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Fast Alpha Particle Distribution in the Presence of Anomalous Spatial Particle Diffusion

Master Thesis in Plasma Physics

Álvaro García Carrasco

Supervisor Prof. Mietek Lisak
Department of Radio and Space Science
Chalmers University of Technology
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Abstract

The high-energy alpha particle population in an ignited D-T fusion plasma may cause thermonuclear wave instabilities, which could lead to anomalous losses of plasma energy and of energetic alpha particles. On the other hand, the high-energy alpha particles have to provide self-sustained heating of the fusion plasma. One of the possible excitation mechanisms of the wave instabilities is associated with an alpha-particle velocity distribution function that is non-monotonically decreasing. The aim of the present master thesis is to study the essential condition for the occurrence of this type of instability, when the alpha particle distribution can become inverted with a larger phase-space density of high-energy than the lower energy particles. Such situation can arise in the presence of anomalous spatial losses of alpha particles or/and in the presence of external fast plasma heating. By solving the generalized Fokker-Planck equation for alpha particles with spatial diffusion, a criterion for achieving an inversion of the velocity alpha particle distribution function is obtained and analyzed.

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Contents

1

Introduction

The potential for reaching ignition in the tokamak configuration has become obvious over the last years, particularly on the strength of plasma confinement results from present day experiments. One of the main objectives of devices such as the existing experiment JET (Joint European Torus) and the planned project ITER (International Thermonuclear Experimental Reactor) is the study of alpha particle production, confinement and alpha particle heating of D-T plasmas.

The energetic alpha particle population ($E_\alpha = 3.5MeV$) in an ignited D-T plasma of the next-generation tokamak will contribute a considerable part of the total plasma pressure (typically 10-20%). Therefore the alpha particles can be expected to have substantial impact on achieving and maintaining high temperatures. The energetic alpha particle population may cause thermonuclear wave instabilities, which could lead to anomalous losses of plasma energy as well as of high-energy alpha particles. On the other hand these instabilities may possibly lead to improvements of the plasma heating due to the anomalous energy transport from alpha particles to the background ions, cf.[1-5]

The theoretical treatment of the thermonuclear instabilities can be divided into three classes depending on different excitation mechanism. The first class considers instabilities that can be excited by alpha particles with an isotropic velocity distribution function that is not monotonically decreasing [6-19]. The second class are the so-called "cone" instabilities, which are activated by the anisotropy in the alpha particle distribution function [27-29]. Finally, the third class includes drift instabilities caused by spatial inhomogeneity of the alpha particle distribution function, cf.[20-26]. In the present thesis, we will concentrate on the first class of the thermonuclear instabilities.

The basic cause of these instabilities connected with the thermonuclear reaction DT-process is the deviation of the distribution function of alpha particles from thermodynamic equilibrium. An essential condition for the occurrence of these instabilities is that the alpha particle distribution function, f_α, is inverted in the velocity space i.e. $\partial f_\alpha/\partial v > 0$. In this situation, free energy is available to drive instabilities. However, starting from the classical, isotropic Fokker-Planck equation describing slowing down of energetic alphas in a background plasma, it can be shown that f_α asymptotically evolves towards the classical slowing down distribution ($f_\alpha \sim v^{-3}$), which is monotonically decreasing. On the other hand it is quite conceivable that other effects could cause an inversion of f_α and give rise to velocity-space instabilities. Such effects can generally involve anomalous spatial losses of alpha particles as they slow down. It is therefore of large interest to formulate an inversion condition for f_α in the presence of anomalous spatial alpha losses. The effect of alpha losses on the evolution of f_α can be modelled by including an anomalous spatial diffusion term into the Fokker-Planck equation, i.e.

$$\frac{\partial f_\alpha}{\partial t} = \frac{1}{\tau_s}\frac{\partial}{\partial v}[(v^3 + v_c^3)f_\alpha] + \frac{1}{r}\frac{\partial}{\partial r}\left(D_\alpha r \frac{\partial f_\alpha}{\partial r}\right) + \frac{S_0}{4\pi v_\alpha^2}\delta(v - v_\alpha)$$

where the first term on the r.h.s. describes friction of alpha particle on background electrons and ions, respectively, the second term corresponds to the anomalous alpha particle spatial diffusion, and the third term represents the source of alpha particles. Furthermore, τ_s is the slowing down time, $m_\alpha v_\alpha^2/2 = 3.5MeV$, and $S_0(r) = n_D n_T < \sigma v >$. Thus, by solving the above equation an requiring that $\partial f_\alpha/\partial v > 0$, it is possible to obtain a condition for D_α determining inversion.

