$$\exists! e \in G : ex = xe = x, \tag{2.3}$$

$$\exists x^{-1} \in G : xx^{-1} = x^{-1}x = e. \tag{2.4}$$

We call a group *Abelian* or *commutative* if for all x,$y \in G$ we have $xy = yx$.

A *subgroup* of a group G is a subset Γ of G such that Γ is itself a group, with the same multiplication.

We can define the *direct product* between two groups,

$$G_1 \times G_2 = \{(g_1,g_2) : g_1 \in G_1, g_2 \in G_2\}, \tag{2.5}$$

where the group multiplication is taken to be component-wise. Note in particular that this is *not* the same thing as writing $G_1G_2 = \{g_1g_2 : g_1 \in G_1, g_2 \in G_2\}$, where G_1 and G_2 are both subsets of some larger group G. In the first case, the two groups do not talk to each other, while in the second case they do so explicitly. We will be making use mostly of the second kind of product, but this should be clear from context.

For a subgroup Γ of G we can for any $g \in G$ define the *left coset*

$$g\Gamma = \{g\gamma : \gamma \in \Gamma\} \tag{2.6}$$

And similarly for the *right coset* Γg.

From this we can define a *quotient set* of a group G with respect to a subgroup Γ. This is the set consisting of all different left (or right) cosets of Γ. That is,

$$G/\Gamma = \{g\Gamma : g \in G\} = \{\{g\gamma : \gamma \in \Gamma\} : g \in G\}, \tag{2.7}$$

and equivalently for the left quotient. This splitting of the group G into cosets can also be seen as introducing an equivalence relation on G, where two elements are considered equal if they form the same coset. This is entirely equivalent to two elements being considered equal if they only differ by a factor in Γ.

Of little importance to us, but still worth mentioning, is the notion of a normal subgroup. Γ is a *normal subgroup* if for any $g \in G$ we have $g\Gamma = \Gamma g$[1]. Iff Γ is a normal subgroup the coset is a group under the following multiplication

$$\forall x\Gamma, y\Gamma \in G/\Gamma, \quad x\Gamma \cdot y\Gamma = xy\Gamma. \tag{2.8}$$

This is called the *quotient group*. The normality condition is absolutely essential, without this the above product is not even well defined.

We will mainly be dealing with simple Lie groups, which by their very definition will not have *any* normal subgroups. We will however be looking at a lot of quotients, these are then always merely cosets, not groups[2].

The way we will be using these quotients and products is inside summations of matrix groups, and then the interpretation is that for a quotient we pick a single representative matrix for each coset, and use only that in the sum. This function that picks a single matrix from the coset will be implicit, but the choice of matrix will be explicit.

Let X be some set, then a *group action* is a map $\phi : G \times X \to X$, which for $x \in X$ and $g \in G$ is typically written $\phi(g,x) = gx$ or $\phi(g,x) = g(x)$, that must satisfy

$$\forall x \in X,\, ex = x \tag{2.9}$$

$$\forall x \in X, g, h \in G,\, g(h(x)) = (gh)(x), \tag{2.10}$$

where e is the identity in G and gh is the group product.

A *representation* of a group is a map from the group into the set of linear transformations acting on some vector space. We will be dealing only with matrix groups which are defined by their so called fundamental representations, e.g. a group of invertible matrices has the identity element as one representation. For any matrix group there are an infinite number of representations, many of these are trivial. For instance, putting our 2×2 matrices as a block in a 3×3 matrix and leaving the third diagonal element as 1.

A representation that is trivial in this way, that can be turned into block diagonal matrices in a suitable basis of the target vector space, is called *reducible*. A representation where this is not possible is called an *irreducible representation*.

A Lie group G is a group which is also a smooth manifold, and where both the group product and inverse are smooth functions. That G is a smooth manifold means that

[1] There is nothing normal about this condition, most subgroups are in fact not normal.

[2] To make matters more confusing, the quotient sets will often end up being subgroups anyway. Normality is only required for the quotient group to have that specific product.

there is a notion of open sets in G (it is a topological group), but also that for any $g \in G$ there is an open set $U \subset G$ containing g and a continuous bijection $\phi : U \longrightarrow V$, where V is some Euclidian vector space, and the map ϕ is smooth (i.e. infinitely differentiable). That the group product and inverse are smooth is just the statement that $\phi(xy)$ is smooth, and so is $\phi(x^{-1})$ (as a function of x).

For us, the Lie groups will tend to be matrix groups and the vector space V will be the corresponding vector space of matrices, with ϕ being the identity. For example we will consider the Lie group of 2×2 complex matrices with determinant 1, $\mathrm{SL}(2,\mathbb{C})$. The vector space V will then be the vector space of complex 2×2 matrices.

2.1.2 The Basics of Lie Algebras

A Lie algebra $\mathfrak{g}$ is a vector space together with a bilinear multiplication which is called the Lie bracket or the commutator. The Lie bracket must be closed, antisymmetric and for all x,y,$z \in \mathfrak{g}$ satisfy the Jacobi identity

$$[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0, \tag{2.11}$$

which says that a cyclic product is zero. The Lie bracket can also be written as

$$\mathrm{ad}_x(y) = [x,y], \tag{2.12}$$

which is useful in particular when one of the elements is fixed or when we are considering repeated applications of the Lie bracket, which can be written as for example $\mathrm{ad}_x^3(y) = [x,[x,[x,y]]]$. The Jacobi identity can also be written in terms of ad as

$$\mathrm{ad}_x([y,z]) = [\mathrm{ad}_x(y),z] + [y, \mathrm{ad}_x(z)]. \tag{2.13}$$

This means that it behaves just like a derivative on the product defined by the Lie bracket.

A *representation* of a Lie algebra is similar to a representation of a group. It is a map from the Lie algebra into a map of linear operators acting on some vector space. As an example we have that ad is a representation of the Lie algebra onto itself, called the adjoint representation.

A *subalgebra* $\mathfrak{s}$ of a Lie-algebra $\mathfrak{g}$ is a vector subspace which is also invariant under the Lie bracket,

$$[\mathfrak{s},\mathfrak{s}] \subset \mathfrak{s}. \tag{2.14}$$

An ideal $\mathfrak{i}$ is a subalgebra which satisfies the stronger condition

$$[\mathfrak{i},\mathfrak{g}] \subset \mathfrak{i}. \tag{2.15}$$

Every Lie algebra has two trivial ideals, the one consisting only of the identity element 0, and the full Lie algebra. A Lie algebra which is not Abelian and does not have any non-trivial ideals is called *simple*. A Lie algebra which is a direct vector space sum of simple Lie algebras is called *semi-simple*. We will in this thesis mostly be concerned with the semi-simple Lie algebras.

A real Lie algebra $\mathfrak{g}(\mathbb{R})$ is said to be a *real form* of a complex Lie algebra $\mathfrak{g}(\mathbb{C})$ if when we replace the scalars in $\mathfrak{g}(\mathbb{R})$ by complex numbers, we get $\mathfrak{g}(\mathbb{C})$.

2.1.3 The Connection Between Lie Algebras and Lie Groups

We have thus far not said a word about how the notions of Lie groups and Lie algebras are connected. Given a Lie group we can define a related Lie algebra. This is a vector space which consists of directions we can move within the Lie group. The way we tend to think of it in physics is that if the Lie group represents some sort of transformation (i.e. it has some action on some space we are interested in), the Lie algebra elements correspond to infinitesimal transformations. We say that the Lie algebra elements generate these transformations, and call them generators. The Lie bracket $[x,y]$ is interpreted as the difference between doing x followed by y, and y followed by x, or just $[x,y] = xy - yx$.

Formally we define the Lie algebra $\mathfrak{g}(G)$ of a Lie group G as the tangent space at the identity element. For general Lie groups this is somewhat complicated to describe, and involves the maps that defined it as a smooth manifold. For us, we will only consider matrix Lie groups, where differentiation is easy to understand.

Consider the Lie group SO(2) consisting of rotations in $\mathbb{R}^2$. This group can be parametrised by for example

$$U(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{2.16}$$

For small values of θ (that is, close to the identity-element at $\theta = 0$) we have

$$U(\mathrm{d}\theta) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathrm{d}\theta + \mathcal{O}(\mathrm{d}\theta^2). \tag{2.17}$$

By rescaling $\mathrm{d}\theta$ we can get any real multiple of the matrix above. Thus the Lie-algebra $\mathfrak{so}(2) = \mathfrak{g}(\mathrm{SO}(2))$ is isomorphic to the vector space $\mathbb{R}$, equipped with the Lie bracket $[x,y] = 0$. This follows directly from the fact that SO(2) is an Abelian group.

For a more complicated example, one can show that the lie algebra $\mathfrak{so}(3)$ of the group of rotations in three dimensions, SO(3), is isomorphic to $\mathbb{R}^3$ with the Lie bracket being given by $[x,y] = x \times y$, the cross product[3].

[3]This is one of the reason why the cross product is so important in classical physics.

We can go back to the Lie group from the Lie algebra using the exponential map[4]

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}. \tag{2.18}$$

From this we can define the notion of the Lie bracket. We want to relate it to commutation in the Lie group, so consider for $x,y \in \mathfrak{g}(G)$

$$\exp(x)\exp(y)\exp(-x) = \exp\Big(\exp(x)y\exp(-x)\Big). \tag{2.19}$$

Now we define the Lie bracket through

$$\exp(\mathrm{ad}_x)y = \sum_{k=0}^{\infty} \frac{\mathrm{ad}_x^k(y)}{k!} = \exp(x)y\exp(-x) \tag{2.20}$$

For matrix Lie groups this leads to

$$\mathrm{ad}_x(y) = xy - yx. \tag{2.21}$$

One thing to note is that not every element of the Lie group can necessarily be written as a matrix exponential of the Lie algebra. One striking example is the negative identity. This cannot be written as a matrix exponential of any real matrix. This phenomenon vanishes for complex Lie groups and that is one of the advantages of working with them. The image of a Lie algebra under the exponential map is the *identity component.*

We can write the Lie bracket as a linear combination of basis elements. Thus, if we choose a basis $T_i \in \mathfrak{g}$ we can write

$$[T_i,T_j] = \sum_k f_{ij}{}^k T_k, \tag{2.22}$$

where the $f_{ij}{}^k$ are called *the structure constants* of the Lie algebra. This is a common way of treating finite-dimensional Lie algebras. For example we have already mentioned the Lie bracket of $\mathfrak{so}(3)$ as the cross product, which can be written as

$$T_i \times T_j = \sum_k \varepsilon_{ijk} T_k, \tag{2.23}$$

where ε_{ijk} is the totaly anti-symmetric Levi-civita tensor.

[4]Once again, these are slightly more complicated to describe when we are not in a complex matrix Lie group, and are instead solutions to differential equations of the form $\frac{\mathrm{d}g}{\mathrm{d}t} = Tg$, with the Lie algebra being the possible elements T such that $\exp(T) = g(1)$ is a group element.

2.1.4 A First Look At $\mathrm{SL}(2,\mathbb{C})$

As our first extended example we are going to look at the Lie group $\mathrm{SL}(2,\mathbb{C})$ and its real forms. A lot of the concepts introduced here will be generalised to other complex semi-simple Lie groups in the coming sections.

$\mathrm{SL}(2,\mathbb{C})$ is defined as the group of complex 2×2 matrices with determinant 1. Explicitly we have that

$$\mathrm{SL}(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a,b,c,d \in \mathbb{C}, ad - bc = 1 \right\} \tag{2.24}$$

and the group multiplication is matrix multiplication. That this forms a group is obvious from basic linear algebra. We only need to remember that a matrix is invertible if its determinant is not zero, and that for any matrices A and B, $\det(AB) = \det(A)\det(B)$.

Now we are interested in the Lie algebra of $\mathrm{SL}(2,\mathbb{C})$. This we defined as the tangent space at the identity, so we want to find the structure of

$$1 + \mathrm{d}g = \begin{pmatrix} 1 + \mathrm{d}a & \mathrm{d}b \\ \mathrm{d}c & 1 + \mathrm{d}d. \end{pmatrix} \tag{2.25}$$

The determinant condition then reads

$$(1 + \mathrm{d}a)(1 + \mathrm{d}d) - \mathrm{d}b\mathrm{d}c = 1 + \mathrm{d}a + \mathrm{d}d + \text{Higher order terms.} \tag{2.26}$$

This has to be 1, which demands that $\mathrm{d}a + \mathrm{d}d = \operatorname{Tr}\mathrm{d}g = 0$. This is the only condition we have, and thus the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ is the space of all complex 2×2 traceless matrices. A nice basis for this space consists of

$$h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \quad e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \quad f = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right). \tag{2.27}$$

Since this is a matrix group the Lie bracket acting on these basis elements can be readily calculated as

$$[e,f] = h, \quad [h,e] = 2e, \quad [h,f] = -2f. \tag{2.28}$$

We will make extensive use of a generalisation of this basis, the Chevalley basis, for more general semi-simple Lie algebras.

Topological Structure of $\mathrm{SL}(2,\mathbb{R})$

Take the basis above and consider the Lie algebra spanned by h,e,f over the real numbers. This gives us the real form $\mathfrak{sl}(2,\mathbb{R})$.

The constraint $ad - bc = 1$ means that the matrix $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ defines a Möbius transformation on $\mathbb{C}$. We can therefore define an action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{C}$ as

$$g(z) = \frac{az+b}{cz+d} = \frac{adz + bcz^* + bd + ac|z|^2}{|cz+d|^2}. \tag{2.29}$$

That this is actually an action can quickly be verified by hand. Now we note that

$$\operatorname{Im} g(z) = \frac{ad-bc}{|cz+d|^2}\operatorname{Im} z = \frac{1}{|cz+d|^2}\operatorname{Im} z, \tag{2.30}$$

so the sign of $\operatorname{Im} z$ is preserved. This means that $\mathrm{SL}(2,\mathbb{R})$ is a symmetry of the upper half plane.

Next, note that

$$g(i) = \frac{adi - bci + ad + bc}{c^2+d^2} = \frac{i + bd + ac}{c^2+d^2}. \tag{2.31}$$

If g was an element of the subgroup $\mathrm{SO}(2)$ we would get

$$g(i) = \frac{i - \sin\theta\cos\theta + \cos\theta\sin\theta}{\sin^2\theta + \cos^2\theta} = i, \tag{2.32}$$

so it turns out that $\mathrm{SO}(2)$ is a so-called stabiliser of i.

A general element of $\mathrm{SL}(2,\mathbb{R})$ can be written in the so called *Iwazawa-decomposition* as

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{2.33}$$

for some choice of x,$y > 0$,$\theta \in [0,2\pi)$. This is a decomposition of the group into a *Borel subgroup* and the maximally compact subgroup. On the Lie algebra level this corresponds to taking $\mathfrak{sl}(2,\mathbb{R}) = \operatorname{Span} e \oplus \operatorname{Span} h \oplus \operatorname{Span}(e-f)$. We will use a generalisation of this a great deal. The matrices $\begin{pmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix}$ make up the split Cartan torus. The entire Cartan torus is the maximal Abelian subgroup, and also contains the negative of these matrices.

We have

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}(i) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}(iy) = x + iy. \tag{2.34}$$

Since $\mathrm{SO}(2)$ is the stabilizer of i we thus have an easy way of finding x and y from an arbitrary matrix; just evaluate the corresponding Möbius transformation at i.

There is therefore a two-way map between the coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ and $H = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ so topologically we have that

$$\mathrm{SL}(2,\mathbb{R}) = \mathrm{SO}(2) \times H. \tag{2.35}$$

Representation Theory

If we choose another basis, such as $\{ih, (e-f), i(e+f)\}$ we end up with another real form of $\mathfrak{sl}(2,\mathbb{C})$. It is not as obvious that this construction is real under the Lie bracket, but it can be readily verified that

$$[ih,(e-f)] = 2i(e+f), \quad [ih,i(e+f)] = -2(e-f), \quad [(e-f),i(e+f)] = 2ih. \tag{2.36}$$

This results in the real form $\mathfrak{su}(2)$ which is familiar from quantum mechanics, and is a compact group. We know that the finite-dimensional representations of this group are labeled by half-integers. This fact follows from compactness, and can be lifted to representations of $\mathfrak{sl}(2,\mathbb{C})$, since a representation of one can be converted into a representation of the other. It therefore also follows for $\mathfrak{sl}(2,\mathbb{R})$.

The reason that we get half-integers labelling the representations is because $[h,e] = 2$. This 2 corresponds to a *simple root* with value 1, which means that the corresponding *fundamental weight* must be 1/2. The representations are all labelled by an integral multiple of the fundamental weights, and the eigenvalues of the representation differ by the possible roots.

We will define these notions more clearly in the coming section.

2.1.5 Cartan Matrix Perspective of Semi-Simple Lie Algebras

Using the structure constants for infinite dimensional Lie algebras quickly becomes daunting if we do not impose some structure that makes the sum contain only a finite number of terms. Even for finite dimensional ones it is a bit clunky.

A better way that works when dealing with complex semi-simple Lie algebras is to construct them from multiple copies of $\mathfrak{sl}(2,\mathbb{C})$, which is the smallest possible complex simple Lie algebra. It turns out that any complex finite-dimensional semi-simple Lie algebra can be constructed by intertwining copies of $\mathfrak{sl}(2,\mathbb{C})$. And many non-complex Lie algebras turn out to be real forms of such a complex Lie algebra.

We have already seen the commutation relations for one copy of $\mathfrak{sl}(2,\mathbb{C})$. Now we want to combine multiple $\mathfrak{sl}(2,\mathbb{C})$ algebras into a larger one. Consider a collection of $r < \infty$ triplets (e_i,f_i,h_i), $i = 1,2,3,\cdots r$, which for any i satisfy the previous commutation relations

$$[e_i,f_i] = h_i, \quad [h_i,e_i] = 2e_i, \quad [h_i,f_i] = -2f_i, \tag{2.37}$$

and for mixing between different indices we introduce the following,

$$[h_i,h_j] = 0, \quad [h_i,e_j] = A_{ij}e_j, \quad [h_i,f_j] = -A_{ij}f_j, \quad [e_i,f_j] = \delta_{ij}h_j. \tag{2.38}$$

The matrix A_{ij} is called the *Cartan matrix*, and now contains all information about the structure of the Lie algebra[5].

The number r, the dimension of the Cartan matrix, is the *rank* of the Lie algebra.

The Cartan matrix must always satisfy for all i that $A_{ii} = 2$. In this thesis we will restrict our attentions to when A is a symmetric positive definite matrix with all off-diagonal elements non-positive.

[5] Note that we are *not* using the Einstein summation convention here, the right hand sides are not sums!

We are now going to spend some time showcasing the details of this construction. Of particular importance for us will be the root system and the bilinear form, the Iwazawa-decomposition (which involves the Borel subgroup and the maximal compact subgroup), and the notion of the Weyl group.

Serre Relations

We have said nothing about commutators of the form $[e_i,e_j]$ and $[f_i,f_j]$. Without imposing any further restrictions they are all new basis elements. This means our Lie algebra is now infinite-dimensional, since we also have to include monsters of the form $\mathrm{ad}_{e_j}^n(e_k)$.

We are of course free to impose additional conditions. But what should they be? We are mostly interested in semi-simple Lie algebras, and it turns out that the above construction contains ideals which we need to get rid of. One can show that $\mathrm{ad}_{e_i}^{1-A_{ij}}(e_j)$ is an ideal element. It is instructive to compute $[f_i, \mathrm{ad}_{e_j}^n(e_k)]$ to see this in action. I suggest that the reader does this and checks what happens when $n \geq 1 - A_{ij}$, in particular when they are equal.

If we impose on our Lie algebra the further conditions[6]

$$\mathrm{ad}_{e_i}^{1-A_{ij}}(e_j) = 0 = \mathrm{ad}_{f_i}^{1-A_{ij}}(f_j), \tag{2.39}$$

we get a semi-simple Lie algebra.

Equations (2.39) are the Serre relations, and together with the information in the Cartan matrix we have the Lie algebra generated by the Cartan matrix, which we denote $\mathfrak{g}(A)$. Such a Lie algebra has a chance of being finite dimensional, but it is by no means guaranteed, since there are many other possible infinite families of commutators. We will only be dealing with Cartan matrices where the Lie algebra turns out to be finite-dimensional.

Basic Root Space Structure

Now that we know how the Lie algebra works in the Chevalley basis we can make some general statements about the Lie algebra.

The set of h_i span the maximal Abelian subalgebra which we will denote $\mathfrak{h}$. This is known as the Cartan subalgebra. The rest of the algebra is spanned by the e_i, f_i and commutators among these.

From the commutation relations we know that for any $h \in \mathfrak{h}$, ad_h is diagonalised[7]. We will call an $\alpha : \mathfrak{h} \to \mathbb{C}$ which for some $x \in \mathfrak{g}$ obeys

$$\forall h \in \mathfrak{h}, \quad [h,x] = \alpha(h)x \tag{2.40}$$

[6] Formally this is the quotient algebra $\mathfrak{g}(A) = \mathfrak{g}/\mathfrak{i}$ of $\mathfrak{g}$ with ideal $\mathfrak{i}$; so it is a proper Lie algebra.

[7] The Jacobi identity guarantees that commutators are also mapped to themselves.

a *root*. We define Δ as the set of all roots. Since α is a map from $\mathfrak{h}$ to the complex numbers, it is a member of the dual space $\mathfrak{h}^*$ consisting of functionals on $\mathfrak{h}$.

It is clear from the construction of the Cartan matrix that for any element e_i or f_i there is a corresponding root. We define a *simple* root as a root corresponding to an e_i and write it as α_i. The root corresponding to f_i is $-\alpha_i$.

For an element of the form $[h,[e_i,e_j]]$ we can use one of the forms of the Jacobi identity to get

$$\mathrm{ad}_h([e_i,e_j]) = [\mathrm{ad}_h(e_i), e_j] + [e_i, \mathrm{ad}_h(e_j)] = (\alpha_i(h) + \alpha_j(h))[e_i,e_j]. \tag{2.41}$$

It is easy to show using induction that for higher commutators we get the same result, the root for this element is just the sum of the corresponding roots. Since all commutators between the e_i and f_j lie in $\mathfrak{h}$ it follows that any root can be written as an integral linear combination of the simple roots, and with the same sign on all integers. This means that we can split up the set of roots Δ into positive roots Δ_+ and negative roots Δ_-, with the sign of the root given by the sign in the sum of the linear combination.

We introduce the root space lattice Q as the span of the simple roots over $\mathbb{Z}$, making it a module. Then any root lies in Q, but not every element of Q is a root.

For each simple root α_i we can define its *co-root* as

$$\alpha_i^\vee = h_i, \tag{2.42}$$

and extend this linearly for any root.

For any root α we can also talk of the root subspace $\mathfrak{g}_\alpha$ consisting of all $g \in \mathfrak{g}$ that for all $h \in \mathfrak{h}$ satisfy

$$[h,g] = \alpha(h)g. \tag{2.43}$$

The dimension of this subspace is called the multiplicity of the root. We can realize that if the Cartan matrix is non-degenerate, which we have assumed, any basis element corresponds to a unique such linear combination. Thus the multiplicity of any root is 1.

Therefore it makes sense to speak of *the* $\mathfrak{sl}(2,\mathbb{C})$ subalgebra belonging to a positive root. First pick an element $E_\alpha \in \mathfrak{g}_\alpha$ and $F_\alpha \in \mathfrak{g}_{-\alpha}$. Now define $H_\alpha = [E_\alpha,F_\alpha]$. Then we can normalise E_α and F_α so that we get the normal $\mathfrak{sl}(2,\mathbb{C})$ commutation relations,

$$[H_\alpha,E_\alpha] = 2E_\alpha, \quad [H_\alpha, F_\alpha] = -2F_\alpha. \tag{2.44}$$

When α is negative we define $E_\alpha = F_{-\alpha}$.