The aim of the project is, first of all, to study the evolution of the alpha particle distribution function in the presence of anomalous spatial alpha diffusion. Thus, the generalized Fokker-Planck equation with $D_\alpha \neq 0$ for high-energy particles will be solved for the general case and for the specific cases with no diffusion $D_\alpha = 0$ and with $D_\alpha = \text{const}$. The results will be used to determine the inversion condition and analyse its sensitivity to the anomalous alpha losses and the variations of temperature.

This thesis is organised as follows, in Chapter 2, it is derived the Fokker-planck equation, which governs the behaviour of high energy alpha particles subject to Coulomb collisions. The Fokker-Planck equation with spatial diffusion is solved in Chapter 3 for its general form and for some particular cases. Finally, Chapter 4 includes a study of the condition for the inversion of the alpha particle distribution function based on the solution previously found.

2

Derivation of the Basic Fokker-Planck Equation

In this chapter, we derive the basic equation which governs the evolution of the high-energy alpha particle velocity distribution function, when considering Coulomb collisions and the presence of alpha particle spatial diffusion. The basic physics of the considered kinetic model for plasma particles can be summarized as follows.

In a strong ionized plasma we have $\lambda_D^3 >> 1$, where λ_D is the Debye length and n is the plasma density, due to the Coulomb forces acting at long distances. The number of particles in the Debye sphere is so large that a test particle interacts at the same time with many particles. Significant change of the test particle momentum (a "real" collision) is caused by an accumulative effect of many "small" collisions, whereas single particle collisions are less important. To see this, let us compare an average time t_1 for a single $\pi/2$-collision with the time t_m for weak multiple collisions. The time t_1 is determined by

$$t_1 = \frac{1}{nv\sigma_t} = \frac{1}{nv\pi b_0^2} \tag{2.1}$$

where σ_t is the cross-section of a single collison, b_0 is the impact parameter and v is the particle velocity. For the "small" collisions the change in the deflection angle ($\theta < \pi/2$) is the change of the particle velocity due to the collision, $|\triangle \mathbf{v}| \approx v\theta$ (for small θ). Then the collision cross-section is $\sigma(\theta) \approx 4b_0/\theta^4$ (Rutherford formula for $\sin\theta/2 \approx \theta/2$). The average of $|\Delta \mathbf{v}|^2$ in time units is determined then by

$$\frac{d|\triangle \mathbf{v}|^2}{dt} = nv \int |\Delta \mathbf{v}|^2 \sigma(\theta)\, d\Omega = 8\pi b_0^2 n v^3 \int_{\theta_{min}}^{\pi/2} \frac{d\theta}{\theta} \tag{2.2}$$

where we used $d\Omega = 2\pi b_0 \sin\theta d\theta \approx 2\pi b_0 \theta d\theta$ and the integration is taken from θ_{min} instead of $\theta = 0$ in order to avoid divergence. The divergence arises due to neglecting the screening of the Coulomb potential. Since the screening is zero at a distance larger that λ_D, the Rutherford formula gives $b = b_0 \cot \frac{\theta}{2}$, at $\theta_{min} = 2b_0/\lambda_D$. For weak collisions, when $|\Delta\mathbf{v}|^2 \sim v$, Eq.(2.2) yields

$$t_m \simeq \frac{1}{8\pi b_0^2 n v \ln(\pi\lambda_D/4b_0)} \tag{2.3}$$

which together with Eq.(2.1) gives

$$\frac{t_1}{t_m} \simeq 8\ln(\lambda/b_0) = 8\Lambda \tag{2.4}$$

where Λ is the Coulomb logarithm. Since Λ is of the order of 10 to 20, we have $t_1/t_m >> 1$, i.e. the cumulative effect of small "small" collisions is much more important than the effect of a single close collision.

This conclusion is helpful in the derivation of a kinetic collision model. We assume that the term $(\partial f/\partial t)_c$ in the Boltzmann equation describes a change in the particle velocity due to other particles in the Debye sphere. Such velocity changes are small and take place continuously for a large number of particles. Thus, we have a so called "random walk" process. Note that the particle distribution function $f(\mathbf{v},\mathbf{r},t)$ is the probability function to find particles at time t with velocity $\mathbf{v}$ and at the position $\mathbf{r}$. If $W(\mathbf{v},\Delta\mathbf{v})$ represents the transition probability that a particle with the velocity $\mathbf{v}$ will experiences a velocity change $\Delta\mathbf{v}$ during the interval time Δt, then

$$f(\mathbf{v},t) = \int f(\mathbf{v}-\Delta\mathbf{v},t-\Delta t)W(\mathbf{v}-\Delta\mathbf{v},\Delta\mathbf{v})\,d(\Delta\mathbf{v}) \tag{2.5}$$

For small $\Delta\mathbf{v}$ and Δt we can Taylor expand the expression under the integral

$$f(\mathbf{v}-\Delta\mathbf{v},t-\Delta t)W(\mathbf{v}-\Delta\mathbf{v},\Delta\mathbf{v}) \simeq f(\mathbf{v},t)W(\mathbf{v},\Delta\mathbf{v}) - \Delta t\frac{\partial f}{\partial t}W(\mathbf{v},\Delta\mathbf{v}) - \left(\Delta\mathbf{v}\frac{\partial}{\partial\mathbf{v}}\right)(fW) + \frac{1}{2}\left(\Delta\mathbf{v}\frac{\partial}{\partial\mathbf{v}}\right)^2(fW) + ... \tag{2.6}$$

Then Eq.(2.5) becomes

$$f(\mathbf{v},t) \simeq f(\mathbf{v},t)\int W(\mathbf{v},\Delta\mathbf{v})\,d(\Delta\mathbf{v}) - \Delta t\frac{\partial f}{\partial t}\int W(\mathbf{v},\Delta\mathbf{v})\,d(\Delta\mathbf{v}) - \frac{\partial}{\partial\mathbf{v}}\left[f\int\Delta\mathbf{v}W(\mathbf{v},\Delta\mathbf{v})\,d(\Delta\mathbf{v})\right] + \frac{1}{2}\int\left(\Delta\mathbf{v}\frac{\partial}{\partial\mathbf{v}}\right)^2 fW(\mathbf{v},\Delta\mathbf{v})\,d(\Delta\mathbf{v}) + ... \tag{2.7}$$

Let us now define

$$<\Delta v_i> \equiv \frac{1}{\Delta t}\int W(\mathbf{v},\Delta\mathbf{v})\Delta v_i\,d(\Delta\mathbf{v}) \equiv a_i \tag{2.8}$$

and

$$< \Delta v_i \Delta v_j > \equiv \frac{1}{\Delta t} \int W(\mathbf{v}, \Delta \mathbf{v}) \Delta v_i \Delta v_j \, d(\Delta \mathbf{v}) \equiv b_{ij} \tag{2.9}$$

Since W is normalized according to

$$\int W(\mathbf{v}, \Delta \mathbf{v}) \, d(\Delta \mathbf{v}) = 1 \tag{2.10}$$

we obtain

$$(\frac{\partial f}{\partial t})_c = -\frac{\partial}{\partial v_i}(a_i f) + \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_j}(b_{ij} f) \tag{2.11}$$

which is the Fokker-Planck collision model. In the literature Eq.(2.11) is often rewritten by introducing

$$A_i = a_i - \frac{1}{2} \frac{\partial b_{ij}}{\partial v_j} \tag{2.12}$$

$$D_{ij} = \frac{1}{2} b_{ij} \tag{2.13}$$

where A_i is the friction coefficient and D_{ij} is the diffusion coefficient in velocity space. The determination of these coeficients in order to obtain the Fokker-Planck operator describing the Coulumb collisions follows closely Ref. [34].