At this point we can also introduce the maps used in the so called Chevalley notation,

$$x_\alpha(z) = \exp(zE_\alpha), \tag{2.45}$$

and

$$h_\alpha(z) = z^{H_\alpha} = \exp(\log(z)H_\alpha). \tag{2.46}$$

Sometimes when clear from context we might also write x_i instead of x_{α_i} to denote this. We get

$$x_\alpha(z)x_\alpha(z') = x_\alpha(z+z') \tag{2.47}$$

and

$$h_\alpha(z)h_\alpha(z') = h_\alpha(zz'). \tag{2.48}$$

We also have

$$h_\alpha(u)x_\beta(v)h_\alpha(1/u) = x_\beta\left(u^{\langle\beta|\alpha\rangle}v\right) \tag{2.49}$$

This, and others, follows directly from the definition of the Lie bracket in Equation (2.20).

Another important object is the Weyl vector, defined as

$$\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha, \tag{2.50}$$

which will appear naturally from time to time, especially in connection with the so-called Weyl group and when dealing with sums of subsets of the roots. It is sometimes a root itself, but this is not true in general.

With the root structure in place we can provide an alternative description of the Cartan matrix. The rows are the actions of the simple roots on the simple coroots.

Invariant Bilinear Form

It is possible to define an invariant bilinear form on a Lie algebra $\mathfrak{g}(A)$. That is, a bilinear form $\langle|\rangle : \mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ which satisfies

$$\langle[x,y]|z\rangle = \langle x|[y,z]\rangle. \tag{2.51}$$

For the finite dimensional simple Lie algebras such a bilinear form is always proportional to the Killing form, which is given by the trace of the adjoint representation,

$$\langle x|y\rangle_K = \mathrm{Tr}[\mathrm{ad}_x\,\mathrm{ad}_y], \tag{2.52}$$

where we interpret ad_x as a matrix, defined by its action on the Lie algebra. The choice of scalar product defining the matrix component is irrelvant, since the trace is invariant under choice of scalar product.

We will find a nicer way of calculating it. We start from the Cartan torus. We already have a bilinear form on the space of co-roots, define for a root α_i and a co-root $\alpha_j^\vee$

$$\langle\alpha_i|\alpha_j^\vee\rangle = \alpha(\alpha_j^\vee) = A_{ij}. \tag{2.53}$$

This can quite trivially be extended to a bilinear form on the Cartan torus through

$$\langle\alpha_i^\vee|\alpha_j^\vee\rangle = \langle\alpha_i|\alpha_j^\vee\rangle = A_{ij}. \tag{2.54}$$

These are, technically, two different bilinear forms, but we will denote both by $\langle|\rangle$.

Since we have assumed that A_{ij} is a symmetric positive definite matrix, this is actually an inner product on the Cartan torus[8]. Thus, we have a new purely geometric interpretation of the Cartan matrix, it encodes the angles between two elements of the Cartan torus.

We extend this to the entirety of the Lie algebra using the invariance and the commutation relations. The invariance and the commutation relations are enough to define the bilinear form uniquely.

To see how they can be calculated let us do an example. For a simple root generator e_i we have on the one hand that

$$\langle e_i|[h_j,h_k]\rangle = \langle e_i|0\rangle = 0. \tag{2.55}$$

But also using invariance of the bilinear form

$$\langle [e_i,h_j]|h_k\rangle = -A_{ji}\langle e_i|h_k\rangle. \tag{2.56}$$

This demands that e_i and h_k are orthogonal, since the case $j = i$ ensures that the factor from the Cartan matrix does not vanish.

Weights

We defined $Q = \text{Span}(\text{Simple roots},\mathbb{Z})$, now consider $\mathfrak{h}^* = \text{Span}(\text{Simple roots},\mathbb{C})$. We extend $\langle|\rangle$ linearly in terms of the simple roots as before. Then in $\mathfrak{h}$ we can find a basis Λ_i of *fundamental weights* such that

$$\langle \alpha_i|\Lambda_j\rangle = \delta_{ij}. \tag{2.57}$$

Now we define the space of weights to be the vector space spanned by the fundamental weights over $\mathbb{C}$. This coincides with the space $\mathfrak{h}^*$. The Weyl vector ρ can be written as the sum of of the fundamental weights;

$$\rho = \sum_{i=1}^{r} \Lambda_i. \tag{2.58}$$

We also define the module Λ consisting of integral linear combinations of the fundamental weights.

A *dominant highest weight* is a an element which is an integral linear combination of the fundamental weights, with all coefficients positive. The weights might appear a bit artificial, but they appear naturally in representation theory, where they are the eigenvalues of the Cartan torus, just like the roots are eigenvalues of the adjoint representation.

[8] But only on the Cartan torus.

The finite-dimensional representations V of a complex semi-simple Lie algebra are all labelled by a dominant highest weight λ, for which there exists a vector $v \in V$ such that for $h \in \mathfrak{h}$

$$V(h)v = \langle \lambda | h \rangle v \tag{2.59}$$

$$V(\mathfrak{n}_+)v = 0. \tag{2.60}$$

That is, the vector v is an eigenvector to all matrices in the Cartan subalgebra and is annihilated by all the positive root generators.

Group Decompositions

There are three subalgebras which behave nicely with the root space structure, $\mathfrak{n}_\pm$, consisting of the positive and negative root generators respectively, and $\mathfrak{h}$. Intuitively we can think of the positive and negative root generators as generating upper and lower unit triangular matrices, and $\mathfrak{h}$ generating diagonal matrices.

Another important subalgebra is the Borel subalgebra, $\mathfrak{b} = \mathfrak{n}_+ \oplus \mathfrak{h}$, which then corresponds to generating upper triangular matrices. This will turn out to be important due to the fact that it appears in the Iwazawa-decomposition, which is our main goal right now.

We can write our Lie algebra as

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_- = \mathfrak{b} \oplus \mathfrak{n}_- \tag{2.61}$$

where $\mathfrak{n}_\pm$ is the sum of the rootspaces of the positive and negative roots, respectively,

$$\mathfrak{n}_\pm = \bigoplus_{\alpha \in \Delta_\pm} \mathfrak{g}_\alpha. \tag{2.62}$$

What we need is a notion of compact subgroups. For matrix groups these correspond to Lie algebras that consist of anti-Hermitian matrices. We want to generalise this, so we need an antilinear involution which generalises the notion of conjugate transpose.

Define the Chevalley involution ω, which acts anti-linearly[9] and on the basis elements as

$$\omega(E_\alpha) = -E_{-\alpha}, \quad \omega(h) = -h. \tag{2.63}$$

For matrix Lie algebras we we have that $g^\dagger = -\omega(g)$. By picking a basis of elements which are all invariant under ω we can generate a real form which generates a compact group. We saw this with SU(2) in (2.36), where we can verify that the basis elements are indeed invariant under ω.

[9] That is $\omega(ax + by) = a^\dagger \omega(x) + b^\dagger \omega(y)$

For a given real form $\mathfrak{g}(\mathbb{R})$ we have that[10]

$$\mathfrak{g}(\mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p}, \tag{2.64}$$

where $\mathfrak{k}$ is the subspace of $\mathfrak{g}(\mathbb{R})$ which is invariant under ω, and $\mathfrak{p}$ is anti-invariant. For the Chevalley basis, $\mathfrak{k}$ is spanned by elements of the form $e_i - f_i$ and $\mathfrak{p}$ by h_i and $e_i + f_i$. This is only a vector space sum, as $\mathfrak{p}$ is not a subalgebra of $\mathfrak{g}$. Instead we have the relations

$$[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}, \quad [\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}. \tag{2.65}$$

This is the *Cartan decomposition*. $\mathfrak{k}$ is known as the compact subalgebra. If $\mathfrak{p}$ contains the Cartan subalgebra we say that the real form $\mathfrak{g}(\mathbb{R})$ is *split*.

If $\mathfrak{g}(\mathbb{R})$ is a split real form there is an *Iwazawa-decomposition*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{h} \oplus \mathfrak{n}_+ = \mathfrak{k} \oplus \mathfrak{b} \tag{2.66}$$

This is a very important decomposition, and one we will make frequent use of.

The Iwazawa decomposition at the group level can be seen as a generalisation of the fact that any matrix can be turned into an upper triangular matrix using rotation matrices.

We will use the notation $N = \exp(\mathfrak{n}_+)$, $B = \exp(\mathfrak{b})$, K as the maximal compact subgroup, which will be $\exp(\mathfrak{k})$ for real and complex groups. We will also use the notation $N_- = \exp(\mathfrak{n}_-)$ and $A = \exp(\mathfrak{h})$ for the Cartan torus. When applicable we will use the notation $G(\mathbb{F})$ to denote subgroups of these where the exponentials are restricted to linear combinations in the set $\mathbb{F}$. For example $N(\mathbb{Z})$ for the subgroup of matrices in N with all components integers.

In all of these subgroups we implicitly include all components, not just the identity component.

The Weyl Group

Now we define the so-called Weyl group. This is an automorphism on the set of roots such that the inner product is invariant. On the group level this can be thought of as a way of changing the orders of rows and columns, but we will be starting from the roots when we define it, since there is a bit more to it than that.

Define for each $i = 1, \cdots, r$ the operator w_i as a Householder transformation under the inner product $\langle|\rangle$ which reflects α_i,

$$w_i(\gamma) = \gamma - 2\frac{\langle\alpha_i|\gamma\rangle}{\langle\alpha_i|\alpha_i\rangle}\alpha_i = \gamma - \langle\alpha_i\gamma\rangle\alpha_i. \tag{2.67}$$

[10] The symbol $\mathfrak{k}$ is actually a lower case fraktur k, believe it or not.

In terms of the simple roots we get

$$w_i(\alpha_j) = \alpha_j - A_{ij}\alpha_i. \tag{2.68}$$

Because we have defined w_i as a Householder transformation we get for free that $w_i(\alpha_i) = -\alpha_i$, $w_i^2 = 1$, $\langle \alpha_i | \gamma \rangle = 0 \Rightarrow w_i(\gamma) = \gamma$ and the bilinear form is invariant, $\langle w_i\alpha | w_i\alpha' \rangle = \langle \alpha | \alpha' \rangle$, since a Householder reflection is always an orthogonal matrix.

What does not follow for free is the fact that $w_i(\alpha_j)$ is actually a root, and not just a point in the root space lattice. Proving this is somewhat technical, so we will not do it.

With this we define the Weyl group $\mathcal{W}$ as the group generated by the fundamental reflections. This still has the property that for any $w \in \mathcal{W}$ we have $\langle w\alpha | w\alpha' \rangle = \langle \alpha | \alpha' \rangle$.

An element of the Weyl group is referred to as a word. The length $l(w)$ of a word w is the number of fundamental reflections in the shortest possible factorisation of the word.

Next we want for each fundamental reflection w_i a Lie group element $\tilde{w}_i$ such that

$$\tilde{w}_i E_\alpha \tilde{w}_i^{-1} = \pm E_{w_i\alpha} \tag{2.69}$$

and

$$\mathrm{Ad}_{\tilde{w}_i} H_\alpha = \pm H_{w_i\alpha}. \tag{2.70}$$

We need the minus signs if we want $\tilde{w}_i$ to be a part of G.

Such an element is given by

$$\tilde{w}_i = x_{-\alpha_i}(1) x_{\alpha_i}(-1) x_{-\alpha_i}(1) = \exp(f_i)\exp(-e_i)\exp(f_i). \tag{2.71}$$

The group elements $\tilde{w}$ are only determined up to multiplication by $\tilde{w}_i^2$. In the future we will never be differentiating between w and $\tilde{w}$, it should be clear from context which space they are acting on.

This might seem a bit artificial, but the Weyl group is of great help when doing calculations. We will be making use of the Bruhat decomposition, which states that a complex Lie group $G(\mathbb{C})$ can be decomposed into

$$G(\mathbb{C}) = \bigcup_{w \in \mathcal{W}} B(\mathbb{C}) w B(\mathbb{C}) \tag{2.72}$$

where $B(\mathbb{C})$ is the Borel subgroup. This also holds for any subfield of $\mathbb{C}$, in particular it holds for $G(\mathbb{Q})$, but it does not hold for $G(\mathbb{Z})$. This is one of the main reasons that we are going to introduce the Adeles, because they allows us to trade a $G(\mathbb{Z})$ symmetry for a $G(\mathbb{Q})$ symmetry.

The Weyl Character Formula*

Let V be a representation of $\mathfrak{g}(A)$, and denote by $\mathfrak{h}_V^*$ the set of weights which appear in this representation (defined as the eigenvalues of $V(H_\alpha)$ in this representation). Then we define the character of the representation as

$$\mathrm{ch}_V(h) = \mathrm{Tr}\, V(h) = \sum_{\alpha \in \mathfrak{h}_V^*} \mathrm{mult}[\alpha]\mathfrak{e}^\mu(h). \tag{2.73}$$

The expression $\mathrm{mult}[\mu]$ is the multiplicity of the weight μ, which is the dimension of the eigenspace of μ in V. The exponential $\mathfrak{e}^\mu$ is a character $\mathfrak{e}^\mu : \mathfrak{h} \longrightarrow \mathbb{C}$ satisfying $\mathfrak{e}^\mu(h) = \mathfrak{e}^{\langle \mu | h \rangle}$ and $\mathfrak{e}^\mu \mathfrak{e}^{\mu'} = \mathfrak{e}^{\mu+\mu'}$. This means that $\mathrm{ch}_V : \mathfrak{h} \longrightarrow \mathbb{C}$.

It is not at all certain that this sum converges, but it always does for finite dimensional representations, since then the number of weights is finite.

If V is a highest weight representation with dominant highest weight Λ there is another way of writing this sum, the Weyl character formula,

$$\mathrm{ch}_V = \frac{\sum_{w \in \mathcal{W}} \epsilon(w) \mathfrak{e}^{w(\Lambda+\rho)-\rho}}{\prod_{\alpha \in \Delta_+} (1 - \mathfrak{e}^{-\alpha})}, \tag{2.74}$$

where $\varepsilon(w) = (-1)^{l(w)}$ is the signature of the word w.

The typical way of defining the character is $\mathfrak{e}^\lambda(h) = e^{\langle h | \lambda \rangle}$. Of course, we are really free to normalise this however we want, so we are not forced to pick e as our base. We can also interpret $\mathfrak{e}^\lambda$ as a function of the Cartan torus, and we write

$$\mathfrak{e}^\lambda(h_\alpha(u)) = |u|^{\langle \alpha | \lambda \rangle}. \tag{2.75}$$

This is multiplicative in u and additive in α, which is what we need for $\mathfrak{e}^\lambda$ to be an exponential on $\mathfrak{h}$.

For $\mathfrak{sl}(2, \mathbb{C})$ we get for a representation V_λ with dominant highest weight λ

$$\mathrm{ch}_{V_\lambda}\left(\begin{smallmatrix} v & 0 \\ 0 & v^{-1} \end{smallmatrix}\right) = \mathrm{Tr}\left(V_\lambda\left(\begin{smallmatrix} v & 0 \\ 0 & v^{-1} \end{smallmatrix}\right)\right) = \frac{|v|^\lambda - |v|^{-\lambda-2}}{1 - |v|^{-2}}. \tag{2.76}$$

2.1.6 Example, $\mathrm{SL}(3, \mathbb{C})$

It is time we did an example. In the end we will be dealing with $\mathrm{SL}(3, \mathbb{R})$ (and Adelisations of this) so we might as well work out the details for $\mathrm{SL}(3, \mathbb{C})$.

Consider the Cartan matrix

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{2.77}$$

We have two basis elements in $\mathfrak{h}$, h_1, and h_2, three elements in $\mathfrak{n}_+$, e_1,e_2 and $[e_1,e_2]$, and corresponding elements f_1,f_2 and $[f_1,f_2]$ in $\mathfrak{n}_-$. All higher commutators are killed by the Serre relations since $1 - A_{12} = 2$. This is an 8-dimensional semi-simple Lie algebra which is isomorphic to $\mathfrak{sl}(3, \mathbb{C})$.

There are two simple roots, α_1 and α_2, which act on elements h of the Cartan torus by

$$\alpha_i(h_j) = \langle \alpha_i | h_j \rangle = A_{ij}. \tag{2.78}$$

There is also a third positive root which we denote $\alpha_3 = \alpha_1 + \alpha_2$ which corresponds to $e_3 = [e_1,e_2]$. This has a corresponding Cartan generator

$$h_3 = [[e_1,e_2],[f_1,f_2]] = h_1 + h_2. \tag{2.79}$$

The fundamental weights are given by the columns of A^{-1} which are

$$\Lambda_1 = \frac{2h_1 + h_2}{3}, \quad \Lambda_2 = \frac{h_1 + 2h_2}{3}. \tag{2.80}$$

The simple co-roots can in turn be written $h_1 = 2\Lambda_1 - \Lambda_2$ and $h_2 = -\Lambda_1 + 2\Lambda_2$.

The Weyl vector is given by

$$\rho = \frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3) = \alpha_3, \tag{2.81}$$

so ρ is actually a root in this case.

In the fundamental representation we have that the Lie algebra basis elements and exponential maps are given by

$$\begin{aligned}
H_{\alpha_1} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_{\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad H_{\alpha_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\
E_{\alpha_1} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{\alpha_3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
h_{\alpha_1}(u) &= \begin{pmatrix} u & 0 & 0 \\ 0 & 1/u & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad h_{\alpha_2}(u) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & 1/u \end{pmatrix}, \quad h_{\alpha_3}(u) = \begin{pmatrix} u & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/u \end{pmatrix}, \\
x_{\alpha_1}(u) &= \begin{pmatrix} 1 & u & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_{\alpha_2}(u) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & u \\ 0 & 0 & 1 \end{pmatrix}, \quad x_{\alpha_3}(u) = \begin{pmatrix} 1 & 0 & u \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned} \tag{2.82}$$

For the negative roots we get a sign change in the H_α and a corresponding inversion in h_α while for E_α and x_α we get a matrix transposition.

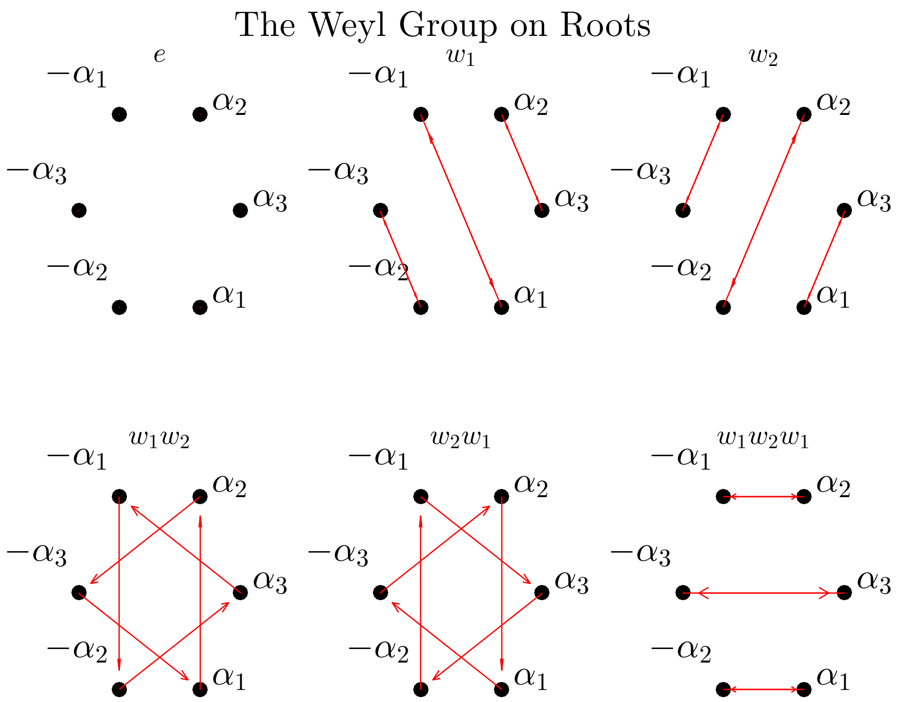

Figure 2.1: The action of the Weyl group when acting on the roots. We have three "reflections" and two "rotations". This Figure is a little misleading in how the roots are placed, the geometry means almost nothing in this picture. See Figure 2.2 for the actual geometry.

Weyl Group

Applying the definition in (2.68) we find that

$$w_1(\alpha_1) = -\alpha_1, \quad w_1(\pm\alpha_2) = \alpha_3 \pm \alpha_3, \quad w_1(\pm\alpha_3) = \pm\alpha_2, \tag{2.83}$$

and similarly

$$w_2(\pm\alpha_1) = \pm\alpha_3, \quad w_2(\pm\alpha_2) = \mp\alpha_2, \quad w_2(\pm\alpha_3) = \pm\alpha_1. \tag{2.84}$$

From this we can work out the rest of the Weyl group. It turns out to have 3 additional non-trivial elements, w_1w_2, w_2w_1 and $w_1w_2w_1 = w_2w_1w_2 = w_3$. This is a non-abelian group of order 6. There is only one such group, S_3, the symmetry group of three objects. This is isomorphic to the symmetry group of an equilateral triangle. The Weyl group of $\mathrm{SL}(n,\mathbb{R})$ is always S_n.

As can be seen in Figure 2.1 the Weyl words w_1, w_2 and w_3 correspond to reflections while w_1w_2 and w_2w_1 are essentially rotations.

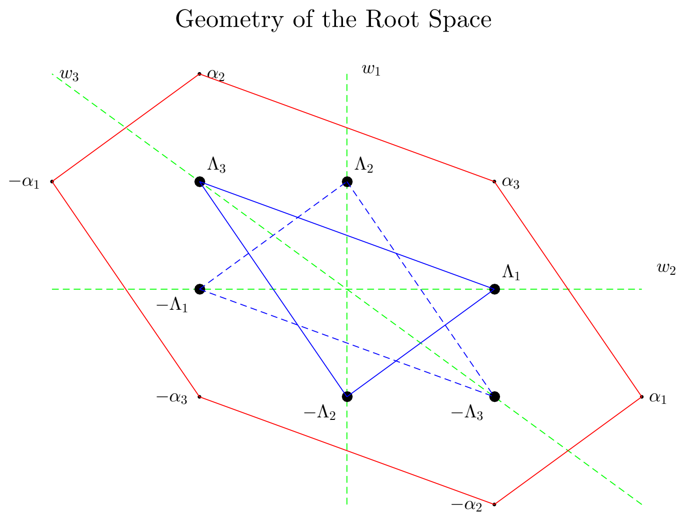

Figure 2.2: Illustration of the structures of $\mathfrak{sl}(3,\mathbb{C})$ in the weight space. All points are written in the basis of fundamental weights. The two triangles are two 3-dimensional representations, corresponding to the fundamental representation, while the hexagon is the 8-dimensional adjoint representation (it also has two zero weights, in addition to the six roots). The striped green lines show the lines through which the Weyl group reflects the rootspace.

Doing the calculation on the fundamental weights we find that

$$w_1(\Lambda_1) = \Lambda_2 - \Lambda_1, \qquad w_1(\Lambda_2) = \Lambda_2, \tag{2.85}$$
$$w_2(\Lambda_1) = \Lambda_1, \qquad w_2(\Lambda_2) = \Lambda_1 - \Lambda_2. \tag{2.86}$$

Thus, if we define $\Lambda_3 = \Lambda_2 - \Lambda_1$ we have two triangles that are left invariant by the Weyl group: $(\Lambda_1, -\Lambda_2, \Lambda_3)$ and $(-\Lambda_1, \Lambda_2, -\Lambda_3)$.