Our present discussion will concern the Fokker-Planck collision model for high-energy alpha particles produced by the thermonuclear D-T reactions. We assume that the background plasma consists of electrons and 50-50% D-T ions and that their distribution functions are isotropic and Maxwellian. The plasma is assumed to be free from impurities and neutrals. The macroscopic electric field giving rise to a plasma potential field, as well as the direct alpha particle drift losses, are here neglected. In these conditions, the behaviour of the distribution function of the alpha particles can be described by the equation, cf. [33].

$$\frac{\partial f_\alpha}{\partial t} = -\frac{1}{\tau_s} \frac{1}{v^2} \frac{\partial}{\partial v} \left[v^2 \left(a f_\alpha - d_\parallel \frac{\partial f_\alpha}{\partial v} \right) \right] + \frac{1}{r} \frac{\partial}{\partial r} \left(r D_\alpha \frac{\partial f_\alpha}{\partial r} \right) + \frac{S_0}{4\pi v_\alpha^2} \delta(v - v_\alpha) \tag{2.14}$$

where

$$a = -\sum_* \frac{4\pi (e_\alpha e^*)^2 n^* L}{m_\alpha m^* v^2} \Phi_1 \left(\frac{v}{v^*} \right) \tag{2.15}$$

and

$$d_\parallel = \sum_* \frac{2\pi (e_\alpha e^*)^2 n^* L}{m_\alpha^2 v} \frac{\Phi_1(\frac{v}{v^*})}{(\frac{v}{v^*})^2} \tag{2.16}$$

here

$$v^* = \left(\frac{2T^*}{m^*}\right)^2 \tag{2.17}$$

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2}\, d\xi \tag{2.18}$$

$$\Phi_1(x) = \Phi(x) - x\frac{d\Phi}{dx} \tag{2.19}$$

The coefficient a is associated to the slowing down (or friction) process, whereas the coefficient $d_{\|}$ is related to the diffusion in the velocity space. Both mechanisms are produced due to the collisions between the alpha particles and the background plasma. The source term in Eq.(2.14) represents the density of alpha particles created per second through thermonuclear reactions. Since each thermonuclear reaction produces alpha particles with almost mono-energetic and isotropic velocity distribution, the source term is approximated by being proportional to the delta function at $v = v_\alpha$. Note, that $S_0 = n_i/\tau_f$, where n_i is the density of background ions and $\tau_f = 4/(ni < \sigma v >)$ is the characteristic time of alpha particle generation as a result of the thermonuclear reactions.

Since the object of the study is the high-energy alpha particle population, some simplifications can be applied in order to achieve simpler relations.

Because $v/v_e << 1$, we have

$$\Phi_1(v/v_e) \simeq \frac{4}{3\sqrt{\pi}}\left(\frac{v}{v_e}\right)^3 \tag{2.20}$$

$$\Phi(v/v_e) - \frac{\Phi_1(v/v_e)}{2(v/v_e)^2} \simeq \frac{4}{3\sqrt{\pi}}\frac{v}{v_e} \tag{2.21}$$

For ions instead, we have $v/v_i >> 1$, which gives

$$\Phi_1(v/v_i) \simeq 1 \tag{2.22}$$

$$\Phi(v/v_i) - \frac{\Phi_1(v/v_i)}{2(v/v_i)^2} \simeq 1 \tag{2.23}$$

Thus the friction coefficient a can be written as

$$a = a_e + a_i \tag{2.24}$$

where

$$a_e = -\frac{v}{\tau_s} \tag{2.25}$$

here τ_s is the Spitzer slowing down time given by

$$\tau_s = \frac{3m_\alpha m_e v_e^3}{16\sqrt{\pi} Z_\alpha^2 e^4 n_e L} \tag{2.26}$$

and

$$a_i = -\frac{v_c^3}{\tau_s v^2} \tag{2.27}$$

with v_c being the critical velocity at which the background electrons and ions contribute equally to the slowing down of the alpha particles. That is

$$v_c^3 = \frac{3\sqrt{\pi}}{4} v_e^3 \frac{m_e}{n_e} \sum_i \frac{Z_i^2 n_i}{m_i} \equiv \frac{3\sqrt{\pi}}{4} v_e^3 \frac{m_e}{n_e} [Z] \tag{2.28}$$