The group elements w_1, w_2 and w_3 then correspond to reflections through the bisectors of the triangles, while $w_1 w_2$ and $w_2 w_1$ are the clockwise and counter-clockwise rotations by $2\pi/3$. This is of course only true in the basis of fundamental weights, since that is an orthonormal basis under $\langle|\rangle$. The basis of simple co-roots, which we will typically use, is not orthonormal under $\langle|\rangle$ and thus the Weyl group also does shearing, as can be seen in Figure 2.2

In matrix form (in the fundamental representation) we have that by applying (2.68) we get that the fundamental reflections act by

$$\tilde{w}_1 = \begin{pmatrix} & 1 & \\ 1 & & \\ & & -1 \end{pmatrix}, \quad \tilde{w}_2 = \begin{pmatrix} -1 & & \\ & & 1 \\ & 1 & \end{pmatrix}. \tag{2.87}$$

The rest of the Weyl group is given by

$$\tilde{w}_1\tilde{w}_2 = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}, \quad \tilde{w}_2\tilde{w}_1 = \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}, \quad \tilde{w}_3 = \begin{pmatrix} & & -1 \\ & -1 & \\ -1 & & \end{pmatrix}. \tag{2.88}$$

In deriving these we have used the fact that we are allowed to multiply by $\tilde{w}_i^2$, which changes the sign of two arbitrary non-zero components.

2.2 The p-adic Numbers

The typical case in physics when studying matrix Lie groups is to consider matrices that have elements in either $\mathbb{R}$, or $\mathbb{C}$. It often happens that one can deduce things about a real Lie group based on the properties of another complex Lie group.

For example the representations of $\mathrm{SL}(2,\mathbb{R})$, the group of real 2×2 matrices with determinant 1 are related to the representations of $\mathrm{SU}(2)$, the group of 2×2 unitary matrices, due to the fact that they are both real forms of $\mathrm{SL}(2,\mathbb{C})$. Thus, even if we are interested only in the real group, it makes sense to study the complex group since it contains additional information about the real group.

However, $\mathbb{C}$ is not the only number field which contains $\mathbb{R}$ as a subfield. We are going to step back and consider the field $\mathbb{Q}$ of rational numbers. We got to $\mathbb{R}$ by demanding that limits make sense under the norm of absolute value, but this is not the only norm we can define on the rational numbers. There are also the p-adic norms which contain number theoretical information that is lost in $\mathbb{R}$.

The Adeles is then a set of objects which combines all of these different number fields into a larger one. This in some sense gets rid of the fact that we have arbitrarily picked $\mathbb{R}$ as our number field, and makes the calculations more natural.

We will extend the notion of Fourier series into the Adelic context, and evaluate a few useful Fourier transforms that will appear when we do our calculations on the (by then Adelic) Lie groups.

2.2.1 The p-adic Norm

Before we construct the p-adic numbers we should remind ourselves how we construct the real numbers. We start with the rational numbers $\mathbb{Q}$ and then we introduce the norm

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases} \tag{2.89}$$

Then we form $\mathbb{R}$ as the completion of $\mathbb{Q}$ with respect to this norm, so that all limits make sense. Explicitly, we consider the space of Cauchy sequences in $\mathbb{Q}$ under this norm, that is, sequences $\{x_n\}_{n=1}^{\infty}$ that satisfy

$$\forall \varepsilon > 0 \in \mathbb{Q}\, \exists N \in \mathbb{N} : \forall n,m > N |x_n - x_m| < \varepsilon. \tag{2.90}$$

Then one defines a real number as a sequence of rational numbers that converges, and two sequences that converge to the same real number are to be considered the same number[11].

The p-adic numbers $\mathbb{Q}_p$, where p is some prime number, are constructed in the exact same way, except we will change the norm $|\cdot|$ into something we will denote $|\cdot|_p$. Now, according to the fundamental theorem of arithmetic any integer can be factorised uniquely into prime factors, and thus for any rational number $x \neq 0$ we can write uniquely

$$x = p^k \frac{n}{m}, \tag{2.91}$$

where k,$n \in \mathbb{Z}$ and $m \in \mathbb{N}$, in such a way that neither n nor m contains any factors of p. Then we define

$$|x|_p = p^{-k}. \tag{2.92}$$

For $x = 0$ we define $|x|_p = 0$.

Of course, not just any function $|\cdot|$ defines a norm. It needs to satisfy for all x,$y \in \mathbb{Q}$

- $|xy|_p = |x|_p|y|_p$
- $|x|_p = 0 \Leftrightarrow x = 0$
- $|x + y|_p \leq |x|_p + |y|_p$.

The first two requirements are trivial. For the third, write $x = p^k \frac{n}{m}$, $y = p^l \frac{n'}{m'}$, and assume without loss of generality that $k \leq l$. Then

$$x + y = \frac{m'np^k + mnp^l}{m'm} = p^k \frac{m'n + mn'p^{l-k}}{m'm}. \tag{2.93}$$

From this we can deduce that we have at least k factors of p in $x + y$, but we can have more since $m'n + mn'p^{l-k}$ might contain additional factors of p if $l - k = 0$, but $m'm$ does not contain factors of p. Thus

$$|x + y|_p \leq p^{-k} \leq p^{-k} + p^{-l} = |x|_p + |y|_p. \tag{2.94}$$

In fact we have something much stronger than the regular triangle inequality,

$$|x + y|_p \leq \max(|x|_p,|y|_p). \tag{2.95}$$

We have equality if x and y have different norms since we must have $l - k = 0$ for the new denominator to contain further factors of p.

We will also have use of the p-adic valuation ν_p, defined by

$$|x|_p = p^{-\nu_p(x)}. \tag{2.96}$$

Then the properties of the norm translate into properties of the p-adic valuation. For x,$y \in \mathbb{Q}_p$ we have

[11] That is, a real number is an equivalence class of Cauchy sequences taking values in $\mathbb{Q}$, where two sequences x_n and y_n are equivalent if for any $\varepsilon > 0$ there is an N such that $\forall n > N, |x_n - y_n| < \varepsilon$.

- $\nu_p(xy) = \nu_p(x) + \nu_p(y)$,
- $\nu(0) = \infty$,
- $\nu(x + y) \geq \min(\nu_p(x),\nu_p(y))$.

Now that we have defined $|\cdot|_p$ we define the p-adic numbers $\mathbb{Q}_p$ as the completion of $\mathbb{Q}$ with respect to the metric $d(x,y) = |x - y|_p$ induced by $|\cdot|_p$, the same way we did with $\mathbb{R}$.

2.2.2 Basic Arithmetic

These numbers are very strange when compared with the reals. A real number is small if it has a larger denominator than numerator, a p-adic number is smaller the more factors of p it has. While a real number can be written as a decimal number, or with any other number as basis, a p-adic number is not very natural to write in any number basis other than p. An arbitrary p-adic number x with norm $|x|_p = p^k$, where $k \in \mathbb{Z}$, can be written as

$$x = a_{-k}p^{-k} + a_{-k+1}p^{-k+1} + \cdots + \frac{a_{-1}}{p} + a_0 + a_1 p + a_2 p^2 + \cdots , \tag{2.97}$$

where the a_i are arbitrary natural numbers between 0 and $p - 1$ (a_{-k} must be non-zero if $|x|_p = p^k$).

We can use the usual notation for representing real numbers also for p-adics, but the decimal expansion continues infinitely to the left, rather than to the right. So for example we could write in the 5-adics,

$$\frac{4}{5^2} + \frac{3}{5} + 2 + 5 + 5^2 + 5^3 + \ldots = \ldots 111.234. \tag{2.98}$$

Notice that we have not allowed there to be a minus sign in this. There is good reason for that. It is not actually meaningful to distinguish between positive and negative p-adic numbers. To see this let $a \in \mathbb{Q}_p$, the sequence $x_n = a + (-1)^n p^n$ converges to a in the p-adics, even though it clearly changes "sign" every other time. To make matters worse, we can find p-adic numbers that are arbitrarily close to rational numbers on both sides of 0. Consider a and $a - p^k$. The number $a + p^k$ lies at a distance p^{-k} from both these numbers (if p is not 2)

This is not actually an issue, let us expand -1 in the p-adics in this way. We have

$$-1 = \frac{p-1}{1-p} = (p-1)(1+p+p^2+p^3+\ldots) = (p-1)+(p-1)p+(p-1)p^2+\ldots. \tag{2.99}$$

This series converges absolutely since the norms of the terms decay like p^{-n}. To see why this works as -1, consider $p = 3$. We then have

$$-1 = 2 + 2 \cdot 3 + 2 \cdot 9 + 2 \cdot 27 + \cdots . \tag{2.100}$$

which means

$$1 + (-1) = (1 + 2) + 2 \cdot 3 + 2 \cdot 9 + \cdots = (1 + 2) \cdot 3 + 2 \cdot 9 + \cdots = (1 + 2) \cdot 9 + \cdots \quad (2.101)$$

and so on. Continuing this process we cancel every term in the series.

2.2.3 Important Subsets

The field $\mathbb{Q}_p$ contains a lot of subsets that will turn out to be useful.

We first define the ring of p-adic integers as

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}. \quad (2.102)$$

This set contains in particular the ordinary integers, but also other numbers. For example $\mathbb{Z}_2$ contains $1/3$, as it has no factors of 2. This is a compact set, and this will have implications later when we deal with Adelic groups.

We will also introduce the multiplicative part of the p-adic integers,

$$\mathbb{Z}_p^* = \{x \in \mathbb{Z}_p : \frac{1}{x} \in \mathbb{Z}_p\} = \{x \in \mathbb{Z}_p : |x|_p = 1\}. \quad (2.103)$$

The condition in the second equation is necesary and sufficient, $|1/x|_p = 1/|x|_p$, so both x and $1/x$ lie in $\mathbb{Z}_p$ exactly when $|x|_p = 1$. This is the unit circle, we have that the circle of radius p^k is given by

$$\{x \in \mathbb{Q}_p : |x|_p = p^k\} = p^{-k}\mathbb{Z}_p^*. \quad (2.104)$$

The important decomposition we will need is that into circles,

$$\mathbb{Q}_p = \bigcup_{k=-\infty}^{\infty} p^k \mathbb{Z}_p^*. \quad (2.105)$$

This is a disjoint decomposition, which will be of great help when we calculate integrals, which will in our case typically only depend on the norm.

2.2.4 Integration Measure

On the p-adics there is a translation invariant integration measure that scales according to what we would expect. That is it has the properties that

$$\mathrm{d}(x + a) = \mathrm{d}x, \quad \text{and} \quad \mathrm{d}(ax) = |a|_p \mathrm{d}x. \quad (2.106)$$

This will be normalized so that

$$\int_{\mathbb{Z}_p} \mathrm{d}x = 1. \quad (2.107)$$

From this we can establish that the closed ball of radius p^k has volume

$$\int_{p^{-k}\mathbb{Z}_p} \mathrm{d}x = \int_{\mathbb{Z}_p} \mathrm{d}(p^{-k}x') = p^k \int_{\mathbb{Z}_p} \mathrm{d}x' = p^k. \tag{2.108}$$

This implies that the full set of p-adics has infinite measure, just like the reals.

There is no theorem to replace the fundamental theorem of calculus. This means that in most cases we are forced to find the values of the integrals using symmetry arguments (e.g. odd functions) or by finding the sizes of certain subsets where the function we are integrating is constant (e.g. function depends only on norm). We will be using both of these techniques.

2.2.5 Special Functions

Now we will introduce some functions on the p-adics that will turn out to be useful.

We start by introducing an important function, the p-adic Gaussian,

$$\gamma_p(x) = \chi_{\mathbb{Z}_p}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Z}_p \\ 0 & \text{otherwise.} \end{cases} \tag{2.109}$$

This is simply the characteristic function of the closed unit ball $\mathbb{Z}_p$. This is related to the real Gaussian $\gamma_\infty(x) = e^{-\pi x^2}$ by the fact that they are both eigenfunctions of the Fourier transform. We will prove this in a bit. There are two related functions

$$\bar{\gamma}_p(x) = 1 - \gamma_p(x) = \begin{cases} 1 & \text{if } x \notin \mathbb{Z}_p \\ 0 & \text{otherwise,} \end{cases} \tag{2.110}$$

which is the characteristic function of $\mathbb{Q}_p \backslash \mathbb{Z}_p$, and

$$\gamma_p^*(x) = \gamma_p(x) - \gamma_p(x/p) = \begin{cases} 1 & \text{if } x \in \mathbb{Z}_p^* \\ 0 & \text{otherwise,} \end{cases} \tag{2.111}$$

which is the characteristic function of $\mathbb{Z}_p^*$.

Another function that will turn out to be useful is

$$J_p(x) = \tilde{x} = \begin{cases} 1 & \text{if } x \in \mathbb{Z}_p \\ x & \text{otherwise.} \end{cases} \tag{2.112}$$

This looks like a mildly uninteresting function at the moment, but it plays an important role in the Iwazawa-decomposition of $\mathrm{SL}(2, \mathbb{Q}_p)$, which will be vital when we look at automorphic forms. It has a few properties that follow straight from the definition, the first being that

$$|\tilde{x}|_p = \max(1, |x|_p) = \gamma_p(x) + \bar{\gamma}_p(x)|x|_p \tag{2.113}$$

and the second that for any $x \in \mathbb{Q}_p$ and $n \in \mathbb{Z}_p$

$$J_p(x+n) = J_p(x). \tag{2.114}$$

What we want to do in the end is evaluate something similar to a Fourier transform. Then we need to know how to generalise $e^{i\omega x}$. This is what is known as a unitary character.

In order to do that we will need $[y]_p$, the p-adic fractional part of y,

$$\left[\sum_{k=k_0}^{\infty} a_k p^k\right]_p = \sum_{k=k_0}^{-1} a_k p^k, \tag{2.115}$$

which is guaranteed to be a rational number.

If x is a rational numbers such that $x \in \mathbb{Z}_p$ for all p, then p is an integer. As a corollary to this, the number $x - \sum_{p<\infty}[x]_p$ is always an integer.

A *unitary character* on a group G is a group homomorphism into the unitary group,

$$\psi : G \to U(1). \tag{2.116}$$

For a p-adic number, with the group operation $+$ we can write such a character as

$$\psi_p^m(x) = e^{2\pi i [mx]_p}. \tag{2.117}$$

Since $[mx]_p$ is always rational this is just the ordinary complex exponential. The number m is the modenumber of the character, and can be any p-adic number. For $p = \infty$ we can simply let the fractional part $[x_\infty]_\infty$ be the entire real number x_∞.

It is easily verified that

$$[x+y]_p \neq [x]_p + [y]_p, \tag{2.118}$$

just take $x = \frac{p-1}{p}$, $y = \frac{1}{p}$. However, we clearly have that

$$[x+y]_p - [x]_p - [y]_p \in \mathbb{Z}, \tag{2.119}$$

so the exponential function is trivial there, thus, even though the fractional part is not linear, the character defined in this way is is still multiplicative.

2.2.6 Fourier Transform

Now that we have introduced norms, p-adic integration and characters we are ready to consider the integrals that we will use in evaluating the Fourier-Whittaker coefficients later. The Fourier transform is an integral of the form

$$\mathcal{F}_x[f(x)](m) = \int_{\mathbb{Q}_p} f(x) e^{-2\pi i [mx]_p} \, \mathrm{d}x. \tag{2.120}$$

This has the normal translation, scaling and linearity properties we are used to, i.e.,

$$\mathcal{F}_x[f(x - x_0)](m) = e^{-2\pi i [m x_0]_p} \mathcal{F}_x[f(x)](m), \tag{2.121}$$

$$\mathcal{F}_x[f(ax)](m) = \frac{1}{|a|_p} \mathcal{F}_x[f(x)][\tfrac{m}{a}], \tag{2.122}$$

$$\mathcal{F}_x[af(x) + bg(x)](m) = a\mathcal{F}_x[f(x)](m) + b\mathcal{F}_x[g(x)](m). \tag{2.123}$$

In this section we will be calculating a number of useful integrals. Our main goal is to evaluate the Fourier transform $\mathcal{F}[|\tilde{x}|_p^s](m)$, since these are the functions that will appear in our Whittaker-coefficients.

The first thing we will do is build up so that we can evaluate the Fourier transform of a constant over $\mathbb{Z}_p^*$. The integrand we are after only depends on the norm. Therefore we decompose $\mathbb{Q}_p$ into circles of the form $p^k \mathbb{Z}_p^*$ and take a sum, like in (2.105). The end result is a geometric series that can be evaluated readily.

The first thing we will do is show that γ_p is a fixed point of the Fourier transform.

Lemma 2.1.

$$\mathcal{F}[\gamma_p(x)](m) = \gamma_p(m). \tag{2.124}$$

Proof. We can derive this from the fact that

$$\mathcal{F}_x[\gamma_p(x)](m) = e^{2\pi i [m]_p} \mathcal{F}_x[\gamma_p(x-1)](m) = e^{2\pi i [m]_p} \mathcal{F}_x[\gamma_p(x)](m). \tag{2.125}$$

Here we have used the translation property, and used the fact that $\mathbb{Z}_p$ is invariant under translation by an integer. This shows that either we must have $e^{2\pi i [m]_p} = 1$, which demands that $m \in \mathbb{Z}_p$ and the remaining integral is 1, or the integral must vanish. This gives us the result. □

From this we can derive the Fourier transform of γ_p^*

Corollary 2.2.

$$\mathcal{F}_x[\gamma_p^*(x)](m) = \gamma_p(m) - \frac{1}{p}\gamma_p(pm). \tag{2.126}$$

Proof. This follows directly from the fact that $\gamma_p^*(x) = \gamma_p(x) - \gamma_p(x/p)$, using the linearity and scaling properties. □

Now we can prove the following about Fourier transforms of functions which only depend on the norm.

Lemma 2.3. *If $\phi : \mathbb{Q}_p \longrightarrow \mathbb{C}$ is a function only of the norm, we have*

$$\mathcal{F}[\phi(x)](m) = \sum_{k=-\infty}^{\nu_p(m)} p^k \left(\phi(p^{-k}) - \phi(p^{-(k+1)})\right). \tag{2.127}$$

Proof. We have

$$\begin{aligned}
\mathcal{F}[\phi(x)](m) &= \int_{\mathbb{Q}_p} \phi(x) e^{-2\pi i [mx]_p} \, \mathrm{d}x &(2.128)\\
&= \sum_{k=-\infty}^{\infty} \int_{p^{-k}\mathbb{Z}_p^*} \phi(x) e^{-2\pi i [mx]_p} \, \mathrm{d}x &(2.129)\\
&= \sum_{k=-\infty}^{\infty} \int_{\mathbb{Z}_p^*} \phi(p^{-k}x') e^{-2\pi i [mp^{-k}x']_p} |p^{-k}|_p \, \mathrm{d}x' &(2.130)\\
&= \sum_{k=-\infty}^{\infty} \phi(p^{-k}) p^k \mathcal{F}[\gamma_p^*(x)](mp^{-k}) &(2.131)\\
&= \sum_{k=-\infty}^{\infty} \phi(p^{-k}) p^k \left(\gamma_p(p^{-k}m) - \frac{1}{p}\gamma_p(p^{-k+1}m)\right), &(2.132)
\end{aligned}$$

where we got rid of x' inside ϕ since it has norm 1. Now shift the second term by $k \mapsto k+1$. This leaves us with

$$\mathcal{F}[\phi(x)](m) = \sum_{k=-\infty}^{\infty} \gamma_p(p^{-k}m) p^k \left(\phi(p^{-k}) - \phi(p^{-(k+1)})\right). \tag{2.133}$$

We can write $m = p^{\nu(m)}$ in γ, which means our sum should only run to $k = \nu_p(m)$ and thus

$$\mathcal{F}[\phi(x)](m) = \sum_{k=-\infty}^{\nu_p(m)} p^k \left(\phi(p^{-k}) - \phi(p^{-(k+1)})\right). \tag{2.134}$$

□

We will later need a Lemma of the form

Lemma 2.4. *If a is a p-adic number such that $|a|_p \leq |m|_p$ we have*

$$\mathcal{F}[\gamma_p(ax)\phi(x)](m) = \mathcal{F}[\phi(x)](m) + \gamma_p^*\left(\frac{a}{m}\right)\frac{1}{|m|_p}\phi\left(\frac{1}{mp}\right). \tag{2.135}$$

Proof. Since γ_p is a function only of the norm we still fulfil the condition in Lemma 2.3. Thus

$$\mathcal{F}[\gamma_p(ax)\phi(x)](m) = \sum_{k=-\infty}^{\nu_p(m)} p^k \left(\gamma_p(ap^{-k})\phi(p^{-k}) - \gamma_p(ap^{-(k+1)})\phi(p^{-(k+1)})\right). \tag{2.136}$$

Since $|a|_p \leq |m|_p$ we have that ap^{-k} is an integer for $k \leq \nu_p(m)$. This means that we can ignore them for all terms except when $k = \nu_p(m)$, where we have an extra $+1$ in the second term. This term appears if and only if $|a|_p = |m|_p$, and thus we have to compensate for it. Therefore

$$\mathcal{F}[\gamma_p(ax)\phi(x)](m) = \mathcal{F}[\phi(x)](m) + \gamma_p^* \left(\frac{a}{m}\right) p^\nu \phi(p^{-(\nu_p(m)+1)}). \tag{2.137}$$

Using the fact that $\frac{1}{m}$ has the same norm as $p^{-\nu_p(m)}$ the result follows. □

Now we are ready to state what we are actually after. These are the Fourier transforms that will appear when we try to find Whittaker-coefficients of Eisenstein series,
Proposition 2.5.

$$\mathcal{F}[|\tilde{x}|_p^s](m) = \gamma_p(m)\frac{1-p^s}{1-p^{s+1}}\left(1 - |pm|_p^{-(s+1)}\right). \tag{2.138}$$

Proof. We can begin by noting that since $|\tilde{x}|_p^s = \gamma_p(x) + \bar{\gamma}_p(x)|x|_p^s$ we need the Fourier transforms of $\gamma_p(x)$ and $\bar{\gamma}_p(x)|x|_p^s$. We already know the first of these, and the second is of the form in the lemma we just proved with $\phi(x) = \bar{\gamma}_p(x)|x|_p^s$. We therefore have

$$\mathcal{F}[\bar{\gamma}_p(x)|x|_p^s](m) = \sum_{k=-\infty}^{\nu_p(m)} p^k \left(\bar{\gamma}_p(p^{-k})p^{ks} - \bar{\gamma}_p(p^{-(k+1)})p^{(k+1)s}\right). \tag{2.139}$$

The $\bar{\gamma}_p$-factor demands that $k \geq 1$ and $k \geq 0$, respectively in each term. Adding the missing $k = 0$ to the first term we end up with

$$\mathcal{F}[\bar{\gamma}_p(x)|x|_p^s](m) = -\gamma_p(m) + \gamma_p(m)\sum_{k=0}^{\nu_p(m)} \left(p^{k(s+1)} - p^{(k+1)(s+1)-1}\right). \tag{2.140}$$

Evaluating this geometric series yields

$$\mathcal{F}[\bar{\gamma}_p(x)|x|_p^s](m) = -\gamma_p(m) + \gamma_p(m)\frac{(1-p^s)(1-|pm|^{-(s+1)})}{1-p^{s+1}}. \tag{2.141}$$

Combining this with the Fourier transform of $\gamma_p(x)$ proves the result. □

2.3 The Adeles

Now that we have the p-adic numbers we are ready to construct the Adeles. An Adele a is a tuple

$$a = (a_\infty, a_2, a_3, a_5, \cdots), \tag{2.142}$$

where $a_p \in \mathbb{Q}_p$, and for all but finitiely many primes we have $x_p \in \mathbb{Z}_p$. It is normal in this context to let $\mathbb{R}$ correspond to an infinite prime number, hence the notation x_∞.

We will often see products either over $p \leq \infty$ or $p < \infty$, where in the first case the product includes $\mathbb{R}$, and in the second it goes over all finite primes. We will thus write

$$\mathbb{A} = \mathbb{R} \times \prod_{p<\infty}' \mathbb{Q}_p = \prod_{p\leq\infty}' \mathbb{Q}_p, \tag{2.143}$$

where we understand that a product with a $'$ is restricted in the above way to keep almost all components in $\mathbb{Z}_p$. The components are often called the *local* parts of the Adele.