The parallel diffusion coefficients are given by

$$d_{\|e} = \frac{1}{\tau_s} \frac{m_e v_e^2}{2m\alpha} \tag{2.29}$$

$$d_{\|i} = \frac{1}{\tau_s} \frac{m_i v_i^2 v_c^3}{2m\alpha v^3} \tag{2.30}$$

Thus, we conclude that

$$\left|\frac{a_e}{a_i}\right| \sim \left|\frac{v^3}{v_c^3}\right| \geq 1 \tag{2.31}$$

$$\left|\frac{d_{\|e}}{d_{\|i}}\right| \sim \left|\frac{v^3}{v_c^3}\right| \geq 1 \tag{2.32}$$

and

$$\left|\frac{d_{\|e}/v}{a_e}\right| \sim \left|\frac{T_e}{E_\alpha}\right| << 1 \tag{2.33}$$

where $E_\alpha = 3.5MeV$. This means that for the velocity range $v_c \leq v \leq v_\alpha$, where $v_\alpha = \sqrt{2E_\alpha/m_\alpha}$, the difussion term can be neglected and the basic equation for high-energy alpha particles becomes

$$\frac{\partial f_\alpha}{\partial t} = \frac{1}{\tau_s}\frac{1}{v^2}\frac{\partial}{\partial v}[(v^3+v_c^3)f_\alpha] + \frac{1}{r}\frac{\partial}{\partial r}\left(rD_\alpha \frac{\partial f_\alpha}{\partial r}\right) + \frac{S_0}{4\pi v_\alpha^2}\delta(v-v_\alpha) \tag{2.34}$$

3

General Solution for the Fokker-Planck Equation

This chapter shows the steps followed toward the achievement of the general solution for the Fokker-Planck equation. At the end of the chapter, besides the general case, two simplified cases are also presented in order to have a better insight into the behave of the distribution function.

Let us consider the Fokker-Planck equation in the presence of spatial diffusion

$$\frac{\partial f_\alpha}{\partial t} = \frac{1}{\tau_s(t)}\frac{1}{v^2}\frac{\partial}{\partial v}[(v^3+v_c^3)f_\alpha] + \frac{1}{r}\frac{\partial}{\partial r}\left(rD_\alpha(r,v)\frac{\partial f_\alpha}{\partial r}\right) + \frac{S_0(r,t)}{4\pi v_\alpha^2}\delta(v-v_\alpha) \tag{3.1}$$

where we assume that $D_\alpha(r,v) = d(v)r^k$. Introducing

$$F = f_\alpha(v^3+v_c^3) \tag{3.2}$$

Eq.(3.1) becomes

$$\frac{\partial F}{\partial t} = \frac{v^3+v_c^3}{\tau_s(t)v^2}\frac{\partial F}{\partial v} + \frac{d(v)}{r}\frac{\partial}{\partial r}\left(r^{k+1}\frac{\partial F}{\partial r}\right) + \frac{S_0(r,t)(v_\alpha^3+v_c^3)}{4\pi v_\alpha^2}\delta(v-v_\alpha) \tag{3.3}$$

From the characteristics of Eq.(3.3), we have

$$\frac{v^2 dv}{v^3+v_c^3} = -\frac{dt}{\tau_s(t)} \tag{3.4}$$

which implies the choice of the variable transformation

$$\begin{cases} z = \left(\frac{v^3+v_c^3}{v_\alpha^3+v_c^3}\right)\exp\left[3\int_0^t \frac{dt'}{\tau_s(t)}\right] \\ \tau = t \end{cases} \tag{3.5}$$

Then, the derivatives in the new variables are obtained applying the chain rule

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau}\frac{\partial \tau}{\partial t} + \frac{\partial}{\partial z}\frac{\partial z}{\partial t} = \frac{\partial}{\partial \tau} + \frac{3}{\tau_s(t)} z \frac{\partial}{\partial z} \tag{3.6}$$

and

$$\frac{\partial}{\partial v} = \frac{\partial}{\partial z}\frac{\partial z}{\partial v} = \frac{\partial}{\partial z}\frac{3v^2}{v_\alpha^3 + v_c^3}\exp\left[3\int_0^t \frac{dt'}{\tau_s(t)}\right] \tag{3.7}$$

which gives

$$\frac{\partial F}{\partial \tau} = \frac{d(v)}{r}\frac{\partial}{\partial r}\left(r^{k+1}\frac{\partial F}{\partial r}\right) + \frac{S_0(r,t)(v_\alpha^3 + v_c^3)}{4\pi v_\alpha^2}\delta(v - v_\alpha) \tag{3.8}$$

where

$$v = \left[z(v_\alpha^3 + v_c^3)\exp\left[-3\int_0^\tau \frac{dt}{\tau_s(t)}\right] - v_c^3\right]^{\frac{1}{3}} \tag{3.9}$$

Eq.(3.8) can be written as a initial value problem noticing that at

$$v = v_\alpha \rightarrow z^* = \exp\left[3\int_0^{\tau^*} \frac{dt}{\tau_s(t)}\right] \tag{3.10}$$

i.e.

$$\exp\left[3\left(\int_\tau^{\tau^*} \frac{dt}{\tau_s(t)}\right)\right] = \frac{v^3 + v_c^3}{v_\alpha^3 + v_c^3} \tag{3.11}$$

so that at

$$\tau^* = \tau \rightarrow v = v_\alpha \tag{3.12}$$

and since

$$\left|\frac{\partial v}{\partial \tau}\right|_{v=v_\alpha} = \left|\frac{z}{3v^2}(v_\alpha^3 + v_c^3)\frac{-3}{\tau_s(\tau)}\exp\left[-3\int_0^\tau \frac{dt}{\tau_s(t)}\right]\right| = \\ = \left|\frac{1}{\tau_s(\tau^*)v_\alpha^2} z^*(v_\alpha^3 + v_c^3)\exp\left[-3\int_0^{\tau^*} \frac{dt}{\tau_s(t)}\right]\right|_{v=v_\alpha} = \frac{v_\alpha^3 + v_c^3}{\tau_s(\tau^*)v_\alpha^2} \tag{3.13}$$

We have

$$\frac{\partial F}{\partial \tau} = \frac{d(z,\tau)}{r}\frac{\partial}{\partial r}(r^{k+1}\frac{\partial F}{\partial r}) \tag{3.14}$$

$$F(r,\tau^*) = \frac{S_0(r,\tau^*)(v_\alpha^3 + v_c^3)}{4\pi v_\alpha^2}\left|\frac{\partial \tau}{\partial v}\right|_{v=v_\alpha} = \frac{S_0(r,\tau^*)\tau_s(\tau^*)}{4\pi} \tag{3.15}$$

In Eq.(3.14) we separate the variables by introducing

$$F(r,\tau) = R(r)T(\tau) \tag{3.16}$$

Then

$$\frac{1}{d(z,\tau)T(\tau)}\frac{dT}{d\tau} = \frac{1}{R(r)r}\frac{d}{dr}\left(r^{k+1}\frac{\partial R}{\partial r}\right) = -\lambda \tag{3.17}$$

i.e.

$$\frac{dT}{d\tau} + \lambda d(z,\tau)T = 0 \tag{3.18}$$

and

$$\frac{1}{r}\frac{d}{dr}\left(r^{k+1}\frac{dR}{dr}\right) + \lambda R = 0 \tag{3.19}$$

The solution of Eq.(3.18) is

$$T(\tau) = C_1 \exp\left[-\lambda \int_{\tau^*}^{\tau} d(z,\tau')\, d\tau'\right] \tag{3.20}$$

while Eq.(3.19) takes the form

$$r^k \frac{d^2R}{dr^2} + (k+1)r^{k-1}\frac{dR}{dr} + \lambda R = 0 \tag{3.21}$$

Let us consider Eq. (3.21) by substituting

$$R = r^{\alpha} y(x); \quad x = ar^{\beta} \tag{3.22}$$

This implies

$$\frac{dR}{dr} = \alpha r^{\alpha-1} y + a\beta r^{\alpha+\beta-1}\frac{dy}{dx} \tag{3.23}$$

$$\frac{d^2R}{dr^2} = \alpha(\alpha-1)r^{\alpha-2}y + \alpha a\beta r^{\alpha+\beta-2}\frac{dy}{dx} + \alpha\beta(\alpha+\beta-1)r^{\alpha+\beta-2}\frac{dy}{dx} + \\ + a^2\beta^2 r^{\alpha+2\beta-2}\frac{dy^2}{dx^2} \tag{3.24}$$