The constraint of letting only finitely many components have arbitrary norm is for technical reasons. We will not dwell on it.

For an Adele a we define the norm

$$|a| = \prod_{p\leq\infty} |a_p|_p, \tag{2.144}$$

which converges since only a finite number of the factors will be greater than 1. Note that this is not a true norm, since $|a|$ can be zero even if $a \neq 0$.

The Adeles clearly contain the real numbers as a subfield, just take the Adeles with everything but the real factor taken to be 1. It also contains the rational numbers, and in more than one way. One way is to let m be a rational number, and then let every component of the Adele be m, since the rationals are contained in every set of p-adic numbers. This works because there is always a largest prime factor in m, and after that $|m|_p = 1$. It can be quickly verified that $|m| = 1$ for all rational numbers (except 0).

This embedding has the nice property that the rational numbers become a discrete subset of $\mathbb{A}$. To see this, note that $\mathbb{Q}$ is topologically discrete if for any point $q \in \mathbb{Q}$ there exists an open set containing no other points of $\mathbb{Q}$ than q itself.

Consider the case $q = 0$. Then we can take the open set to be

$$V = (-\varepsilon, +\varepsilon) \times \prod_{p<\infty} \mathbb{Z}_p. \tag{2.145}$$

We remember that $\mathbb{Z}_p$ is the closed unit ball, but it is also an open set due to the discrete nature of the p-adic norm (we can define $\mathbb{Z}_p$ as the *open* ball of a radius slightly larger than 1.) The only rational number that lies in every single $\mathbb{Z}_p$ are the actual integers, but for $\varepsilon < 1$ no integer other than 0 lies in the real open set. Thus this set contains only the rational number 0. This can be generalized to any rational number by simply shifting the open sets. This will be important because it means that subgroups consisting of rational Adeles will be discrete subgroups.

Suppose that we have a function f_p defined for each prime p. Then we define the Adelisation of f_p as

$$f_{\mathbb{A}}(a) = \prod_{p\leq\infty}' f_p(a_p). \tag{2.146}$$

We can then get back a function on $\mathbb{R}$ by considering $f_{\mathbb{A}}$ evaluated at a convenient constant for all primes except ∞. The functions f_p for $p < \infty$ is the local part of f.

We will be considering integrals of such functions. We will not say more about integration other than that if $f_{\mathbb{A}}(a)$ is a direct product of functions we have

$$\int_{\mathbb{A}} f_{\mathbb{A}}(x)\,\mathrm{d}x = \prod_{p\leq\infty} \int_{\mathbb{Q}_p} f_p(x_p)\,\mathrm{d}x_p. \tag{2.147}$$

2.3.1 The Riemann ξ-Function

The Riemann ξ-function is a function that is related to the famous Riemann ζ-function,

$$\zeta(s) = \sum_{n=0}^{\infty} n^{-s} = \prod_{p<\infty} \frac{1}{1-p^{-s}}. \tag{2.148}$$

We see that this has almost been written as an Adelic product, but it is missing the infinite part. This turns out to make ζ somewhat messy. ζ satisfies the functional relation

$$\zeta(s) = \zeta(1-s)2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s). \tag{2.149}$$

By adding in the missing infinite part of the product this gets simplified.

The ξ function is defined as[12]

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-s/2}\Gamma(s/2)\prod_{p<\infty}\frac{1}{1-p^{-s}}, \tag{2.150}$$

and we can see that if we define $\xi_p(s) = \zeta_p(s) = \frac{1}{1-p^{-s}}$ for finite primes and $\xi_\infty(s) = \pi^{-s/2}\Gamma(s/2)$ we have

$$\xi(s) = \prod_{p\leq\infty} \xi_p(s). \tag{2.151}$$

This function is actually a simple Adelic integral,

$$\xi(s) = \int_{\mathbb{A}} \gamma(x)|x|^{s-1}\,\mathrm{d}x, \tag{2.152}$$

where $\gamma(x) = e^{-\pi x^2} \times \prod_{p<\infty} \gamma_p(x)$.

This function will turn out to appear quite frequently in the context of automorphic forms, and it appears quite naturally as the integrals of norms.

[12]The exact normalisation varies. Sometimes ξ is written with a factor of $s(1-s)$ in front, which satisfies the same functional relation.

2.4 Adelic and p-adic Lie Groups

Now that we have discussed Lie-groups and the Adeles we are ready to discuss Adelic Lie-groups. The generalisations are in most cases obvious, we just replace the entries which were real or complex numbers by p-adic or Adelic numbers.

One thing that changes is that in the p-adic context the exponential exp is changed, because the series no longer converges sufficiently; $\frac{1}{n!}$ is a large p-adic number. This is not something we will be worried about, because we never evaluate the exponential in this way. The definition of the Lie bracket is unchanged despite this.

The most important thing that changes is that the maximal compact subgroup $K(\mathbb{Q}_p)$ is *not* the orthogonal one generated by $E_\alpha - E_{-\alpha}$, but rather we have that since $\mathbb{Z}_p$ is compact, $K(G(\mathbb{Q}_p)) = G(\mathbb{Z}_p)$. The Iwazawa-decomposition is still somewhat valid with this change, we have that any element $g \in G(\mathbb{Q}_p)$ can be written as

$$g = nak, \tag{2.153}$$

with $n \in N(\mathbb{Q}_p)$, $a \in A(\mathbb{Q}_p)$ and $k \in G(\mathbb{Z}_p)$. This decomposition is not unique like in the real case. The obvious problem is that $b = na$ is only determined up to multiplication by $B(\mathbb{Z}_p)$ on the right. Thus we have to be careful when we use it. We must ensure that our results do not depend on the choice of decomposition.

A generic element $g \in G(\mathbb{A})$ can be written as

$$g = (g_\infty, g_2, g_3, \cdots), \tag{2.154}$$

with $g_p \in G(\mathbb{Q}_p)$.

There is an integration measure on $N(\mathbb{A})$. The measure $\mathrm{d}n$ is the Haar measure of the group $N(\mathbb{A})$, which for all $g \in N(\mathbb{A})$ satisfies $\mathrm{d}(gn) = \mathrm{d}n$ and is normalised so that

$$\int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \mathrm{d}n = 1. \tag{2.155}$$

2.4.1 Multiplicative Characters on $A(\mathbb{A})$

We will be interested in multiplicative characters $\chi : A(\mathbb{A}) \longrightarrow \mathbb{C}$ such that we can extend them to functions $\chi_G : G(\mathbb{A}) \longrightarrow \mathbb{C}$ through

$$\chi(a) = \chi_G(a) = \chi_G(nak) \tag{2.156}$$

in a well-defined way as a function on the entirety of $G(\mathbb{A})$. As we stated before we have to be careful here, and must have that χ is trivial on $A(\mathbb{Q})$.

However, notice that any Cartan generator is part of an $\mathrm{SL}(2,\mathbb{A})$ subgroup. For $\mathrm{SL}(2,\mathbb{A})$ we know that $A(\mathbb{Z}_p) = A(\mathbb{Z}_p^*)$ since for $a \in A(\mathbb{A})$ we have

$$a = \begin{pmatrix} h & 0 \\ 0 & \frac{1}{h} \end{pmatrix}. \tag{2.157}$$

If $h \notin \mathbb{Z}_p^*$ only one of these entries can be in $\mathbb{Z}_p$. Thus the entries of a are well defined up to their norm, so as long as χ only depends on the norm of the entries everything is fine.

Now we will define our characters χ in terms of a choice of root s. For any root α

$$\chi(h_\alpha(x)) = |h_\alpha(x)|^s = |x|^{\langle s|\alpha\rangle}. \tag{2.158}$$

Any such character χ can be written in this form, and we extend the function $|\cdot|^s$ as a character χ_G in the way described above. This is the notation we will use throughout.

2.4.2 Unitary Characters On $N(\mathbb{A})$

We introduce now the concept of a unitary character on $N(\mathbb{A})$.

$$\psi : N(\mathbb{A}) \longrightarrow U(1), \tag{2.159}$$

which for n,$n' \in N(\mathbb{A})$ satisfies

$$\psi(nn') = \psi(n)\psi(n'). \tag{2.160}$$

This can be achieved by considering first a unitary character from the Adeles, $\psi_{\mathbb{A}}^m$: $\mathbb{A} \longrightarrow \mathbb{C}$ with

$$\psi_{\mathbb{A}}^m(x) = e^{2\pi i m_\infty x_\infty} \times \prod_{p<\infty} e^{-2\pi i [m_p x_p]_p}, \tag{2.161}$$

Then we just take

$$\psi(x_\alpha(u)) = \psi_{\mathbb{A}}(u), \tag{2.162}$$

with a new possible choice of m for each root. The reason for the sign change is that we want to have $\psi_{\mathbb{A}}^m(x)$ to be trivial on $\mathbb{Q}$ for $m \in \mathbb{Q}$. This follows because we then get something of the form $mx_\infty - \sum_p [mx_p]_p$, which we established in Section 2.2.5 is always an integer.

Note that such a character ψ cannot actually depend on the entire subgroup $N(\mathbb{A})$, only the parts of $N(\mathbb{A})$ defined by the simple roots. Consider such a ψ acting on a general element of $[N(\mathbb{A}),N(\mathbb{A})]$. Such ah element can be written as $xyx^{-1}y^{-1}$ for some x and y in $N(\mathbb{A})$. Then

$$\psi(xyx^{-1}y^{-1}) = \psi(x)\psi(y)\psi(x^{-1})\psi(y^{-1}) = \psi(xx^{-1})\psi(yy^{-1}) = 1. \tag{2.163}$$

where we have used that $\psi(\cdot) \in \mathrm{U}(1)$ is Abelian. This means that ψ should be taken to be trivial on $[N(\mathbb{A}),N(\mathbb{A})]$[13].

We will almost never write out the dependence on the modenumber m explicitly, and instead reserve that position for the notation

$$\psi^a(n) = \psi(ana^{-1}), \tag{2.164}$$

where a is some element of $A(\mathbb{A})$. We know from Equation (2.49) that ana^{-1} is actually in $N(\mathbb{A})$, so this is fine.

[13] It is also possible to consider characters that are not defined on the entirety of N, in which case it might be impossible to break $\psi(xyx^{-1}y^{-1})$ into factors. This results in the so-called non-Abelian Whittaker coefficients, and are necessary if you want to consider the dependence on the entire group. We will not consider these.

3

Automorphic Forms In String Theory

We will now be looking at an extended introductory example. This will allow us to show a bit more concretely how automorphic forms appear in string theory, as well as provide context for some of the concepts we have just introduced. It will also serve as a primer for what we will do in the coming chapter.

First we will look at how automorphic forms arise in type IIB string theory, and then we will work through an example on the corrections to gravity that arise from interactions, and determine the corresponding automorphic form.

3.1 Gravity Corrections in Type IIB String Theory

In string theory one considers the dynamics of one-dimensional objects called strings. In the non-quantum picture these can be thought of as curves moving through space, and they can be either open or closed. As they move they trace out a surface in space-time, the so-called *world-sheet*. String theory can more accurately be described as the theory of these world-sheets. Just like particle physics can be thought of as the study of the trajectories of particles, the *world lines*.

Consider first a classical string with no quantum mechanical behaviour moving in flat Minkowski spacetime. In order to do anything we first need to parametrise our world sheet in a D-dimensional space-time. Let σ and τ be two parameters (σ can be thought of as an angle around the string while τ can be thought of as time). Then our surface can be parametrised by D functions $X^{\mu}(\tau,\sigma)$ where $\mu = 1,2,\cdots D$.

The dynamics of the world-sheets are then defined in terms of an action. The possible world-sheets are those that extremise the action. The simplest example of this is taking the action to be the world-sheet area in space-time. This is not quite the area of normal Euclidian space, but it is an equivalent to the fact that particles have an action given by their proper time; a particle with a given momentum will always travel along a straight line, since these are the shortest possible curves connecting two points in space-time.

Since the parameters σ and τ are completely arbitrary nothing physically real depends on our choice of parametrisation. We should also be free to change the parametrisation of space-time, since this too is arbitrary. This results in an extremely large symmetry-group consisting of all possible changes of variable on the world sheet and on space-time. In the end this results in string theory being conformally invariant.

Within a string theory there is a set number of scalar particles. These parametrise the so-called moduli-space. The quantum corrections in the theory will be related to automorphic forms on the moduli-space.

We will be looking at type IIB string theory. This is a chiral super-symmetric string theory with only closed strings. In order for the theory to be super-symmetric it must contain anti-commuting variables $\psi^{\mu}_{\alpha}(\tau,\sigma)$ on the world-sheet, in addition to the world-sheet coordinates $X^{\mu}(\tau,\sigma)$. That the theory is super-symmetric then says that we can rotate ψ^{μ}_{α} and X^{μ} coordinates in a particular fashion.

It can be shown that the moduli-space (which we recall is the space spanned by the scalars of the theory) of a maximally supersymmetric string theory is always a symmetric space of the form $\mathcal{M} = G(\mathbb{R})/K(G(\mathbb{R}))$, where $G(\mathbb{R})$ is some Lie group and $K(G(R))$ is its maximally compact subgroup [8].

All super-symmetric string theories demand that the number of space-time dimensions is 10, because it is not possible to have a representation of the Lorentz group acting on quantum super-strings unless the number of dimensions is 10. (For Bosonic strings the number of dimensions has to be 26.)

We can reduce this by assuming that space-time is a manifold of the form $\mathbb{R}^d \times T_{10-d}$ for some number of dimensions d, where T_n is an n-dimensional torus. Then we can expand our fields in terms of the size of the torii, and keep only the constant term. The part of the integral defining the action on these torii can be evaluated and we get a new, lower-dimensional action.

When reducing the number of dimensions we introduce new scalars in the theory, because a scalar is something that does not transform under coordinate transformations, and with a lower number of dimensions there are fewer transformation. The torii are nice in that they preserve the maximal supersymmetry, so the moduli space must still be a symmetric space. This means that while we reduce the number of space-time symmetries, we increase the moduli-space ones.

We are going to be looking at the corrections of the uncompactified type IIB string

D	G	K	$G(\mathbb{Z})$
10	$\mathrm{SL}(2,\mathbb{R})$	$\mathrm{SO}(2)$	$\mathrm{SL}(2,\mathbb{Z})$
9	$\mathrm{SL}(2,\mathbb{R}) \times O(1,1)$	$\mathrm{SO}(2)$	$\mathrm{SL}(2,\mathbb{Z}) \times \mathbb{Z}_2$
8	$\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$	$\mathrm{SO}(3) \times \mathrm{SO}(2)$	$\mathrm{SL}(3,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$
7	$\mathrm{SL}(5,\mathbb{R})$	$\mathrm{SO}(5)$	$\mathrm{SL}(5,\mathbb{Z})$
6	$\mathrm{Spin}(5,5,\mathbb{R})$	$\mathrm{Spin}(5)^2/\mathbb{Z}_2$	$\mathrm{Spin}(5,5,\mathbb{Z})$
5	$E_6(\mathbb{R})$	$\mathrm{USp}(8)/\mathbb{Z}_2$	$E_6(\mathbb{Z})$
4	$E_7(\mathbb{R})$	$\mathrm{SU}(8)/\mathbb{Z}_2$	$E_7(\mathbb{Z})$
3	$E_8(\mathbb{R})$	$\mathrm{Spin}(16)/\mathbb{Z}_2$	$E_8(\mathbb{Z})$

Table 3.1: Moduli spaces of compactifications of type IIB string theory, and their corresponding compact and discrete subgroups of dualities. The discrete subgroups enter the picture through the fact that they correspond to charge conservation, so the theory retains an unbroken discrete symmetry [9].

theory. In type IIB string theory we start with $\mathcal{M} = \mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)$ in ten dimensions, and reducing the number of dimensions one finds the compactified moduli-spaces in Table 3.1. There we have also written down discrete subgroups $G(\mathbb{Z})$, which will be relevant in a bit.

If we do a series expansion in the two perturbative parameters α' (related to the fundamental length of a piece of string) and g_s (the coupling constant between the strings) we find that to lowest order the theory is described by an effective action which reproduces Einstein gravity in 10 dimensions, and additional terms for the other fields in the theory. This comes from the conformal symmetry still present on the world-sheet. Explicitly the lowest order effective action is given by

$$S = \frac{1}{\alpha'^4} \int \sqrt{G} e^{-2\phi} \Big[R + \cdots \Big] \, \mathrm{d}^{10} x. \tag{3.1}$$

Here the subscript on S indicates lowest order in both α' and g_s. $\sqrt{G}$ is the square root of the determinant of the metric, which makes the action conformal in space-time. R is the Ricci scalar defined as a contraction of the Riemann-tensor, which is a combination of derivatives of the metric. ϕ is the dilaton field, which in a way determines the strength of the coupling between the strings. The elipsis contains things such as the two scalar fields ϕ and χ and any p-forms the theory contains, but we are not interested in these.

We see that S is very similar to the Einstein-Hilbert action which results in Einstein gravity. By rescaling the metric we obtain

$$S_{\mathrm{EH}} = \frac{1}{\alpha'^4} \int \sqrt{G} \Big[R + \cdots \Big] \, \mathrm{d}^{10} x, \tag{3.2}$$

which is just the Einstein-Hilbert action and is manifestly (at least in the terms we show) invariant under $\mathrm{SL}(2,\mathbb{R})$. This holds also for the rest of the fields [8].

In the classical theory we would get general relativity out of this action. However, we have to add quantum corrections to this. The next-to-lowest order in α' is the third order action,

$$S' = \frac{1}{\alpha'^4}\int \sqrt{G}e^{-2\phi}\Big[R + \alpha'^3\mathcal{R}^4 + \cdots\Big]\,\mathrm{d}^{10}x. \tag{3.3}$$

The quantity $\mathcal{R}^4$ is a specific scalar constructed out of the Riemann-tensor. Its exact form is not important. So far we have not looked at corrections from g_s, these should in principle appear as an expansion on the $\mathcal{R}^4$ term and be of the form

$$S'' = \frac{1}{\alpha'^4}\int \sqrt{G}e^{-2\phi}\Big[R + \alpha'^3\sum_{g=0}^{\infty} c_g e^{2(g-1)\phi}\mathcal{R}^4 + \cdots\Big]\,\mathrm{d}^{10}x, \tag{3.4}$$

where g denotes the genus of the world-sheet, the "number of loops" and therefore the number of factors g_s. These are difficult to compute. For more details see [10].

In the classical type IIB theory in 10 dimensions, the part of the action containing gravity is invariant under the entirety of $\mathrm{SL}(2,\mathbb{R})$, as we've mentioned. This is spontaneously broken in the quantum theory by the corrections. However due to charge conservation we retain the discrete $\mathrm{SL}(2,\mathbb{Z})$ symmetry. This is a general phenomenon, and this is where the discrete subgroups in Table 3.1 enter the picture.

This means that the full correction must be a function on $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$, rather than $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. We can avoid having to calculate all the different perturbative contributions for different genus, by instead packing them into one function: A unique automorphic form on $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. This has the distinct advantage that it also captures any non-perturbative contributions.

Instead of writing out a series we can write the action as

$$S'' = \frac{1}{\alpha'^4}\int \sqrt{G}e^{-2\phi}\Big[R + \alpha'^3 f(\tau)\mathcal{R}^4 + \cdots\Big]\,\mathrm{d}^{10}x, \tag{3.5}$$

where $f(\tau)$ is a function which is invariant under $\mathrm{SL}(2,\mathbb{Z})$ Möbius transformations on the complex scalar axio-dilaton field $\tau = \chi + ie^{-\phi}$, consisting of the real scalar axion field χ and dilaton field ϕ. (Recall that $\mathrm{SL}(2,\mathbb{R})$ had an action through Möbius transformation on the upper half-plane.)

From super-symmetry arguments one can further derive (see [11, 12]) that the function f must be an eigenfunction of the Laplacian on $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ with eigenvalue 3/4:

$$\Delta f(g) = y^2(\partial_x^2 + \partial_y^2)f = \frac{3}{4}f(g), \tag{3.6}$$

where x and y are defined in the parametrisation $g(i) = x + iy$. This parametrisation is not suitable for string theory, but makes the math easier. We will get back to interpreting this later.

So what do we know?

- The function we are after should be a function from the symmetric space $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ into the complex numbers.
- From dualities coming from charge conservation we know that the function we are after should be invariant under the discrete subgroup $\mathrm{SL}(2,\mathbb{Z})$.
- From supersymmetry we know that the function satisfies an analytical constraint, it is an eigenfunction to the Laplacian on $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$.
- Due to physicality considerations the function must also satisfy certain growth conditions. For instance, when the coupling tends to zero we should get back approximately the lowest order corrections.

As we shall see, these conditions are exactly the conditions we will put on an automorphic form. This automorphic form is unique up to normalisation, so now we only have to find one such nontrivial function [1].

This generalises to other corrections where the moduli space is a different symmetric space. So in principle all automorphic forms corresponding to entries in Table 3.1 are interesting. $\mathrm{SL}(3,\mathbb{R})$ does not appear on its own, but it is the simplest non-trivial extension of $\mathrm{SL}(2,\mathbb{R})$.

The discussion here above is closely mirrored by Chapter 8 of D. Perssons PhD thesis [13], albeit with more details.

3.2 Constructing the Automorphic Form

Now to actually find the automorphic form. First of all, since we know from before that $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ is topologically the same as the upper half plane we can introduce $z = g(i)$ and work in terms of that, with $\mathrm{SL}(2,\mathbb{R})$ acting through Möbius transformations. We start by finding eigenfunctions to the Laplacian Δ which do not depend on x. This gives us the family of solutions

$$\chi_s(z) = y^s = \mathrm{Im}(z)^s. \tag{3.7}$$

This is however obviously not invariant under the entirety of $\mathrm{SL}(2,\mathbb{Z})$ on the left, so we form what is known as the (formal) Eisenstein series

$$E(z,s) = \sum_{\gamma\in\mathrm{SL}(2,\mathbb{Z})} \chi_s(\gamma z), \tag{3.8}$$

with $z = g(i)$. However, χ_s already has some invariances. Recalling equation (2.30) we have

$$\chi_s\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)z\right) = \frac{\mathrm{Im}(z)^s}{|cz+d|^{2s}}. \tag{3.9}$$

For $c = 0, d = 1$ nothing changes so we have invariance. This forces $a = 1$ from the determinant condition. This translates to invariance under the subgroup $N(\mathbb{Z}) = \{\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) : n \in \mathbb{Z}\}$, the subgroup of unit upper triangular matrices, which is the same as real translations.

In our Eisenstein series we should therefore only include these terms once. This means we have to divide away all of these elements, which results in

$$E(z,s) = \sum_{\gamma \in N(\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{Z})} \mathrm{Im}(\gamma z)^s. \tag{3.10}$$

This takes care of all infinite symmetries, there is also the case $c = 0, a = -1, d = -1$, but this is only a finite contribution so it is fine. This will only change the overall normalisation by a factor of 2.

Now we will need to figure out how to describe a general element in the coset $N(\mathbb{Z})\backslash \mathrm{SL}(2, \mathbb{Z})$. Any element in the coset can be written as

$$N(\mathbb{Z})\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left\{\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \begin{pmatrix} a+c\xi & b+d\xi \\ c & d. \end{pmatrix} : \xi \in \mathbb{Z}, \right\}, \tag{3.11}$$

with $ad - bc = 1$. It is clear that any choice of c and d results in a unique coset, but what we can do with a and b is less obvious. It turns out that a and b provide no information. Let us prove this.