Thus, Eq.(3.21) becomes

$$a^2\beta^2 r^{2\beta}\frac{d^2y}{dx^2} + a\beta(2\alpha+\beta+k)r^{\beta}\frac{dy}{dx} + [\lambda r^{2-k} + \alpha(\alpha+k)]y = 0 \tag{3.25}$$

Since $x = ar^{\beta}$, it is obtained

$$\beta^2 x^2 \frac{d^2y}{dx^2} + \beta(2\alpha+\beta+k)x\frac{dy}{dx} + \left[\lambda\left(\frac{x}{a}\right)^{\frac{2-k}{\beta}} + \alpha(\alpha+k)\right]y = 0 \tag{3.26}$$

Imposing

$$\begin{cases} \frac{2\alpha+\beta+k}{\beta} = 1 \\ \frac{2-k}{\beta} = 2 \\ \frac{\lambda}{\beta^2} = a^{\frac{2-k}{\beta}} \end{cases} \tag{3.27}$$

gives

$$\begin{cases} \alpha = -k/2 \\ \beta = \frac{2-k}{2} \\ a = \frac{\sqrt{\lambda}}{|\beta|} \end{cases} \tag{3.28}$$

Consequently, Eq.(3.18) becomes a Bessel equation of order ν with

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \mu^2)y = 0 \tag{3.29}$$

$$R = r^{-k/2}y(x); \quad x = \frac{2\sqrt{\lambda}}{2-k}r^{\frac{2-k}{2}} \tag{3.30}$$

$$\nu = \frac{k}{2-k} \tag{3.31}$$

where $k \neq 2$ to avoid singularities in the order of the Bessel equation. The solution of Eq.(3.29) is

$$y(x) = C_2 J_\nu(x) = C_2 J_{\frac{k}{2-k}}(\gamma_j(r)) \tag{3.32}$$

where

$$\gamma_j(r) = \frac{2}{2-k}\sqrt{\lambda_j}r^{\frac{2-k}{2}} \tag{3.33}$$

the Bessel function of second kind has not been considered for being singular at $r = 0$. The total solution of Eq.(3.1) should obey, besides being bounded $\forall r \in [0,r_0]$, the boundary conditions

$$f_\alpha = 0 \quad \text{if} \quad \begin{cases} v > v_\alpha \\ v < v_f(t) \\ r = r_0 \end{cases} \tag{3.34}$$

where

$$v_f(t) = \sqrt[3]{(v_\alpha^3 + v_c^3)\exp\left[-3\int_0^t \frac{dt'}{\tau_s(t')}\right] - v_c^3} \tag{3.35}$$

and represents the velocity front of the velocity distribution function evolving toward thermal energies. Then, the solution should be proportional to

$$H(v_\alpha - v) - H(v_f(t) - v) \tag{3.36}$$

where $H(x)$ is a step function defined by

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \tag{3.37}$$

On the other hand, the boundary conditions impose

$$J_{\frac{k}{2-k}}(\gamma_j(r_0)) = 0 \tag{3.38}$$

that is

$$\lambda_j = \mu_{j\nu}^2 \frac{(2-k)^2}{4} r_0^{k-2} \tag{3.39}$$

where $\mu_{j\nu}$ is the jth root of the bessel function of $\nu = \frac{k}{k-2}$ order, $J_{\frac{k}{k-2}}(\mu_{j\nu}) = 0$. The total solution can be written as

$$f_\alpha(v,r,t) = \frac{H(v_\alpha - v) - H(v_f(t) - v)}{v^3 + v_c^3} \sum_{j=1}^{\infty} C_j r^{-k/2} J_{\frac{k}{2-k}}(\gamma_j(r)) \exp\left[-\lambda_j \int_{\tau^*}^{\tau} d(z,\tau')\, d\tau'\right] \tag{3.40}$$

where

$$\int_{\tau^*}^{\tau} d(z,\tau')\, d\tau' = -\tau_s \int_{v_\alpha}^{v} d(v') \frac{v'^2}{v'^3 + v_c^3}\, dv' \tag{3.41}$$

i.e.

$$f_\alpha(v,r,t) = \frac{H(v_\alpha - v) - H(v_f(t) - v)}{v^3 + v_c^3} \sum_{j=1}^{\infty} C_j r^{-k/2} J_{\frac{k}{2-k}}(\gamma_j(r)) \exp\left[\lambda_j \tau_s(t) \int_{v_\alpha}^{v} d(v') \frac{v'^2}{v'^3 + v_c^3}\, dv'\right] \tag{3.42}$$