There is a result in number theory known as Bêzout's identity which states that over the integers we have

$$ax + by \propto \gcd(a,b). \tag{3.12}$$

For us this implies that $\gcd(c,d) = 1$, through the condition on the determinant. So c and d must be coprime (this also applies to a and b, obviously). Further, the possible solutions of a and b are all of the form

$$a = a_0 + \xi \cdot c, \quad b = b_0 + \xi \cdot d, \tag{3.13}$$

where a_0 and b_0 is some solution and ξ is an arbitrary integer. This means for any choice of c and d there is only one choice of a and b, up to multiplication by an element in $N(\mathbb{Z})$ [1, 14].

Thus each coset is uniquely defined by a pair of coprime integers, c and d, and there is a solution for each coprime c and d. Thus we find

$$E(z,s) = \sum_{\gcd(c,d)=1} \frac{y^s}{|cz + d|^{2s}}. \tag{3.14}$$

Summing over coprime integers is hard. Instead we define the similar function:

$$\mathcal{E}(z,s) = \sum_{(m,n)\in\mathbb{Z}^2} \frac{y^s}{|mz + n|^{2s}}. \tag{3.15}$$

Terms that obviously do not work, $(m,n) = (0,0)$, are excluded. Now we show that this is proportional to $E(z,s)$.

We make the substitutione $m = kc$ and $n = kd$, where we take $k = \gcd(m,n)$. This makes c and d coprime. Such a factorisation is clearly unique, and any choice of k, c and d yields a valid choice of m and n. Doing this substitution we get

$$\mathcal{E}(z,s) = \sum_{k=1}^{\infty} \frac{1}{k^{2s}} \sum_{\gcd(c,d)=1} \frac{y^s}{|cz+d|^{2s}} = \zeta(2s)E(z,s), \tag{3.16}$$

where we have used the definition of the Riemann-ζ function.

Thus these two functions are equal modulo a constant factor, and therefore $\mathcal{E}(z,s)$ is also an automorphic form, and is also proportional to the function f we were originally after. We will not distinguish between them, as our discussion will not care about the normalisation.

Evaluating the Sum

Now we want to write this function in a more useful form. The end goal is to find a Fourier expansion of the function, but for now we start by evaluating the sums.

We have

$$\mathcal{E}(z,s) = \sum_{m,n} \frac{y^s}{|mz+n|^{2s}} = y^s \sum_n \frac{1}{|n|^{2s}} + y^s \sum_{m\neq 0} \sum_n \frac{1}{|mz+n|^{2s}}. \tag{3.17}$$

The first sum in n is easy since it is again just a ζ function

$$\mathcal{E}(z,s) = 2y^s\zeta(2s) + y^s \sum_{m\neq 0} \sum_n \frac{1}{|mz+n|^{2s}}. \tag{3.18}$$

To simplify the remaining sum we will use a trick. Using the definition of the Γ function we have

$$\Gamma(s) = \int_0^\infty e^{-x} x^s \frac{\mathrm{d}x}{x} = [x = \xi/t] = \xi^s \int_0^\infty \frac{e^{-\frac{\xi}{t}}}{t^{s+1}}\,\mathrm{d}t \tag{3.19}$$

so

$$\frac{1}{\xi^s} = \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{-\frac{\xi}{t}}}{t^{s+1}}\,\mathrm{d}t. \tag{3.20}$$

Taking $\xi = \pi|mz+n|^2$ (the extra π will be convenient in a bit) we get

$$\sum_n \frac{1}{|mz+n|^{2s}} = \sum_n \frac{\pi^s}{\Gamma(s)} \int_0^\infty \frac{e^{-\frac{\pi}{t}|mz+n|^2}}{t^{s+1}} \tag{3.21}$$

$$= \frac{\pi^s}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, \frac{e^{-\frac{\pi y^2 m^2}{t}}}{t^{s+1}} \sum_n e^{-\frac{\pi}{t}(mx+n)^2} \tag{3.22}$$

The last sum in n we will simplify using Poisson re-summation.
Theorem 3.1 (Poisson Re-summation)**.** *Let $f :\to \mathbb{C}$ be a Schwarz function. Then*

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{k\in\mathbb{Z}} \hat{f}(k), \tag{3.23}$$

where $\hat{f}$ is the Fourier transform of f.

We can use this on $f(n) = \exp(-\frac{\pi}{t}(mx+n)^2)$, for which the Fourier transform is $\hat{f}(k) = \sqrt{t}\exp(-\pi k^2 t + 2\pi ikmx)$. It is here that the previous mysterious π is handy: We now get something which has period 1 in x.

At this point we have

$$\mathcal{E}(z,s) = 2y^s\zeta(2s) + \frac{y^s\pi^s}{\Gamma(s)}\sum_{m\neq 0}\sum_{k} e^{2\pi ikmx}\int_0^\infty \mathrm{d}t\, \frac{e^{-\frac{\pi y^2 m^2}{t} - \pi k^2 t}}{t^{s+1/2}}. \tag{3.24}$$

For the final two sums the integral will evaluate to different things when k is zero and when it is nonzero. For $k = 0$ we get, when $\operatorname{Re} s > 1/2$,

$$\int_0^\infty \mathrm{d}t\, \frac{e^{-\frac{\pi m^2}{t}}}{t^{s+1/2}} = \frac{1}{(\pi m^2 y^2)^{s-1/2}}\Gamma(s-1/2) \tag{3.25}$$

and when $k \neq 0$ we get

$$\int_0^\infty \mathrm{d}t\, \frac{e^{-\frac{\pi m^2}{t} - \pi k^2 t}}{t^{s+1/2}} = 2\left|\frac{k}{my}\right|^{s-1/2} K_{s-1/2}(2\pi|km|y), \tag{3.26}$$

where K_ν is the modified Bessel function of the second kind.

Putting this together we have that

$$\begin{aligned}\mathcal{E}(z,s) =& 2y^s\zeta(2s) + \frac{\pi^s y^s}{\Gamma(s)}\frac{\Gamma(s-1/2)}{\pi^{s-1/2}y^{2s-1}}\sum_{m\neq 0}\frac{1}{m^{2s-1}} + \\ & \frac{\pi^s y^s}{\Gamma(s)}\frac{2}{y^{s-1/2}}\sum_{k\neq 0, m\neq 0}\left|\frac{k}{m}\right|^{s-1/2} K_{s-1/2}(2\pi|km|y)e^{2\pi ikmx},\end{aligned} \tag{3.27}$$

and consequently

$$\begin{aligned}\mathcal{E}(z,s) =& 2y^s\zeta(2s) + 2\zeta(2s-1)\frac{\Gamma(s-1/2)}{\Gamma(s)}\sqrt{\pi}y^{1-s} + \\ & \frac{2\pi^s\sqrt{y}}{\Gamma(s)}\sum_{k\neq 0, m\neq 0}\left|\frac{k}{m}\right|^{s-1/2} K_{s-1/2}(2\pi|km|y)e^{2\pi ikmx}.\end{aligned} \tag{3.28}$$

Now we want to change variable to $n = km$, and sum over n and m instead of k and m. For a given n we must have that m is a divisor of n, which we denote $m|n$, so that k is an integer. Thus

$$\begin{aligned}
&\sum_{k\neq 0, m\neq 0} \left|\frac{k}{m}\right|^{s-1/2} K_{s-1/2}(2\pi|km|y)e^{2\pi ikmx} \\
&= \sum_{n\neq 0, m>0: m|n} \left|\frac{n}{m^2}\right|^{s-1/2} K_{s-1/2}(2\pi|n|y)e^{2\pi inx} \\
&= \sum_{n\neq 0} |n|^{s-1/2} 2\mu_{1-2s}(n) K_{s-1/2}(2\pi|n|y)e^{2\pi inx},
\end{aligned} \tag{3.29}$$

where we have defined

$$\mu_s(n) = \sum_{m|n, m>0} m^s. \tag{3.30}$$

This is the so-called divisor sum, or the instanton-measure, depending on if you are in a mathematics or physics context, respectively [1]. We will talk about how this is related to instantons in a little bit.

The final answer is thus

$$\begin{aligned}
\mathcal{E}(z,s) =& 2\zeta(2s)y^s + 2\zeta(2s-1)\frac{\Gamma(s-1/2)}{\Gamma(s)}\sqrt{\pi}y^{1-s} + \\
&+ \frac{4\pi^s\sqrt{y}}{\Gamma(s)} \sum_{n\neq 0} |n|^{s-1/2} \mu_{1-2s}(n) K_{s-1/2}(2\pi|n|y)e^{2\pi inx}.
\end{aligned} \tag{3.31}$$

If we normalise this by dividing it by a factor of $2\zeta(2s)$ and using the fact that the Riemann-ξ is $\xi(s) = \zeta(s)\pi^{-s/2}\Gamma(s/2)$ we have

$$\mathcal{E}(z,s) = y^s + \frac{\xi(2s-1)}{\xi(2s)} y^{1-s} + \frac{2\sqrt{y}}{\xi(2s)} \sum_{n\neq 0} |n|^{s-1/2} \mu_{1-2s}(n) K_{s-1/2}(2\pi|n|y)e^{2\pi inx}. \tag{3.32}$$

Getting here was quite the mouthful. We had several difficult steps, which would probably only get worse if we tried to do the same thing for more complicated Lie groups. Perhaps the most complicated part of the calculation was that we had to find the structure of $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})$. This involved non-trivial results from number theory. Then we actually had to evaluate the non-trivial sum, and used several tricks on the way.

We will redo this calculation later, but at that point we will consider the Adelic group $\mathrm{SL}(2,\mathbb{A})$ instead. This means we can skip most of the above complications. In particular, we will not have to know anything about $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})$, because we will be able to trade it for the finite Weyl group.

3.2.1 Interpretation

Now that we know the automorphic form on $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ we can think about what it means in string theory. We have two real scalar fields in the theory; the axion field χ, and the dilaton field ϕ. These parametrise our moduli space through the complex axio-dilaton field

$$\tau = \chi + ie^{-\phi}. \tag{3.33}$$

In this parametrisation we have that $e^{\langle\phi\rangle} = g_s$ can roughly be interpreted as the coupling strength between the strings [13].

This means that in terms of fields we have (noting that $s = 3/2$ leads to the desired eigenvalue 3/4)

$$f(\tau,3/2) = g_s^{-3/2} + \frac{\xi(2)}{\xi(3)} g_s^{1/2} + \frac{2g_s^{-1/2}}{\xi(3)} \sum_{n\neq 0} |n|\mu_{-2}(n) K_1(2\pi|n|\tfrac{1}{g_s}) e^{2\pi i n\chi}. \tag{3.34}$$

This is not necessarily normalised in the correct way, but we will ignore that.

Consider weak coupling, when $g_s \ll 1$. The only nontrivial function we have to expand is K_1, which has an asymptotic behaviour

$$K_1(x) = \sqrt{\frac{\pi}{2x}} e^{-x}(1 + \mathcal{O}(1/x)). \tag{3.35}$$

Thus

$$f(\tau,3/2) = g_s^{-3/2} + \frac{\xi(2)}{\xi(3)} g_s^{1/2} + \frac{2g_s^{-1/2}}{\xi(3)} \sum_{n\neq 0} |n|\mu_{-2}(n) \sqrt{\frac{\pi g_s}{4\pi|n|}} e^{2\pi\left(-|n|\frac{1}{g_s} + in\chi\right)} (1 + \mathcal{O}(g_s)). \tag{3.36}$$

We note that the sum contains a factor of $e^{-2\pi|n|\frac{1}{g_s}}$ which is not perturbative; its series expansion does not just contain positive powers of g_s.

If we define the instanton fields

$$S(n) = -2\pi i|n|\tau = -2\pi i|n|(\chi + ie^{-\phi}), \tag{3.37}$$

where n denotes the charge of the instanton, we can rewrite our sum as

$$f(\tau,3/2) = \Big(\cdots\Big) + \frac{1}{\xi(3)} \sum_{n=1}^{\infty} \sqrt{|n|}\mu_{-2}(n) \left(e^{-S(n)} + e^{-S(n)^\dagger}\right)(1 + \mathcal{O}(g_s)), \tag{3.38}$$

where $S(n)^\dagger = 2\pi i|n|(\chi - ie^{-\phi})$ is the anti-instanton with charge n.

Here we see that $\mu_s(n)$ is in some way related to the size of the contribution from the instantons, hence the name. To see why, we have to understand what the charge n is for an instanton. The charge is given by the product of the wrapping number m and the rotational wave-number k. The wrapping number counts how many loops the string has, while the rotational wave-number counts how fast the string is spinning around itself. The total number of different possible configurations of instantons with a given charge n is then related to the number of ways the charge can be factorised into $n = mk$. The instanton measure is thus a measure of how many such instanton configurations there are for a given charge n [13, 15].

3.2.2 Alternative Forms of the Instanton Measure

Now we are going to look at an alternative way of writing the instanton measure. This will reveal to us how the prime numbers enter into this.

We have defined

$$\mu_s(n) = \sum_{d|n,d>0} d^s. \tag{3.39}$$

This function is actually multiplicative as long as the arguments are coprime. To see this, suppose that $n = km$, where $\gcd(k,m) = 1$. Then

$$\mu_s(km) = \sum_{d|km,d>0} d^s. \tag{3.40}$$

Since the two variables are co-prime any such d can be written uniquely as $d = d_k d_m$, where d_k is a divisor of k and d_m is a divisor of m. Furthermore any choice of d_k and d_m results in a valid divisor of km. Thus we can sum over d_k and d_m instead. This nets us

$$\mu_s(km) = \sum_{d_k|k,d_m|m,d_k,d_m>0} (d_k d_m)^s = \sum_{d_k|k,d_k>0} d_k^s \sum_{d_m|m,d_m>0} d_m^s = \mu_s(k)\mu_s(m). \tag{3.41}$$

Now it suffices to evaluate μ_s at powers of primes. We have

$$\mu_s(p^\nu) = 1 + p^s + p^{2s} + \cdots + p^{\nu s} = \frac{1 - p^{-s(\nu+1)}}{1 - p^s}. \tag{3.42}$$

If we write $n = \prod_{p<\infty} p^{\nu_p(n)}$ we get

$$\mu_s(n) = \prod_{p<\infty} \mu_s(p^{\nu_p}) = \prod_{p<\infty} \frac{1 - |pn|_p^s}{1 - p^s}. \tag{3.43}$$

This form of the instanton measure is the one we will see later when we do the Adelic treatment.

The factors in the Euler product for $\mu_s(n)$ look very similar to the characters of representations of $\mathrm{SL}(2,\mathbb{R})$,

$$\mathrm{ch}_\lambda(h(a)) = |a|^\lambda \frac{1 - |a|^{-2\lambda-2}}{1 - |a|^{-2}}. \tag{3.44}$$

If we take $a = p^{-s}$ and $\lambda = \nu(m)$ we get

$$\mathrm{ch}_{\nu_p(m)}(h(p^{-s})) = |m|_p^{-s} \frac{1 - |pm|_p^{2s}}{1 - p^{2s}}. \tag{3.45}$$

This means that we can write

$$\mu_{1-2s}(n) = |n|_\infty^{1/2-s} \prod_{p<\infty} \mathrm{ch}_{\nu_p(n)}(h(p^{-(1/2-s)})), \tag{3.46}$$

where we have used that for rational numbers $1/|n|_\infty = \prod_{p<\infty} |n|_p$ (this follows from the fact that the total Adelic norm is 1). Therefore the instanton measure is related to the characters of representations of $\mathrm{SL}(2, \mathbb{R})$, and the representations that are represented depends on the prime factorisation of n.

The fact that the instanton measure, which as we will see later is the part of the Fourier-expansion that comes from the p-adics, can be written as a product of characters over representations is a general phenomenon. The Casselman-Shalika formula, which we will look at later, can be put in this form [16].

4

Automorphic Forms

We are often interested not just in the Lie groups, but also functions on the Lie group. In particular, we are typically interested in functions that behave nicely with respect to the group multiplication. One particular class of functions we are interested in is the so-called *automorphic forms*. In essence, automorphic forms are well-behaved eigenfunctions to some second order differential equation, has a discrete symmetry of some kind and plays nicely with the Lie group.

In this chapter we are going to introduce the notion of an automorphic form and show how one can construct some of them using Eisenstein series. This will be similar to what we just did for $\mathrm{SL}(2, \mathbb{R})$ in the previous chapter.

Then we are going to introduce the Fourier-Whittaker coefficients, which will be the corresponding notion of (some of the) terms in the Fourier-series expansion of the Eisenstein series. We will not be looking at all possible Fourier-coefficients, only those which depend on the subgroup $N(\mathbb{A})$. In general one has to consider all unipotent subgroups of $G(\mathbb{A})$.

We will then simplify the integrals defining the Fourier-Whittaker coefficient and turn them into a more useful form. We will then see that they actually turn into integrals over each prime separately, which is quite remarkable and won't be obvious from the start.

The last thing we will do is to derive some general formulas for the Fourier-Whittaker coefficients; Langlands constant term formula, which is the part of the Fourier series containing the constant term, when all modenumbers are 0; and the Casselman-Shalika formula which deals with the case when all modenumbers are 1. In particular the proof of Langlands formula will be important in the next chapter.

To get these we will need to find the Fourier-Whittaker coefficients of $\mathrm{SL}(2, \mathbb{A})$ manually, so we will also do that calculation.

4.1 Automorphic Forms and Eisenstein Series

An automorpic form f is a smooth function from a Lie group G into the complex numbers, that satisfies (the somewhat vague conditions)

- There is some discrete subgroup Γ of G where we have for $\gamma \in \Gamma$, $g \in G$ that $f(\gamma g) = a(\gamma)f(g)$, where $a(\gamma)$ is some fixed function,
- it is an eigenfunction of some invariant operators on G,
- and it does not explode into infinities in some nasty way. (This can be restrictions on growth or the size of some spaces.)

The exact criteria vary depending on application and setting [1].

In this thesis, we will only consider automorphic forms on an Adelic Lie group $G(\mathbb{A})$ that are smooth functions $\phi : G(\mathbb{A}) \longrightarrow \mathbb{C}$ and satisfy:

- For any $\gamma \in G(\mathbb{Q})$ and $g \in G(\mathbb{A})$ we have $\phi(\gamma g) = \phi(g)$.
- For any $k \in K(G(\mathbb{A}))$ and $g \in G(\mathbb{A})$ we have $\phi(gk) = \phi(g)$, where $K(G(\mathbb{A}))$ is the maximal compact subgroup.
- It is an eigenfunction to some $G(\mathbb{A})$-invariant second order differential operators on $G(\mathbb{A})$.
- We have that for any norm $\|\cdot\|$ on $G(\mathbb{A})$ there are constants C and n such that for all $g \in G(\mathbb{A})$ $|\phi(g)| \leq C\|g\|^n$.

These conditions can be translated right into the real or p-adic case, with some minor changes such as a new discrete subgroup.

Now on how to actually construct an automorphic form. One way, and the method we will consider, is to create an Eisenstein series.

Suppose you are able to find a function $f : A(\mathbb{A}) \to \mathbb{C}$ which is an eigenfunction to your differential operator and which is multiplicative. Then it can be extended to a character on the entire group as we did in Section 2.4.1 and thus have for any $a \in A(\mathbb{A})$

$$f(a) = |a|^s \tag{4.1}$$

for some choice of root s. This is already invariant on $A(\mathbb{Q})$ by construction, since the Adelic norm of any rational number is 1. To make it invariant on the entirety of $G(\mathbb{Q})$ we form the Eisenstein series as the sum of the orbit over $G(\mathbb{Q})$

$$E(g,s) = \sum_{\gamma \in B(\mathbb{Q}) \backslash G(\mathbb{Q})} |\gamma g|^s. \tag{4.2}$$

Where this converges absolutely it is an automorphic form. It is still a solution to our $G(\mathbb{A})$-invariant differential operator, since we can then evaluate this operator term by term. It is also by construction invariant under $G(\mathbb{Q})$ on the left. We will not show that it satisfies the growth condition, but it does [1, 17].

We will also not show where this series converges, however it is known due to Godement [18, 19] that the Eisenstein series converges on the so-called Godements domain for sufficiently "large" s, in particular $s > \rho$ (component-wise). Furthermore, if we consider s to be in the complexification of $\mathfrak{h}^*$ it is known due to Langlands [17] that the Eisenstein series above can be extended by analytic continuation to a holomorphic function on the entire complexified rootspace, except at a few isolated poles. This follows from the possibility of deriving a functional relation for the Eisenstein series, which we will not do.

The reason we take the quotient with $B(\mathbb{Q})$ is that $|\cdot|^s$ is already invariant under $B(\mathbb{Q})$ on the left. Since this is an infinitely large group we would get an infinite number of terms with the same value, one for each element in $B(\mathbb{Q})$. This would diverge. Therefore we only consider terms which are different with respect to $B(\mathbb{Q})$.

What we are really after in the end is the real Eisenstein series

$$E_{\mathbb{R}}(g,s) = \sum_{\gamma \in B(\mathbb{Z}) \backslash G(\mathbb{R})} |\gamma g|^s. \tag{4.3}$$

Naively one would say that the group $G(\mathbb{Q})$ clearly contains the subgroup $G(\mathbb{Z})$, so when we restrict to the real case by only varying g_∞ we get something that is invariant under $G(\mathbb{Z})$. There is one hangup, however. The subgroup $G(\mathbb{Z}) \subset G(\mathbb{A})$ is not the same thing as the subgroup $G(\mathbb{Z}) \subset G(\mathbb{R})$. The former is made up by the Adelic groups of the form $(g,g,g,\cdots)$, with $g \in G(\mathbb{Z})$, while the latter is of the form $(g,1,1,\cdots)$. However, one can show that there is a bijection between our definition of the Eisenstein series for real g and the real Eisenstein series

$$E_\infty(g,s) = \sum_{\gamma \in B(\mathbb{Z}) \backslash G(\mathbb{Z})} |\gamma g|^s. \tag{4.4}$$

Thus the restriction to the real case still gives us what we want [1].

4.2 Fourier-Whittaker Coefficients

Given an Eisenstein series E (and thus implicitly a choice of root) on a group $G(\mathbb{A})$ we have that the Eisenstein series is invariant under $G(\mathbb{Q})$, which in particular means that it is invariant under $N(\mathbb{Q})$. Thus, E is periodic in $N(\mathbb{A})$ with "period" $N(\mathbb{Q})\backslash N(\mathbb{A})$, and can be expanded in a Fourier series.

We define the Fourier-Whittaker coefficient with respect to the unitary character $\psi : N(\mathbb{A}) \longrightarrow U(1)$ as

$$W^\circ_\psi(g,s) = \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(ng,s)\overline{\psi(n)}\,\mathrm{d}n. \tag{4.5}$$

The circle is there to illustrate that this is a spherical coefficient, that is, it is invariant under K on the right. Since E is invariant on $N(\mathbb{Q})$ ψ also has to be invariant under the change of variables $n \mapsto N(\mathbb{Q})n$. As we have seen in Section 2.4.2 this means that the mode-numbers in ψ all have to be rational numbers.

The Whittaker-coefficient can be thought of as the normal Fourier-series coefficient, but with the mode baked in. Just like for a normal Fourier series we can restore the original function by summing over the Fourier-modes, however there are some details involved that we will not cover. We have considered only unitary characters defined on $N(\mathbb{A})$, while we have to consider unitary characters defined on arbitrary unipotent subgroups in order to get the full Fourier-series. We have

$$E(g,s) = \sum_\psi W^\circ_\psi(g,s) + \cdots , \tag{4.6}$$

where the terms in $\cdots$ contain Fourier-coefficients for unipotent subgroups of $G(\mathbb{A})$ other than $N(\mathbb{A})$, such as $[N(\mathbb{A}),N(\mathbb{A})]$ [1].

Expanding the sum in E we get

$$W^\circ_\psi(s,g) = \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \sum_{\gamma\in B(\mathbb{Q})\backslash G(\mathbb{Q})} |\gamma ng|^s\overline{\psi(n)}\,\mathrm{d}n. \tag{4.7}$$

We are now going to spend quite some time massaging the Fourier-Whittaker coefficient into something much simpler, which results in the following proposition.
Proposition 4.1. *The Fourier-Whittaker coefficients can be written as*

$$W^\circ_\psi(s,nak) = \psi(n) \sum_{w\in\mathcal{W}_\psi} |a|^{w^{-1}(s-\rho)+\rho} \prod_{p\leq\infty} I_{w,\psi,p}(s,a), \tag{4.8}$$

where

$$I_{w,\psi,p}(s,a) = \int_{N_{w-}(\mathbb{Q}_p)} |wn|_p^s\overline{\psi_p^{a_p}(n)}\,\mathrm{d}n \tag{4.9}$$

and $N_{w\pm}$ are the set of elements in N generated by the positive roots that get mapped by w to positive roots and negative roots, respectively. $\mathcal{W}_\psi$ is the subset of $\mathcal{W}$ such that ψ is trivial on N_{w+}.

The truly remarkable thing here is that the Fourier-Whittaker coefficient factorises into local integrals. This is a highly non-trivial statement, since both E and the integration domain can not be written as local products. It is only their combination which can be written as a product.

This proposition says that if ψ is nontrivial on a set of positive roots, the Whittaker coefficient is a sum over the Weyl words which map all of these positive roots to negative numbers.

There are two important special cases of this. When ψ is trivial we get the so-called constant term (which is often treated separately). In this case ψ is trivial on all N_{w+}, so the sum ranges over all Weyl words.

Corollary 4.2. *The constant term corresponding to the trivial character $\psi = 1$ is given by*

$$W_1^\circ(s,nak) = \sum_{w\in\mathcal{W}} |a|^{w^{-1}(s-\rho)+\rho} \int_{N_{w-}(\mathbb{Q}_p)} |wn|_p^s \, \mathrm{d}n. \tag{4.10}$$

When ψ is generic, which we recall means that it is not trivial on any root generator, we get a contribution only from the word which maps all positive roots to negative ones. This is only done by the longest Weyl word.

Corollary 4.3. *The Fourier-Whittaker coefficient for a generic character ψ is given by*

$$W_\psi^\circ(s,nak) = \psi(n)|a|^{w_0} \int_{N(\mathbb{Q}_p)} |w_0 n|_p^s \psi_p^{a_p}(n) \, \mathrm{d}n, \tag{4.11}$$

where w_0 is the longest Weyl word.

For other characters one can simplify further, but we will not need this so we will cut the calculation short [1].

The rest of this section will be dedicated to proving Proposition 4.1

4.2.1 Rewriting As A Sum Over The Weyl Group

The first step is to get in a sum over the full Weyl group. To do this we want to use the Bruhat decomposition, which we introduced in Section 2.1.5, which states that

$$G(\mathbb{Q}) = \bigcup_{w\in\mathcal{W}} B(\mathbb{Q})wB(\mathbb{Q}). \tag{4.12}$$

This will allow us to sum over the Weyl group instead of over cosets of $G(\mathbb{Q})$. This is where most of the magic happens. All the steps up to now can be done for a real group, but in the real case the discrete subgroup is $G(\mathbb{Z})$ so we cannot do this decomposition because the Bruhat decomposition does *not* hold for $G(\mathbb{Z})$. This is because $\mathbb{Z}$ is not a field, unlike $\mathbb{Q}$ [1, 20].

In order to rewrite this as a sum over the Weyl group we need to factorise out $B(\mathbb{Q})$ also on the right in the sum. Let

$$\gamma = \alpha\delta \tag{4.13}$$

where α runs over $B(\mathbb{Q})\backslash G(\mathbb{Q})/B(\mathbb{Q})$. Just thinking naively, we would expect the sum over δ to run over $B(\mathbb{Q})$, but it is not necessarily the whole $B(\mathbb{Q})$ since that might lead to overcounting. Just denote the set of δ by Δ for now. We have

$$W^\circ_\psi(s,g) = \sum_{\alpha \in B(\mathbb{Q})G(\mathbb{Q})/B(\mathbb{Q})} \sum_{\delta \in \Delta} \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} |\alpha\delta n g|^s \overline{\psi(n)} \, \mathrm{d}n. \tag{4.14}$$

Then δ and $\delta' = \alpha^{-1}B(\mathbb{Q})\alpha\delta$ (of course, granted that this latter element is actually in $B(\mathbb{Q})$) yields the same element γ in $B(\mathbb{Q})\backslash \mathrm{SL}(2,\mathbb{Q})$. Thus we need to divide away this factor $\alpha^{-1}B(\mathbb{Q})\alpha$, and we have

$$\Delta = \alpha^{-1}B(\mathbb{Q})\alpha \cap B(\mathbb{Q})\backslash B(\mathbb{Q}). \tag{4.15}$$

This gets rid of all extra terms, since this is the only symmetry of $|\cdot|^s$ that applies. The other one would be spherical symmetry on the right, but that has nothing to do with $B(\mathbb{Q})$ [1].

Now, using the Bruhat decomposition we can identify α with elements of $\mathcal{W}$ to get

$$W^\circ_\psi(s,g) = \sum_{w \in \mathcal{W}} \sum_{w^{-1}B(\mathbb{Q})w \cap B(\mathbb{Q})\backslash B(\mathbb{Q})} \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} |w\delta n g|^s \overline{\psi(n)} \, \mathrm{d}n. \tag{4.16}$$

What we want to do is change variable from n to δn, but δ potentially lies in the full Borel subgroup $B(\mathbb{Q})$, not just $N(\mathbb{Q})$, which is the only part of the group where ψ is defined.

However, since the Cartan torus $A(\mathbb{A})$ is left invariant by the Weyl group, it is clear that $w^{-1}B(\mathbb{Q})w$ contains the set $A(\mathbb{Q})$, and thus thus we can factorise that away at once, which yields

$$w^{-1}B(\mathbb{Q})w \cap B(\mathbb{Q})\backslash B(\mathbb{Q}) = w^{-1}N(\mathbb{Q})w \cap N(\mathbb{Q})\backslash N(\mathbb{Q}). \tag{4.17}$$

Now we have that δ lies in $N(\mathbb{Q})$, so we can introduce $m = \delta n$. Then the Haar measure $\mathrm{d}n$ is invariant, and ψ is invariant since δ is in $N(\mathbb{Q})$ where ψ is trivial. The domain of integration changes to $\delta(N(\mathbb{Q})\backslash N(\mathbb{A}))$. We can move the sum inside the integration limit, and simplify the quotient and products, to get

$$W^\circ_\psi(s,g) = \sum_{w \in \mathcal{W}} \int_{w^{-1}N(\mathbb{Q})w \cap N(\mathbb{Q})\backslash N(\mathbb{A})} |wmg|^s \overline{\psi(m)} \, \mathrm{d}m. \tag{4.18}$$

We can break out the dependence on n to rewrite this as

$$W^\circ_\psi(s,nak) = \psi(n) \sum_{w \in \mathcal{W}} \int_{w^{-1}N(\mathbb{Q})w \cap N(\mathbb{Q})\backslash N(\mathbb{A})} |wma|^s \overline{\psi(m)} \, \mathrm{d}m, \tag{4.19}$$

which is often more convenient.

We will write this sum as

$$W^\circ_\psi(s,g) = \psi(n) \sum_{w \in \mathcal{W}} C_{\psi,w}(s,a). \tag{4.20}$$

4.2.2 Simplifying the Integrals

Now we want to simplify $C_{\psi,w}(a)$. Consider at the coset

$$wN(\mathbb{Q})w^{-1} \cap N(\mathbb{Q})\backslash N(\mathbb{A}). \tag{4.21}$$

Let $N_{w\pm}$ be the subset generated by the positive roots that are mapped to positive/negative roots, respectively. Then if we split up $N(\mathbb{A}) = N_{w+}(\mathbb{A})N_{w-}(\mathbb{A})$ only $N_{w+}(\mathbb{A})$ will be affected by the quotient with $wN(\mathbb{Q})w^{-1} \cap N(\mathbb{Q})$. Therefore we have

$$C_{\psi,w}(s,a) = \int\limits_{N_{w+}(\mathbb{Q})\backslash N_{w+}(\mathbb{A})} \mathrm{d}n_+ \int\limits_{N_{w-}(\mathbb{A})} \mathrm{d}n_- |wn_+n_-a|^s.\overline{\psi(n_+n_-)}, \tag{4.22}$$

This can be split up into two separate integrals using the multiplicative properties (noting that by construction $wn_+w^{-1} \in N(\mathbb{A})$, where χ is trivial)

$$C_{\psi,w}(s,a) = \int\limits_{N_{w+}(\mathbb{Q})\backslash N_{w+}(\mathbb{A})} \overline{\psi(n_+)}\,\mathrm{d}n_+ \int\limits_{N_{w-}(\mathbb{A})} |wn_-a|^s\overline{\psi(n_-)}\,\mathrm{d}n_-. \tag{4.23}$$

Doing the change of variables $n_+ \mapsto n_+ + n'$, with $n' \in N_{w+}(\mathbb{A})$ we find that

$$\int\limits_{N_{w+}(\mathbb{Q})\backslash N_{w+}(\mathbb{A})} \overline{\psi(n_+)}\,\mathrm{d}n_+ = \psi(n') \int\limits_{N_{w+}(\mathbb{Q})\backslash N_{w+}(\mathbb{A})} \overline{\psi(n_+)}\,\mathrm{d}n_+. \tag{4.24}$$

This proves that if $\psi(n')$ is not trivial on the entirety of $N_{w+}(\mathbb{A})$ the integral must vanish. If $\psi(n_+)$ is trivial the remaining integral is just 1, due to our chosen normalisation. Thus

$$C_{\psi,w}(s,a) = \int\limits_{N_{w-}(\mathbb{A})} |wn_-a|^s\overline{\psi(n_-)}\,\mathrm{d}n_- \tag{4.25}$$

when ψ is trivial on $N_{w+}(\mathbb{A})$, and zero otherwise.

Now we want to isolate the dependence on a. Note that from (2.49) we know that when bringing through a this induces only a rescaling of the parameters in n_-, the size of which is for the generator of a root α just $|a|^\alpha$. Therefore, we get that

$$C_{\psi,w}(s,a) = |waw^{-1}|^s|a|^{\sum_{w\alpha<0}\alpha} \int\limits_{N_{w-}(\mathbb{A})} |wn_-|^s\psi^a(n_-)\,\mathrm{d}n_-. \tag{4.26}$$

Now we need a small Lemma.

Lemma 4.4. *For any Weyl word w*

$$\sum_{\alpha>0:w\alpha<0} \alpha = \rho - w^{-1}\rho. \tag{4.27}$$

Proof. We have that by definition of ρ

$$\sum_{\alpha>0:w^{-1}\alpha>0} \alpha + \sum_{\alpha>0:w^{-1}\alpha<0} \alpha = 2\rho. \tag{4.28}$$

If we act on this with w^{-1} we get

$$\sum_{\alpha>0:w^{-1}\alpha>0} w^{-1}\alpha + \sum_{\alpha>0:w^{-1}\alpha<0} w^{-1}\alpha = 2w^{-1}\rho. \tag{4.29}$$

Now, if $w^{-1}\alpha = \pm\alpha'$ for some positive root α' we clearly have $\alpha = \pm w\alpha'$. Thus if we change variable in the sums from $\alpha \mapsto \pm w\alpha'$ we get

$$\sum_{w\alpha'>0:\alpha'>0} \alpha' - \sum_{w\alpha'>0:\alpha'<0} \alpha' = 2w^{-1}\rho. \tag{4.30}$$

Subtracting the two equations we get the result. □

With this we can simplify our integral to

$$C_{\psi,w}(s,a) = |waw^{-1}|^s |a|^{\rho - w^{-1}\rho} \int\limits_{N_{w-}(\mathbb{A})} |wn_-|^s \psi^a(n_-)\, \mathrm{d}n_-. \tag{4.31}$$

We then have, using that w acting on a is the same thing as $w^T = w^{-1}$ acting directly on s, that we can simplify this whole thing into

$$C_{\psi,w}(s,a) = |a|^{w^{-1}(s-\rho)+\rho} \int\limits_{N_{w-}(\mathbb{A})} |wn_-|^s \psi^a(n_-)\, \mathrm{d}n_-. \tag{4.32}$$

Again, this holds when ψ is trivial on $N_{w+}(\mathbb{A})$. When ψ is non-trivial the whole thing will vanish. Notice that everything here can be be factorised so we can write this as

$$C_{\psi,w}(s,a) = |a|^{w^{-1}(s-\rho)+\rho} \prod_{p\leq\infty} \int\limits_{N_{w-}(\mathbb{Q}_p)} |wn|_p^s \psi_p^{a_p}(n)\, \mathrm{d}n. \tag{4.33}$$

Introducing

$$I_{w,\psi,p}(s,a) = \int\limits_{N_{w-}(\mathbb{Q}_p)} |wn|_p^s \psi_p^{a_p}(n)\, \mathrm{d}n \tag{4.34}$$

Proposition 4.1 now follows, and so this concludes the proof.

4.3 Whittaker coefficients for $\mathrm{SL}(2,\mathbb{A})$

We are now going to calculate the contributions $I_{w,\psi,p}(s,a)$ for $\mathrm{SL}(2,\mathbb{A})$ using the machinery we established in the previous section. This serves two purposes. First of all, it serves as a good example to see what is actually going on here. Secondly, and perhaps most importantly, our strategy for deriving the corresponding results in other semi-simple Lie groups will be to reduce the problem to multiple integrals over $\mathrm{SL}(2,\mathbb{A})$ subgroups. Of primary importance to us will be the following two propositions and their corollaries.

Proposition 4.5. *The integral $I_{w,\psi,p}(s,a)$ for* SL(2, $\mathbb{Q}_p$) *is for finite primes p given by*

$$\int_{\mathbb{Q}_p} |x_{-\alpha}(u)|^{(1+\langle s-\rho|\alpha\rangle)\frac{\alpha}{\langle\alpha|\alpha\rangle}} e^{2\pi i[mu]_p}\,\mathrm{d}u = \gamma_p(m)\frac{\xi_p(\langle s-\rho|\alpha\rangle)}{\xi_p(1+\langle s-\rho|\alpha\rangle}\left(1-|pm|_p^{\langle s-\rho|\alpha\rangle}\right), \tag{4.35}$$

and the result holds also for $p=\infty$ if $m=0$.

Remark 4.6. *Since the rootspace of* SL(2, $\mathbb{A}$) *is just 1-dimensional the bilinear form is just multiplication with an inserted 2. We have therefore $s = \langle s|\alpha\rangle\frac{\alpha}{\langle\alpha|\alpha\rangle}$ and $2s-1 = \langle s-\rho|\alpha\rangle$.*

Corollary 4.7. *Performing the product for $w=0$ we find*

$$\int_{\mathbb{A}} |x_{-\alpha}(u)|^{(1+\langle s-\rho|\alpha\rangle)\frac{\alpha}{\langle\alpha|\alpha\rangle}}\,\mathrm{d}u = \frac{\xi(\langle s-\rho|\alpha\rangle)}{\xi(1+\langle s-\rho|\alpha\rangle)}. \tag{4.36}$$

Proposition 4.8. *For any positive root α and any $u \in \mathbb{Q}_p$ we have the Iwazawa-decomposition*

$$x_{-\alpha}(u) = x_\alpha(\tfrac{1}{u})h_\alpha(\tfrac{1}{u})k. \tag{4.37}$$

where $k \in G(\mathbb{Z}_p)$.

Corollary 4.9. *The above formulas also works for any* SL(2, $\mathbb{A}$) *subgroup of a larger group $G(\mathbb{A})$.*

Both of these propositions will be proved in the process of finding the Fourier-Whittaker-coefficients of SL(2, $\mathbb{A}$).

The integrals we are after are

$$I_{\psi,w,p}(s,a_p) = \int_{N_{-,w-}(\mathbb{Q}_p)} |wn|^s\overline{\psi_p^{a_p}(n)}\,\mathrm{d}n \tag{4.38}$$

$$= \int_{\mathbb{Q}_p} |x_{-\alpha}(-u)|^s e^{\pm 2\pi i[m|a|_p^\alpha u]_p}\,\mathrm{d}u, \tag{4.39}$$

with $\psi^{a_p}(x_\alpha(u)) = e^{\pm 2\pi i[m|a_p|^\alpha u_p]_p}$, with the plus for the real prime and minus for the finite primes.

For SL(2, $\mathbb{A}$) the Weyl group consists of two elements, the identity and the one fundamental reflection. For the identity element we find that $N_{w-}(\mathbb{A})$ is empty, so the result is just 1[1].

For the nontrivial Weyl word we have that since the only root is mapped to the negative one, $N_{w-}(\mathbb{A}) = N(\mathbb{A})$. We have to find the Iwazawa-decomposition of something in $N_-(\mathbb{A})$, and then carry out the integral. This will be different for finite and infinite primes so we split the problem up in two parts.

[1]This is not zero, because this case means that the entire integral has been carried out in the integral over N_{w+}, which is 1.

4.3.1 Contribution From the Infinite Prime

First we get the Iwazawa-decomposition, which we can find in the fundamental representation. We have

$$x_{-\alpha}(u) = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}. \tag{4.40}$$

If we multiply by a rotation matrix $k = \begin{pmatrix} \xi & \mp\sqrt{1-\xi^2} \\ \pm\sqrt{1-\xi^2} & \xi \end{pmatrix}$ we get

$$n_{-}k = \begin{pmatrix} \xi & * \\ \pm\sqrt{1-\xi^2} + u\xi & * \end{pmatrix}. \tag{4.41}$$

The two right-most components are irrelevant. Setting the lower left corner to zero gives us

$$\xi = \pm\frac{1}{\sqrt{1+u^2}}, \tag{4.42}$$

where the sign depends on the sign of u, and is irrelevant. Thus

$$I_{\psi,w,\infty}(s,a_\infty) = \int_{\mathbb{R}} (1+u^2)^{-s} e^{2\pi i m a_\infty^2 u}\, \mathrm{d}u. \tag{4.43}$$

Remember that the modenumber m has to be a rational number, so it is the same for all primes (and we neglect to write m_∞). Also note that we get an extra factor of 2 from taking $\langle s|\alpha\rangle$ in the definition of $|\cdot|^s$.

For $m = 0$ this evaluates to

$$I_{1,w,\infty}(s,a_\infty) = \sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)} = \frac{\pi^{-(s-1/2)}\Gamma(s-1/2)}{\pi^{-s}\Gamma(s)} = \frac{\xi_\infty(2s-1)}{\xi_\infty(2s)}. \tag{4.44}$$

This proves the infinite part of Proposition 4.5. For $m \neq 0$ the integral evaluates to

$$I_{\psi,w,\infty}(s,a_\infty) = \frac{2}{\xi_\infty(2s)}|m|_\infty^{s-1/2}|a_\infty|_\infty^{2s-1}K_{s-1/2}(2\pi|ma_\infty^2|_\infty), \tag{4.45}$$

where K_s is a modified Bessel function of the second kind.

4.3.2 Contribution from the Finite Primes

Now we have again that in the fundamental representation

$$x_{-\alpha}(u) = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}. \tag{4.46}$$

Here we will find use for the function

$$\tilde{u} = \begin{cases} 1 & u \in \mathbb{Z}_p, \\ u & u \notin \mathbb{Z}_p. \end{cases} \tag{4.47}$$

What we want to do is multiply by an element in $B(\mathbb{Q}_p)$ so that the result is in $K(\mathbb{Q}_p) = \mathrm{SL}(2, \mathbb{Z}_p)$. Because if $g = bk$ then $b^{-1}g = k$. This product can be written

$$\begin{pmatrix} v & x \\ 0 & 1/v \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} = \begin{pmatrix} v + ux & x \\ u/v & 1/v \end{pmatrix}. \tag{4.48}$$

Here we see that for example $v = \tilde{u}$ and $x = -1$ is a solution, since we have both $u/\tilde{u}$ and $u - \tilde{u}$ in $\mathbb{Z}_p$ for any choice of u. Taking the inverse of this element in $B(\mathbb{A})$ we find that an Iwazawa-decomposition is

$$\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} = \begin{pmatrix} 1/\tilde{u} & 1 \\ 0 & \tilde{u} \end{pmatrix} k = \begin{pmatrix} 1 & 1/\tilde{u} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\tilde{u} & 0 \\ 0 & \tilde{u} \end{pmatrix} k. \tag{4.49}$$

This proves Proposition 4.8.

This means that what we need is the integral

$$I_{w,\psi,p}(s,a_p) = \int_{\mathbb{Q}_p} |\tilde{u}|_p^{-2s} e^{-2\pi i [m a_p^2 u]} \, \mathrm{d}u = \mathcal{F}[|\tilde{u}|_p^{-2s}](a_p^2 m). \tag{4.50}$$

Once again we get a factor of 2 from taking the inner product. This Fourier transform has already been found in Proposition 2.5 to be

$$I_{w,\psi,p}(s,a_p) = \gamma_p(a_p^2 m)(1 - p^{-2s}) \frac{1 - |p a_p^2 m|_p^{2s-1}}{1 - p^{1-2s}} \tag{4.51}$$

$$= \gamma_p(a_p^2 m) \frac{\xi_p(2s-1)}{\xi_p(1 + (2s-1))} (1 - |p a_p^2 m|_p^{2s-1}) \tag{4.52}$$

This proves Proposition 4.5.

4.3.3 Assembling the Global Formula

With these integrals calculated we can now assemble the full function. The exponent that appears in the Whittacker coefficient is $w(2s - \rho) + \rho$ which for the trivial Weyl word 1 is just $2s$ while for the nontrivial Weyl word -1 it is $2\rho - 2s$. Thus we have

$$W_1^\circ(s, \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} a & 0 \\ 0 & 1/a \end{smallmatrix}\right) k) = |a|^{2s} + \frac{\xi(2s-1)}{\xi(2s)} |a|^{2-2s} \tag{4.53}$$

and

$$W_\psi^\circ(s,\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)\begin{pmatrix}a & 0\\ 0 & 1/a\end{pmatrix}k) = |a|^{2-2s}|a|_\infty^{2s-1}\frac{2}{\xi(2s)}|m|_\infty^{s-1/2}K_{s-1/2}(2\pi|ma_\infty^2|_\infty)e^{2\pi imx} \\ \times\prod_{p<\infty}\gamma_p(a_p^2m)\frac{1-|pa_p^2m|_p^{2s-1}}{1-p^{1-2s}}e^{-2\pi i[mx_p]_p}. \tag{4.54}$$

To get the Eisenstein series for the real case we can set $g_p = 1$ for $p < \infty$. This nets us a factor of $\prod_{p<\infty}\gamma_p(m_p)$ which according to our discussion in Section 2.2.5 means m must actually be an integer in order for the term not to vanish. Thus we have proved that the Eisenstein series of $\mathrm{SL}(2,\mathbb{R})$ is given by

$$E\left(s,\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sqrt{y} & 0\\ 0 & \frac{1}{\sqrt{y}}\end{pmatrix}\right) = y^s + \frac{\xi(2s-1)}{\xi(2s)}y^{1-s} \\ + \frac{4\sqrt{y}}{\xi(2s)}\sum_{m\neq 0}|m|_\infty^{s-1/2}\mu_{1-2s}(m)K_{s-1/2}(2\pi|m|_\infty y)e^{2\pi imx}, \tag{4.55}$$

where we have used the alternate form of the *instanton measure*

$$\mu_s(m) = \prod_{p<\infty}\frac{1-|pm|_p^{-s}}{1-p^s} \tag{4.56}$$

we found back in (3.43). This matches precisely with the expression we had in our previous discussion, Equation (3.32).

4.4 Langlands Constant Term Formula

We are now going to derive the Whittaker coefficient for trivial $\psi = 1$ on an Lie group $G(A)$ defined by a Cartan matrix, known as Langlands constant term formula. As we have already hinted, this will be done by reducing the problem to integrals over multiple $\mathrm{SL}(2,\mathbb{A})$ subgroups, for which we now know the result. The reason we do this is that when we derive our Whittaker coefficient for $\mathrm{SL}(3,\mathbb{Q}_p)$ we will use the same method, but keep track of the changes of variable.

Theorem 4.10 (Langlands formula [1, 17])**.** *The Whittaker-coefficient for the trivial character is given by*

$$W_1^\circ(s,nak) = \sum_{w\in\mathcal{W}}|a|^{w(s-\rho)+\rho}M(w,s-\rho), \tag{4.57}$$

where $M(w,s)$ is given by

$$M(w,s) = \prod_{\alpha>0:w\alpha<0}\frac{\xi(\langle s|\alpha\rangle)}{\xi(1+\langle s|\alpha\rangle)}. \tag{4.58}$$

Remark 4.11. *The proof we give will be valid for Lie algebras which are simply laced, that is, the off-diagonal elements of the Cartan matrix are 0 or* -1*. This in particular means that all roots have the same norm. Langlands formula is valid even without this assumption. We follow the proof as done in [1].*

Proof. What we have left to evaluate thanks to Corollary 4.2 is the integral

$$I_{w,1}(s,a) = \int_{N_{w-}(\mathbb{A})} |wn|^s \, \mathrm{d}n, \tag{4.59}$$

To do this we need to parametrise n in a suitable way. We first write w in reduced form in terms of the fundamental reflections as

$$w = w_{i_1} w_{i_2} \cdots w_{i_l}, \tag{4.60}$$

where l is the length of the Weyl word. Then we let

$$n = x_{\gamma_1}(u_1) x_{\gamma_2}(u_2) \cdots x_{\gamma_l}(u_l). \tag{4.61}$$

The γ_i is some choice of the ordering of the positive roots which get mapped to negative ones by w.

In this parametrisation we get

$$I_{w,1}(s,a) = \int_{\mathbb{A}^l} \left| w \prod_{j=1}^{l} x_{\gamma_j}(u_j) \right|^s \mathrm{d}u = \int_{\mathbb{A}^l} \left| \prod_{j=1}^{l} w x_{\gamma_j}(u_j) w^{-1} \right|^s \mathrm{d}u, \tag{4.62}$$

where the product is an *ordered product*, that is, factors have to be put in left to right in order, and we have used the fact that $w \in K(\mathbb{A})$, so it can be put in on the right with no issue. Now, by definition the adjoint action of w acts on x_α by changing the root so we get

$$I_{w,1}(s,a) = \int_{\mathbb{A}^l} \left| \prod_{j=1}^{l} x_{w\gamma_j}(u_j) \right|^s \mathrm{d}u, \tag{4.63}$$

where we have undone the fact that some of the parameters change sign.

Now we start evaluating this character. We write the right-most factor in its Iwazawa decomposition as

$$x_{w\gamma_l}(u_l) = b(u_l) k(u_l). \tag{4.64}$$

Important to note is that $x_{w\gamma_l}(u_l)$ lies in the $\mathrm{SL}(2,\mathbb{A})$ subgroup of $G(\mathbb{A})$ generated by $-w\gamma_l$ and thus the Iwazawa-decomposition is the same as it was for $\mathrm{SL}(2,\mathbb{A})$, and most importantly does not involve any other generators.

We need to bring the n and a out to the left, which means bringing it through every single remaining factor. Thus we get

$$I_{w,1}(s,a) = \int_{\mathbb{A}^l} |b(u_l)|^s \left| \prod_{j=1}^{l-1} b^{-1}(u_l) x_{w\gamma_j}(u_j) b(u_l) \right|^s \mathrm{d}u_l. \tag{4.65}$$

Now we choose $\gamma_l = \alpha_{i_l}$, which is always a root that gets mapped to negative ones by w, since the last factor is w_{i_l}. The effect of bringing through $b(u_l)$ is a factor of $|b_{w\gamma_l}(u_l)|^{w(\gamma_1+\gamma_2+\cdots+\gamma_{l_1})}$ corresponding to the rescaling of the variables, and a unimodular shift of the remaining parameters $u_1, \cdots u_{l-1}$. In particular note that this means that α_{i_l} is a simple root, with norm $\langle\alpha_{i_l}|\alpha_{i_l}\rangle = 2$ and $\langle\rho|\alpha_{i_l}\rangle = 1$. For details see [1].

Reusing Lemma 4.4 we then have

$$\gamma_1 + \gamma_2 + \cdots + \gamma_{l-1} = \rho - w^{-1}\rho - \alpha_{i_l} \tag{4.66}$$

and thus

$$\begin{aligned} I_{w,1}(s,a) &= \int_{\mathbb{A}} |b_{w\gamma_l}(u_l)|^{s+w(\rho-w^{-1}\rho-\alpha_{i_l})}\, \mathrm{d}u_l I_{ww_l^{-1}} \\ &= \int_{N_{w\gamma_l}(\mathbb{A})} |wn_{\gamma_l -}|^{s+w\rho-\rho-w\alpha_{i_l}}\, \mathrm{d}n_{\gamma l-} I_{ww_l^{-1}}. \end{aligned} \tag{4.67}$$

This can be evaluated, since this outermost integral can be evaluated as in Proposition 4.5. To do that we must project unto the positive root $-w\alpha_{i_l}$. We have

$$\langle s - \rho + w\rho - w\gamma_l| - w\alpha_{i_l}\rangle = \langle s - \rho| - w\alpha_{i_l}\rangle - \langle\rho - \alpha_{i_l}|\alpha_{i_l}\rangle. \tag{4.68}$$

where we have gotten rid off the w in the second term due to the fact that it is orthogonal under $\langle|\rangle$. Now note that since α_{i_l} is a simple root we know that $\langle\rho|\alpha_{i_l}\rangle = 1$ and $\langle\alpha_{i_l}|\alpha_{i_l}\rangle = 2$. Thus

$$I_{w,1}(s,a) = \int_{N_{w\gamma_l}(\mathbb{A})} |wn_{\gamma_l -}|^{(\langle s-\rho|-w\alpha_{i_l}\rangle+1)\frac{-w\alpha_{i_l}}{\langle w\alpha_{i_l}|w\alpha_{i_l}\rangle}}\, \mathrm{d}n_{\alpha_{i_l}-} \int \cdots \tag{4.69}$$

This first integral is now a standard $\mathrm{SL}(2,\mathbb{A})$ contribution as in Corollary 4.5. Now we are going to get rid of w_{i_l} in w. Note that none of the roots $\gamma_1, \cdots \gamma_{l-1}$ can be α_{i_l}. Therefore the action of w_{i_l} is just to permute the remaining positive roots and thus we can now get rid of that factor in w, and write

$$I_{w,1}(s,a) = \frac{\xi(\langle s-\rho| - w\gamma_l\rangle + 1 - 1)}{\xi(\langle s-\rho| - w\gamma_l\rangle + 1)} I_{ww_l^{-1}} = \frac{\xi(\langle s-\rho| - w\gamma_l\rangle)}{\xi(1 + \langle s-\rho| - w\gamma_l\rangle)} I_{ww_l^{-1}}. \tag{4.70}$$

and the remaining integrals in $I_{ww_l^{-1}}$ can be treated in the exact same way.

By induction we can take a product over all the positive roots that get mapped to negative roots by w, with each integral being of the form in Corollary 4.7, i.e.

$$I_{w,1}(s,a) = \prod_{\alpha>0: w\alpha<0} \frac{\xi(\langle s-\rho| - w\alpha\rangle)}{\xi(1 + \langle s-\rho| - w\alpha\rangle)}. \tag{4.71}$$

The prefactor from breaking out a has a dependence on w^{-1}. We would like to get rid of that through a change of variables. To do that we need to replace the dependence on w here by w^{-1}. We can get rid of the factor of w by using the same argument as in Lemma 4.4. If $w\alpha = -\alpha'$ for some positive root α' then in turn $w^{-1}\alpha' = -\alpha$. Therefore we can write

$$I_{w,1}(s,a) = \prod_{\alpha>0:w^{-1}\alpha<0} \frac{\xi(\langle s-\rho|\alpha\rangle}{\xi(1+\langle s-\rho|\alpha\rangle)}. \tag{4.72}$$

The full Whittaker coefficient is then (after changing variable in the Weyl sum from $w \mapsto w^{-1}$)

$$W_1^\circ(s,nak) = \sum_{w\in\mathcal{W}} |a|^{w(s-\rho)+\rho} \prod_{\alpha>0|w\alpha<0} \frac{\xi(\langle s-\rho|\alpha\rangle)}{\xi(1+\langle s-\rho|\alpha\rangle)}, \tag{4.73}$$

as claimed. □

4.5 Casselman-Shalika*

Now we are going to consider the unramified characters $\psi(n)$ which are built up by the unramified characters $\psi(x) = e^{2\pi i x_\infty} \prod_{p\leq\infty} e^{-2\pi i [x]_p}$. All the modenumbers are 1, which means that ψ is only trivial on $N(\mathbb{Q})$. In this special case it is possible to evaluate the contributions from finite primes p.

The only purpose this serves is to showcase one approach of evaluating the integral, which appears in the appendix of alternative approaches. This has no bearing on the next chapter except for as a means of comparing the result.

Theorem 4.12 (The Casselman-Shalika formula)**.** *The local Whittaker-coefficient for the unramified character ψ^1 is given by [1, 16]*

$$W_{\psi^1,p}^\circ(s,nak) = \psi(n) \sum_{w\in\mathcal{W}} |a|^{w(s-\rho)+\rho} \prod_{\alpha>0} \frac{\xi_p(-\langle w(s-\rho)|\alpha\rangle)}{\xi_p(1+\langle s-\rho|\alpha\rangle)}. \tag{4.74}$$

Remark 4.13. *This is a remarkable formula, because this sum over the Weyl-group is a new sum. The one we had in Proposition 4.1 has already been carried out; only w_0 maps all positive roots to negative ones, and the unramified character is never trivial.*

Remark 4.14. *With some further work one can show that this is in fact related to the characters of representations of G, which we saw for* $\mathrm{SL}(2,\mathbb{A})$ *in the previous chapter.*

The proof of this formula is very different from the proof of Langlands formula. Instead of attacking the integral directly we will perform one of the integrals and then derive a functional relation for the Whittaker coefficient. From this we can create a function that is Weyl-invariant and we write this new function as a Weyl orbit. From this we get the Casselman-Shalika formula. We will once again be following the proof in [1], which is a much simplified version of the original proof in [16].

4.5.1 Constructing Weyl Invariant Function

The functional relation we are after is
Proposition 4.15. *For any Weyl word w*

$$W^{\circ}_{\psi^1,p}(w(s+\rho),g) = \prod_{\alpha>0:w\alpha<0}\left[\frac{\xi_p(1+\langle s|\alpha\rangle)}{\xi_p(1-\langle s|\alpha\rangle)}\right]W^{\circ}_{\psi^1,p}(s+\rho,g). \tag{4.75}$$

Proof. We have

$$W^{\circ}_{\psi^1,p}(s,g) = \int_{N(\mathbb{Q}_p)} |wn_ig|^s\psi(n)\,\mathrm{d}n_i, \tag{4.76}$$

where we have used Corollary 4.3. Now define the function

$$V_i(s,g) = \int_{N_{i-}(\mathbb{A})} |n_{i-}g|^s\psi_-(n_{i-})\,\mathrm{d}n_{i-}, \tag{4.77}$$

where $N_{i-}(\mathbb{A})$ is the part of the lower unipotent subgroup generated by the simple root α_i. By defining $\hat{N}_-(\mathbb{Q}_p) = N_{i-}(\mathbb{Q}_p)\backslash N_-(\mathbb{Q}_p)$ we can then write

$$W^{\circ}_{\psi^1,p}(s) = \int_{\hat{N}_-(\mathbb{Q}_P)} V_i(s,\hat{n}_-g)\psi_-(\hat{n}_-)\,\mathrm{d}\hat{n}_-. \tag{4.78}$$

The function $V_i(s,g)$ is then invariant under

$$V_i(s,\hat{n}gk) = V_i(s,g), \tag{4.79}$$

where $\hat{n}\in\hat{N}(\mathbb{Q}_p)$ and $k\in K(\mathbb{Q}_p) = G(\mathbb{Z}_p)$. This is because the commutator of $N_{i-}(\mathbb{Q}_P)$ with $\hat{N}(\mathbb{Q}_p)$ just results in a factor in $N(\mathbb{Q}_p)$ under which $|\cdot|^s$ is invariant. The k invariance is trivial.

Now we write $\hat{n}_-g = \hat{n}\hat{a}g_ik$. Putting this in V_i we find that we are after

$$V_i(s,\hat{a}g_i) = \int_{N_{i-}(\mathbb{Q}_p)} |n_{i-}\hat{a}g_i|^s\psi_-(n_-)\,\mathrm{d}n_{i-}. \tag{4.80}$$

Breaking out a we find

$$V_i(s,\hat{a}g_i) = |\hat{a}|^{s-\alpha_i}\int_{N_{i-}(\mathbb{Q}_p)} |n_{-i}g_i|^s\psi_-^{\hat{a}}(n_-)\,\mathrm{d}n_{i-}. \tag{4.81}$$

This remaining integral is just a standard $\mathrm{SL}(2,\mathbb{Q}_p)$ integral, which we can evaluate, and from this we we can derive that

$$V_i(w_is,g) = V_i(s,g)\frac{\xi_p(1+\langle w_i(s-\rho)|\alpha_i\rangle)}{\xi_p(1-\langle w_i(s-\rho)|\alpha_i\rangle)}. \tag{4.82}$$

From this we know that

$$W^{\circ}_{\psi^1,p}(w_i s,g) = \frac{\xi_p(1+\langle s-\rho|\alpha_i\rangle)}{\xi_p(1-\langle s-\rho|\alpha_i\rangle)} W^{\circ}_{\psi^1,p}(s,g). \tag{4.83}$$

Writing this in the form in the proposition we have that for a general Weyl word $w = w_{i_1} w_{i_2} \cdots w_{i_l}$ we have

$$W^{\circ}_{\psi^1,p}(w(s+\rho)) = \frac{\xi_p(1+\langle w's|\alpha_i\rangle)}{\xi_p(1-\langle w's|\alpha_i\rangle)} W^{\circ}_{\psi^1,p}(w'(s+\rho),g). \tag{4.84}$$

where $w' = ww_{i_l}^{-1}$. We get the result by noting that as we carry out the product we hit every such root exactly once. □

Now, we can combine this with a function which has the opposite transformation under w. Such a function is

$$\zeta(s) = \prod_{\alpha>0} \xi_p(1+\langle s-\rho|\alpha\rangle), \tag{4.85}$$

which can be verified by hand [1]. The function

$$\zeta(s) W^{\circ}_{\psi^1,p}(s,g) \tag{4.86}$$

is therefore Weyl invariant. We will not consider the details but it can be shown from the holomorphicity of $|\cdot|^s$ that the the dependence on a can be extracted so that we may write

$$\zeta(s+\rho) W^{\circ}_{\psi^1,p}(s+\rho,a) = \sum_{w\in\mathcal{W}} f(ws)|a|^{w^{-1}s+\rho}. \tag{4.87}$$

for some function f [16].

4.5.2 Determining the Weyl Orbit

Now we want to determine the function f. When w is the longest Weyl word we know that it is the coefficient in front of $|a|^{w_0^{-1}(s-\rho)+\rho}$ in $W^{\circ}_{\psi^1,p}(s)$, multiplied by $\zeta(s)$. This part of $W^{\circ}_{\psi^1,p}(s,1)$ is exactly the constant term in the polynomial resulting from the integral

$$\int_{N(\mathbb{Q}_p)} |w_0 n|^{s+\rho} \psi^a(n)\, \mathrm{d}n, \tag{4.88}$$

due to the form of Proposition 4.1.

We only need this for one special case, so let $a = 0$ (this is somewhat formal as $a = 0$ is not an element of $A(\mathbb{A})$) in ψ^a, which makes ψ trivial, so we find that the contribution is exactly the same as the contribution to the constant term

$$\int_{N(\mathbb{Q}_p)} |w_0 n|^{s+\rho}\, \mathrm{d}n = M(w_0,s) = \prod_{\alpha>0} \frac{\xi_p(\langle s|\alpha\rangle)}{\xi_p(1+\langle s|\alpha\rangle)}. \tag{4.89}$$

Therefore

$$f(w_0 s) = \zeta(s) M(w,s) = \prod_{\alpha>0} \xi_p(\langle s|\alpha\rangle) \tag{4.90}$$

which means

$$f(s) = \prod_{\alpha>0} \xi_p(\langle w_0^{-1} s|\alpha\rangle) = \prod_{\alpha>0} \xi_p(-\langle s|\alpha\rangle), \tag{4.91}$$

where we have noted that w_0 maps all positive roots to all negative ones.

Changing the variable in the sum from $w \mapsto w^{-1}$ we find

$$\zeta(s) W^{\circ}_{\psi^1,p}(s,a) = \sum_{w\in\mathcal{W}} |a|^{w(s-\rho)+\rho} \prod_{\alpha>0} \xi_p(-\langle w(s-\rho)|\alpha\rangle). \tag{4.92}$$

From this the Casselman-Shalika formula 4.12 now follows.

5

Whittaker Coefficients for $\mathrm{SL}(3, \mathbb{Q}_p)$

Now we have finally arrived at the core result of this thesis. The goal here will be to calculate the contribution from finite primes using direct integration, rather than Casselman-Shalika. In this we will for simplicity restrict our attention to the case where $g_p = 1$. A non-constant only really yields a known contribution and a rescaling of the parameters in ψ, so we will not include them explicitly. We are also typically only interested in the real Eisenstein series, where g_p is set to 1.

The way that turns out to work is to use the same parametrization of the integral as we did in deriving Langlands formula for the constant term, but now we have to keep track of the changes of variable since they affect the arguments of ψ. Despite these changes of variables the result turns out to be not so bad, and the Fourier transforms can be readily evaluated.

In this chapter we have suppressed the subscripts p on the absolute value and γ, since it should be clear that we always mean $|\cdot|_p$ and γ_p.

5.1 The Problem

Starting from Corollary 4.3 we know that all we need is the following Proposition.
Proposition 5.1. *The integral*

$$I = I_{w,\psi,p} = \int_{N(\mathbb{Q}_p)} |wn|^s \psi(n) \,\mathrm{d}n, \tag{5.1}$$

where

$$\psi\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = e^{-2\pi i [m_1 x + m_2 y]_p}, \tag{5.2}$$

is given by

$$\begin{aligned} I =& \gamma(m_1)\gamma(m_2)M(w, s-\rho) \\ &\Big(1 - |pm_1|^{s_2} - |pm_2|^{s_1} + |pm_1|^{s_2}|pm_2|^{s_3} + |pm_1|^{s_3}|pm_2|^{s_1} - |pm_1|^{s_3}|pm_2|^{s_3}\Big). \end{aligned} \tag{5.3}$$

The rest of this section will be dedicated to proving this.

We are going to prove this in a slightly more convoluted way than is perhaps necessary, just to further illustrate what happens in the derivation of Langlands constant term. We will be carrying w around for quite some time, despite the fact that we can just get rid of it from the start. We would just have to remember the order we should place the roots in. All that would change is that it would appear like we are not only integrating over the images of simple roots under w.

Before we begin let us note the following lemma.

Lemma 5.2. *Unless the constants m_1,m_2 defining ψ are p-adic integers the coefficient W°_ψ, and thus also I, vanishes.*

Proof. The Whittaker-coefficient is defined by the integral

$$I = \int_{\mathbb{Q}_p^3} |wn|^s \overline{\psi(n)} \, \mathrm{d}n. \tag{5.4}$$

Since the character $|\cdot|^s$ is invariant under $K = \mathrm{SL}(3, \mathbb{Z}_p)$ on the right we can change variable from n to $n' = mk$, with $k \in N(\mathbb{Z}_p)$. This results in

$$I = \psi(k) \int_{\mathbb{Q}_p^3} |wn|^s \overline{\psi(n)} \, \mathrm{d}n. \tag{5.5}$$

This proves that either $\psi(k)$ is 1 or the integral vanishes, which means m_1 and m_2 must be p-adic integers for the integral not to vanish. □

5.1.1 Evaluating the Character

The first step is to evaluate the character $|\cdot|^s$ inside the integral. This involves changing variables a few times and will therefore introduce changes in ψ, according to the following.

Lemma 5.3. *The integral $I = \int_{N(\mathbb{Q}_p)} |wn|^s \psi(n) \, \mathrm{d}n$ when w is the longest Weyl word and $\psi(x_1(u)x_2(v)) = e^{-2\pi i [m_1 u + m_2 v]_p}$ is for $\mathrm{SL}(3, \mathbb{Q}_p)$ given by*

$$I = \int_{\mathbb{Q}_p^3} |\tilde{x}|^{-(1+s_2)} |\tilde{y}|^{-(1+s_1)} |\tilde{z}|^{-(1+s_3)} \exp\left(-2\pi i \left[m_1 x + m_2 \frac{\tilde{z}}{\tilde{x}} y + m_2 z\right]_p\right) \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z, \tag{5.6}$$

where $s_i = \langle s - \rho | \alpha_i \rangle$.

Proof. We have

$$I = \int_{N(\mathbb{Q}_p)} |wn|^s \psi(n)\,\mathrm{d}n. \tag{5.7}$$

The longest Weyl word can be factored as $w = w_1 w_2 w_1$, so we should start with α_1. Thus we parametrise our integral as

$$I = \int_{\mathbb{Q}_p^3} |w x_{\alpha_2}(n_y) x_{\alpha_3}(n_z) x_{\alpha_1}(n_x)|^s \psi(n_x, n_y)\,\mathrm{d}n_x\,\mathrm{d}n_y\,\mathrm{d}n_z. \tag{5.8}$$

Acting with w on all the maps x_α by inserting $w^{-1}w$ in between all factors we get

$$|x_{-\alpha_1}(-n_y) x_{-\alpha_3}(-n_z) x_{-\alpha_2}(-n_x)|^s. \tag{5.9}$$

Here we have used the matrices in (2.88) for w.

We know from Proposition 4.8 that an Iwazawa decomposition of a negatively generated element is

$$x_{-2}(n_x) = x_2(1/\tilde{n}_x) h_2(1/\tilde{n}_x) k. \tag{5.10}$$

Commuting this through the other factors we find that in the integral we have (these can be found by multiplying matrices in the fundamental representation)

$$|h_2(1/\tilde{n}_x)|^s |x_{-1}(n_y \tilde{n}_x - n_z)) x_{-3}(n_z/\tilde{n}_x)|^s. \tag{5.11}$$

This can be restored by taking

$$n_z \mapsto n_z \tilde{n}_x, \tag{5.12}$$

followed by

$$n_y \mapsto n_y/\tilde{n}_x + n_z \tag{5.13}$$

which gives us (after pulling out w again, and undoing also this sign change)

$$I = \int_{\mathbb{Q}_p^3} |h_2(1/\tilde{n}_x)|^s |w x_2(n_y) x_3(n_z)|^s \psi(n_x, \tfrac{n_y}{\tilde{n}_x} + n_z)\,\mathrm{d}n_x\,\mathrm{d}n_y\,\mathrm{d}n_z. \tag{5.14}$$

The next step in the procedure is to get rid of the right-most factor in w, which in this case is w_1. The action of w_1 is to swap places between α_2 and α_3 so we get

$$I = \int_{\mathbb{Q}_p^3} |h_2(1/\tilde{n}_x)|^s |w' x_3(n_y) x_2(n_z)|^s \psi(n_x, \tfrac{n_y}{\tilde{n}_x} + n_z)\,\mathrm{d}n_x\,\mathrm{d}n_y\,\mathrm{d}n_z, \tag{5.15}$$

where $w' = w_1 w_2$. What luck, the $x_2(n_z)$ is already placed to the right[1]. Now we act with $w' = w_1 w_2$ which sends α_3 to α_{-1} and α_2 to $-\alpha_3$[2]. Therefore we have to simplify the character

$$|x_{-1}(n_y) x_{-3}(n_z)|^s \tag{5.16}$$

[1]It is almost like someone planned that.

[2]Here we are being extremely convoluted. We could have gotten the same result by just not pulling out w, and removing the rightmost factor of w_1.

and we write just as before

$$x_{-3}(n_z) = x_3(1/\tilde{n}_z)h_3(1/\tilde{n}_z)k. \tag{5.17}$$

Commuting this through we get

$$|h_3(1/\tilde{n}_z)|^s|x_2(n_y/\tilde{n} - z)x_{-1}(n_y/\tilde{n}_z)|^s = |h_3(1/\tilde{z})|^s|x_{-1}(n_y/\tilde{n}_z)|^s. \tag{5.18}$$

This can be restored by taking $n_y \mapsto n_y\tilde{n}_z$ so our integral becomes

$$I = \int_{\mathbb{Q}_p^3} |h_2(1/\tilde{n}_x)h_3(1/\tilde{n}_z)|^s|w'x_3(n_y)|^s\psi(n_x, n_y\tfrac{\tilde{n}_z}{\tilde{n}_x} + n_z)\,\mathrm{d}n_x|\tilde{n}_z|\,\mathrm{d}n_y\,\mathrm{d}n_z. \tag{5.19}$$

Note that we have picked up an extra factor of $|\tilde{n}_z|$.

Just for completeness we do the last argument in the same way. We get rid of the rightmost letter in w', which is w_2. This word takes α_3 to α_1 so we get inside the integral just

$$|w_1x_1(-n_y)|^s = |h_1(1/\tilde{n}_y)|^s. \tag{5.20}$$

Thus our full integral is (changing variables to x,y,z now that there is no confusion)

$$I = \int_{\mathbb{Q}_p^3} |\tilde{x}|^{-(1+s_2)}|\tilde{y}|^{-(1+s_2)}|\tilde{z}|^{-(1+s_3)}\psi(x, y\tfrac{\tilde{z}}{\tilde{x}} + z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{5.21}$$

We have gone ahead and inserted $s_i = \langle s - \rho|\alpha_i\rangle$. To do this we have used the fact that $\langle\rho|\alpha_3\rangle = 2$ rather than 1, which conveniently cancels the extra $|\tilde{n}_z|$ we picked up in step two. Putting in the definition of $\psi(x,y) = e^{-2\pi i[m_1x+m_2y]_p}$ and changing all signs gives us the result. □

Remark 5.4. *We chose to derive this using $w = w_1w_2w_1$, but since w is also equal to $w_2w_1w_2$ we can derive an alternative expression for ψ, which reveals a symmetry between m_1 and m_2. Then one gets instead*

$$\psi = \exp\left(-2\pi i\left[m_2y + m_1\frac{\tilde{z}}{\tilde{y}}x + m_1z\right]_p\right). \tag{5.22}$$

5.1.2 Evaluating the Fourier Transforms

Now that we have reduced the problem to just an ordinary integral over $\mathbb{Q}_p^3$ we can integrate ahead. Continuing from Lemma 5.3 we have

$$I = \int_{\mathbb{Q}_p^3} |\tilde{x}|^{-(s_2+1)}|\tilde{y}|^{-(s_1+1)}|\tilde{z}|^{-(s_3+1)}\exp\left(-2\pi i\left[m_1x + m_2y\frac{\tilde{z}}{\tilde{x}} + m_2z\right]_p\right)\mathrm{d}x\,\mathrm{d}x\,\mathrm{d}z. \tag{5.23}$$

We are all but forced to integrate y first, since it is the only variable that does not appear in a convoluted way in ψ.

Using Proposition 4.5 we obtain

$$I = \frac{\xi_p(s_1)}{\xi_p(1+s_1)} \int_{\mathbb{Q}_p^2} \gamma\left(m_2 \tfrac{\tilde{z}}{\tilde{x}}\right) \left(1 - \left|p\frac{m_2\tilde{z}}{\tilde{x}}\right|^{s_1}\right) |\tilde{x}|^{-(s_2+1)}|\tilde{z}|^{-(s_3+1)} e^{-2\pi i[m_1x+m_2z]_p} \,\mathrm{d}y\,\mathrm{d}z. \tag{5.24}$$

Note that since we know that m_2 must be an integer, we can replace $\tilde{z}$ by z in γ, since this will not affect its value. This puts it in the form in Lemma 2.4, and thus we can remove this factor as long as we put in a proper counter-term, resulting in

$$I = \frac{\xi_p(s_1)}{\xi_p(1+s_1)} \int_{\mathbb{Q}_p^2} \left(1 - \left|p\frac{m_2\tilde{z}}{\tilde{x}}\right|^{s_1}\right) |\tilde{x}|^{-(1+s_2)}|\tilde{z}|^{-(1+s_3)} e^{-2\pi i[m_1x+m_2z]_p} \,\mathrm{d}x\,\mathrm{d}z + I_0. \tag{5.25}$$

The counterterm I_0 turns out to be zero, which we will prove after the real calculation is done.

I splits into two terms, $I = \frac{\xi_p(s_1)}{\xi_p(1+s_1)}(I_1' - |pm_2|^{s_1} I_2')$, with

$$I_1' = \int_{\mathbb{Q}_p^2} |\tilde{x}|^{-(1+s_2)}|\tilde{z}|^{-(1+s_3)} e^{-2\pi i[m_1x+m_2z]_p} \,\mathrm{d}x\,\mathrm{d}z. \tag{5.26}$$

and

$$I_2' = \int_{\mathbb{Q}_p^2} |\tilde{x}|^{-(1+s_2+s_1)}|\tilde{z}|^{-(1+s_3-s_1)} e^{-2\pi i[m_1x+m_2z]_p} \,\mathrm{d}x\,\mathrm{d}z. \tag{5.27}$$

Note that $s_2 + s_1 = s_3$ and $s_3 - s_1 = s_2$. This is exactly the action of w on α_2 and α_3. In fact, if we define $s_i(w) = \langle s - \rho | w\alpha_i \rangle$ we have that by introducing

$$I'(w) = \int_{\mathbb{Q}_p^2} |\tilde{x}|^{-(1+s_2(w))}|\tilde{z}|^{-(1+s_3(w))} e^{-2\pi i[m_1x+m_2z]_p} \,\mathrm{d}x\,\mathrm{d}z \tag{5.28}$$

we can write $I = \frac{\xi_p(s_1)}{\xi_p(1+s_1)}(I'(e) - |pm_2|^{s_1} I'(w_1))$.

These two remaining integrals are completely decoupled, so we can evaluate them both without issue to find

$$I'(w) = \gamma(m_1)\gamma(m_2)\frac{\xi_p(s_2(w))}{\xi_p(1+s_2(w))}\frac{\xi_p(s_3(w))}{\xi_p(1+s_3(w))}(1-|pm_1|^{s_2(w)})(1-|pm_2|^{s_3(w)}). \tag{5.29}$$

Note here that for the ξ_p factors it does not matter if w is e or w_1, since w_1 just swaps s_2 and s_3.

Assembling the full I from this we get

$$I = \frac{\xi_p(s_1)}{\xi_p(1+s_1)}\frac{\xi_p(s_2)}{\xi_p(1+s_2)}\frac{\xi_p(s_3)}{\xi_p(1+s_3)}\gamma(m_1)\gamma(m_2) \tag{5.30}$$

$$\Big(1 - |pm_1|^{s_2} - |pm_2|^{s_3} + |pm_1|^{s_2}|pm_2|^{s_3} \tag{5.31}$$

$$- |pm_2|^{s_1} + |pm_1|^{s_3}|pm_2|^{s_1} + |pm_2|^{s_1+s_2} - |pm_1|^{s_3}|pm_2|^{s_3}\Big) \tag{5.32}$$

which can be simplified to the desired form in Proposition 5.1.

We still have to prove that the counterterm I_0 is zero. In terms of Lemma 2.4 the integral

$$\int_{\mathbb{Q}_p^2} \gamma(\tfrac{m_2}{\tilde{x}} z) \left(1 - \left|p \frac{m_2 \tilde{z}}{\tilde{x}}\right|^{s_1}\right) |\tilde{x}|^{-(1+s_2)} |\tilde{z}|^{-(1+s_3)} e^{-2\pi i [m_1 x + m_2 z]_p} \, \mathrm{d}x \, \mathrm{d}z \tag{5.33}$$

has (when viewed as a Fourier transform in z) $\phi(z) = \left(1 - \left|p \frac{m_2 \tilde{z}}{\tilde{x}}\right|^{s_1}\right) |\tilde{x}|^{-(1+s_2)} |\tilde{z}|^{-(1+s_3)}$, $a = \frac{m_2}{\tilde{x}}$ and $w = m_2$ and thus gives rise to the counterterm

$$I_0 = \int_{\mathbb{Q}_p} \gamma^*(\tfrac{1}{\tilde{x}}) \frac{1}{|m_2|} \phi(\tfrac{1}{p m_2}) e^{-2\pi i [m_1 x]_p} \, \mathrm{d}x \tag{5.34}$$

when we remove $\gamma(\frac{m_2}{\tilde{x}} z)$.

Since m_2 is an integer we know that $\frac{1}{p m_2}$ is guaranteed to not be in $\mathbb{Z}_p$. Therefore all factors $\tilde{z}$ spit it right out. Furthermore, the factor $\gamma^*(\frac{1}{\tilde{x}})$ ensures that $|\tilde{x}| = 1$. This means we get

$$I_0 = \gamma^*(\tfrac{1}{\tilde{x}}) \frac{1}{|m_2|} \phi(\tfrac{1}{p m_2}) = \gamma^*(x) \frac{1}{|m_2|} \left(1 - \left|\frac{p m_2}{p m_2}\right|^{s_1}\right) |p m_2|^{1+s_3} = 0. \tag{5.35}$$

Therefore the entire integral vanishes, and the counterterm I_0 is zero as promised.

This concludes the proof of Proposition 5.1.

5.2 Discussion

So we have managed to evaluate the sought-after Fourier-Whittaker coefficients. What did it entail? First we had to find the Iwazawa-decomposition of $\left(\begin{smallmatrix} 1 & 0 & 0 \\ y & 1 & 0 \\ z & x & 1 \end{smallmatrix}\right)$, this turned out to be more naturally expressed with the variables changed according to

$$\begin{aligned} x &\mapsto x', \\ y &\mapsto y' \frac{\tilde{z}'}{\tilde{x}'} + z', \\ z &\mapsto z' \tilde{x}'. \end{aligned} \tag{5.36}$$

This netted us

$$\left| \begin{pmatrix} 1 & 0 & 0 \\ y' \frac{\tilde{z}'}{\tilde{x}'} + z' & 1 & 0 \\ z' \tilde{x}' & x' & 1 \end{pmatrix} \right|^s = |\tilde{x}|^{-(1+s_1)} |\tilde{y}|^{-(1+s_2)} |\tilde{z}|^{-(1+s_3)}. \tag{5.37}$$

This change of variable accomplished some subtle things. First of all, it made ψ aware of z'. This was crucial, because it meant that ψ could feel all variables, which meant that all the integrals could have the same form: A Fourier-transform of $|\tilde{x}|^{s'}$, for some root s'. In particular it turns into iterations of the same integral as for $\mathrm{SL}(2, \mathbb{Q}_p)$

These roots s' that arose were also always in the Weyl orbit of the original root s, and this was ensured by the specific form $\frac{\tilde{z}'}{\tilde{x}'}$. This specific factor was what made it so that after integrating out y we still only had the projections s_1,s_2, and s_3; Not something like $s_1 - s_2$ or $s_1 + s_3$. However, after the y integral is done we did not pick up any more products that could be interpreted as Weyl words. The rest of the changes of variable did not seem to do anything like that. This is likely restored in the case $m_1 = m_2$.

The other non-trivial thing we did was to show that the counterterm I_0, which arose from an extra factor of γ in the integral, vanished. This demanded special properties of the Fourier transform of $|\tilde{x}|^s$ in response to the changes of variables we did.

To generalise this treatment one would have to find the Weyl group structure inside the change of variables, as well as show that the integral can be carried out through iterated integrals such as in Equation (5.28). Also one would have to show that the counterterm is zero in general. It clearly cannot remain since it does not contain the necessary ratio of functions ξ_p.

The next step would be to write this result in terms of characters of representations of $\mathrm{SL}(3, \mathbb{Q}_p)$, just like was done for $\mathrm{SL}(2, \mathbb{Q}_p)$ with the instanton measure. This has been done by D. Bump, but for completeness it would be nice to have included the derivation.

After that we could look to generalise the calculation for arbitrary $\mathrm{SL}(n, \mathbb{Q}_p)$, or even arbitrary Lie groups. The difficulty on this depends on the difficulty in understanding the structure of the commutators $[E_\alpha, E_{-\beta}]$, and the changes of variable they induce.

A

Alternative Approaches

Since the goal of this thesis is to explore methods of evaluating the p-adic integrals that appear in the Fourier-Whittaker coefficients of Eisenstein series it feels natural that one should also include the methods that do not work. This is perhaps not a very interesting part of the thesis, but it is included for completion. Below we will go through the approaches that we attempted before we used the approach back in Chapter 5, and discuss why they fail.

The first approach is to try and use the same method as in the proof of Casselman-Shalika, introduce a new projected version of the character and perform the integral over that first. Then try to restore the argument and repeat the procedure for all the roots. The second approach is to try and find an explicit Iwazawa-decomposition in the same way we did for $\mathrm{SL}(2, \mathbb{Q}_p)$ back in Proposition 4.8.

A.1 After The First Step Of Casselman-Shalika

The first thing we are going to be looking at is using the approach we used in the proof of the Casselman-Shalika formula. The idea is that since this is what is used to calculate the integral for the general case, it should also work in the specific case of $\mathrm{SL}(3, \mathbb{Q}_p)$. The problem is of course that in the proof of Casselman-Shalika we used a symmetry-argument that we do not want to use here, but one might expect it to be possible to evaluate the rest of the integral anyway. The advantage would be that one does not have to determine any Iwazawa-decompositions of the integration variables, but this is perhaps hoping for too much.

We start from Proposition 4.1. We want to evaluate the integral

$$W^\circ_\psi(s,1) = I = \int\limits_{N(\mathbb{Q}_p)} |wn|^s \psi(n)\,\mathrm{d}n = \int_{N_-(\mathbb{Q}_p)} |n_-|^s \psi_-(n_-)\,\mathrm{d}n_-. \tag{A.1}$$

We then let

$$X_1(g) = \int\limits_{N_1(\mathbb{Q}_p)} |wn_1 g|^s \psi(n_1)\,\mathrm{d}n_1. \tag{A.2}$$

This lets us write our desired integral as

$$W^\circ_\psi(s,1) = \int\limits_{\hat{N}(\mathbb{Q}_p)} X_1(n_2 n_3)\psi(n_2)\,\mathrm{d}n_2\,\mathrm{d}n_3, \tag{A.3}$$

where $\hat{N}(\mathbb{Q}_p) = N_1(\mathbb{Q}_p)\backslash N(\mathbb{Q}_p)$.

Now we want to determine $X_1(g)$, to do that we decompose

$$g = \hat{n}_- \hat{a} g_1 k, \tag{A.4}$$

where $\hat{n}_- = w^{-1}\hat{n}w$. This leaves us with

$$X_1(\hat{n}_-\hat{a}g_1k) = \int\limits_{N_1(\mathbb{Q}_p)} |wn\hat{n}_-\hat{a}_- g_1|^s \psi(n_1)\,\mathrm{d}n_1. \tag{A.5}$$

From here we recall from the general proof that $\hat{n}_-$ can be brought through n without issue since they belong to different roots. The factor $\hat{a}$ we can fix in the standard way from Proposition 4.1 to get

$$X_1(\hat{n}_-\hat{a}g_1k) = |\hat{a}|^{w^{-1}(s-\rho)+\rho} \int\limits_{N_1(\mathbb{Q}_p)} |ng_1|^s \psi^{\hat{a}}\psi(n_1)\,\mathrm{d}n_1. \tag{A.6}$$

The integral that remains is now a standard $\mathrm{SL}(2,\mathbb{Q}_p)$ integral which we can do using Proposition 4.5 to get

$$X_1(\hat{n}_-\hat{a}g_1k) = |\hat{a}a_1|^{w^{-1}(s-\rho)+\rho}\psi^{\hat{a}}(n_1)\gamma_p(m_1|\hat{a}a_1|^{\alpha_1})\frac{\xi_p(\langle s-\rho|\alpha_1)}{\xi_p(1+\langle s-\rho|\alpha_1)}(1-|pm|\hat{a}a_1|^{\alpha_1}|_p^{\langle s-\rho|\alpha_1\rangle}). \tag{A.7}$$

Here comes the problem. What we need to do now in order to continue without having to worry about how we did the Iwazawa-decomposition is to restore all factors; we need to bring in the missing factors everywhere. This fails at multiple points.

- In all expressions $|\hat{a}a_1|$ we can bring in n_1 and k without issue, but we would need to bring in a factor of $\hat{n}_-$ on the left. This is impossible to do because it lies in a lower subgroup, where $|\cdot|^s$ is not invariant, $|\hat{a}a_1|$ is simply not in general equal to $|\hat{n}|$. This can be verified when everything except n_- equals 1. (Of course, this is not something that n_2n_3 can actually be, but this should not matter.)

- In $\psi^{\hat{a}}(n_1)$ we need to fix both n_1 and the fact that there is a lone $\hat{a}$ there. Both are likely impossible to fix. In particular, in order to fix n_1 the only real possibility is that this turns out to equal $\psi(w_1 n_2 n_3 w_1^{-1})$ which is not realistic.

We can however see some structures present that were also there in in our previous calculation. For instance, after having carried out the integral over the first root we have something that contains $\psi^{\hat{a}}(n_1)$. This looks an awful lot like it could be something of the form $\exp\left(-2\pi i \left[n_x \frac{\tilde{n}_z}{\tilde{n}_y}\right]_p\right)$ which is a factor we actually had in our integral. However, remember that we had to carry out this integral *first.* Now we have not done so, and that is most likely why this expression we have for $X_1(g)$ is so complicated.

Of course, one could continue integrating from here if one knew the Iwazawa-decomposition of $n_2 n_3$ in terms of the lower Borel subgroup, but then one runs into the same problem that we solved using the parametrisation from the proof of Langlands formula, so one might as well start from there.

A.2 Direct Iwazawa-Decomposition By Ansatz

Another approach that might work is to do like we did in finding the Iwazawa-decomposition of $\mathrm{SL}(2,\mathbb{Q}_p)$. There we multiply our matrix wnw^{-1} by an element in $B(\mathbb{Q}_p)$ on the left such that the result is in $K = \mathrm{SL}(3,\mathbb{Z}_p)$. Then we have that the relevant $B(\mathbb{Q}_p)$ matrix is the inverse of this. We therefore have to find parameters v_1,v_2 and a,b,c such that the following matrix

$$\begin{pmatrix} v_1 & a & c \\ 0 & \frac{v_2}{v_1} & b \\ 0 & 0 & \frac{1}{v_2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix} = \begin{pmatrix} v_1 + ax + cz & a + cy & c \\ bz + \frac{v_2}{v_1}x & by + \frac{v_2}{v_1} & b \\ \frac{z}{v_2} & \frac{y}{v_2} & \frac{1}{v_2} \end{pmatrix} \tag{A.8}$$

have only integer components, which means we have 9 inequalities on the norms of these entries. We can solve the inequality $|a+cy|_p \leq 1$ by simply shifting $a \mapsto a - cy$ we then have to solve the six inequalities

$$|v_1 + ax + cz - cxy|_p \leq 1, \quad |bz + \tfrac{v_2}{v_1}x|_p \leq 1, \quad |by + \tfrac{v_2}{v_1}|_p \leq 1, \tag{A.9}$$

$$|\tfrac{z}{v_2}|_p \leq 1, \quad |\tfrac{y}{v_2}|_p \leq 1, \quad |\tfrac{1}{v_2}|_p \leq 1, \tag{A.10}$$

under the constraints that a,b,c are p-adic integers. We are free to change variables x, y and z, and we know that we will be forced to do that if we want to find nice expressions for v_1 and v_2. Thus let

$$\begin{aligned} x &\mapsto \xi x + x_0, \\ y &\mapsto \eta y + y_0, \\ z &\mapsto \zeta z + z_0. \end{aligned} \tag{A.11}$$

Which gives us

$$|v_1 + a(\xi x + x_0) + c(\zeta z + z_0) - c(\xi x + x_0)(\eta y + y_0)|_p \leq 1, \tag{A.12}$$

$$|b(\zeta z + z_0) + \tfrac{v_2}{v_1}(\xi x + x_0)|_p \leq 1, \quad |b(\eta y + y_0) + \tfrac{v_2}{v_1}|_p \leq 1, \tag{A.13}$$

$$|\tfrac{\zeta z + z_0}{v_2}|_p \leq 1, \quad |\tfrac{\eta y + y_0}{v_2}|_p \leq 1, \quad |\tfrac{1}{v_2}|_p \leq 1. \tag{A.14}$$

If we guess that

$$v_1 = \tilde{x}\tilde{z}, \quad v_2 = \tilde{y}\tilde{z}, \tag{A.15}$$

which basically corresponds to taking

$$b = nh_1(\tilde{x})h_2(\tilde{y})h_3(\tilde{z}), \tag{A.16}$$

which we know from before is the correct answer, we find

$$|\tilde{x}\tilde{z} + a(\xi x + x_0) + c(\zeta z + z_0) - c(\xi x + x_0)(\eta y + y_0)|_p \leq 1 \tag{A.17}$$

$$|b(\zeta z + z_0) + \tfrac{\tilde{y}}{\tilde{x}}(\xi x + x_0)|_p \leq 1, \quad |b(\eta y + y_0) + \tfrac{\tilde{y}}{\tilde{x}}|_p \leq 1, \tag{A.18}$$

$$|\tfrac{\zeta z + z_0}{\tilde{y}\tilde{z}}|_p \leq 1, \quad |\tfrac{\eta y + y_0}{\tilde{y}\tilde{z}}|_p \leq 1, \quad |\tfrac{1}{\tilde{y}\tilde{z}}|_p \leq 1. \tag{A.19}$$

The bottom three equations are now almost trivial, they provide some constraints on the growth of ζ,η,y_0, and z_0 but nothing hard. There are three remaining inequalities

$$|\tilde{x}\tilde{z} + aX + c(Z - XY)|_p \leq 1, \tag{A.20}$$

$$|bZ + \tfrac{\tilde{y}}{\tilde{x}}X|_p \leq 1, \tag{A.21}$$

$$|bY + \tfrac{\tilde{y}}{\tilde{x}}|_p \leq 1, \tag{A.22}$$

were we have introduced $X = \xi x + x_0$ and so forth just to make it easier to read.

Unfortunately these do not become trivial even when we provide a nice guess for v_1 and v_2. We know that it should be possible to put it in this form, because that is what we found when we did the calculation previously. It is also the result we must have in order for Langlands constant term to be correct if we remove ψ (this also demands that the Jacobian for the change of variables gives us a factor of $|\tilde{z}|$ extra). Without some way to show that these have a solution a,b,c for some choice of the parameters ξ,η,ζ,x_0,y_0,z_0 we cannot proceed.

The two bottom inequalities are especially hard to solve, since they demand we find a b that solves both inequalities simultaneously.

These three can be solved for the case $\mathrm{SL}(3, \mathbb{Q}_p)$, of course, but if we want to apply it to larger groups some method of finding these changes of variables X,Y,Z, that make the inequalities have solutions has to be found. It is possible that such methods exist, or can be found, but we are not aware of them.

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