In order to satisfy the initial condition (3.15) we take the solution for $F(z,\tau)$ at $\tau = \tau^*$ and using the orthogonality properties of the Bessel function, obtain the relation

$$\begin{aligned}\sum_{j=1}^{\infty} \int_0^{r_0} r^{1-k/2} J_{\frac{k}{2-k}}(\gamma_j(r)) \frac{S_0(r,\tau^*)\tau_s(\tau^*)}{4\pi}\, dr &= \\ = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \int_0^{r_0} r^{1-k} J_{\frac{k}{2-k}}(\gamma_j(r)) J_{\frac{k}{2-k}}(\gamma_i(r)) C_j \delta_{ij}\, dr &= \\ = \sum_{j=1}^{\infty} C_j \int_0^{r_0} r^{1-k} J^2_{\frac{k}{2-k}}(\gamma_j(r))\, dr \end{aligned} \tag{3.43}$$

which gives

$$C_j = \frac{\int_0^{r_0} \frac{S_0(r,\tau^*)\tau_s(\tau^*)}{4\pi} r^{\frac{2-k}{2}} J_{\frac{k}{2-k}}(\gamma_j(r))\, dr}{\int_0^{r_0} r^{1-k} J^2_{\frac{k}{2-k}}(\gamma_j(r))\, dr} \tag{3.44}$$

Thus, Eq. (3.42) becomes

$$f_\alpha(v,r,t) = [H(v_\alpha - v) - H(v_f(t) - v)]\,\frac{\tau_s(\tau^*)}{4\pi(v^3+v_c^3)}\sum_{j=1}^{\infty} J_{\frac{k}{2-k}}(\gamma_j(r))r^{-k/2}$$
$$\exp\left[\lambda_j\tau_s(t)\int_{v_\alpha}^{v} d(v')\frac{v'^2}{v'^3+v_c^3}\,dv'\right]\frac{\int_0^{r_0} S_0(r,\tau^*)r^{\frac{2-k}{2}}J_{\frac{k}{2-k}}(\gamma_j(r))\,dr}{\int_0^{r_0} r^{1-k}J^2_{\frac{k}{2-k}}(\gamma_j(r))\,dr} \quad (3.45)$$

It should be taken into account that the distribution function given by Eq.(3.45) only represents the high-energy part of the alpha particle distribution. To this solution should be added a low energy thermal Maxwellian distribution representing the accumulating alpha particle ashes. Furthermore, this solution is only valid for $k < 2$. In the case $k > 2$, the function $J_{\frac{k}{2-k}}(\gamma_j(r))r^{-k/2}$ is singular at $r = 0$, i.e. the solution is unbounded at r=0.

In order to have a better insight into the general solution (3.45), some specific simplified cases can be considered.

3.1 Case $D_\alpha = 0$

In absence of spatial diffusion i.e. when $D_\alpha = 0$, it can be found from (3.45) that

$$f_\alpha(v,r,t) = [H(v_\alpha - v) - H(v_f(t) - v)]\,\frac{\tau_s(\tau^*)S_0(r,\tau^*)}{4\pi(v^3+v_c^3)} \quad (3.46)$$

where τ^* is determined by

$$\int_{\tau}^{\tau^*}\frac{dt}{\tau_s(t)} = \frac{1}{3}\ln\left(\frac{v^3+v_c^3}{v_\alpha^3+v_c^3}\right) \quad (3.47)$$

Equations (3.46) and (3.47) represent the well-known slowing-down solution [33].

3.2 Case $D_\alpha = const.$

In this case, the solution (3.45) simplifies into

$$f_\alpha(v,r,t) = [H(v_\alpha - v) - H(v_f(t) - v)]\,\frac{\tau_s(\tau^*)}{4\pi(v^3+v_c^3)}$$
$$\sum_{j=1}^{\infty}\frac{J_0(r\sqrt{\lambda_j})\int_0^{r_0} S_0(r,\tau^*)J_0(r\sqrt{\lambda_j})r\,dr}{\int_0^{r_0} J_0^2(r\sqrt{\lambda_j})r\,dr}\left(\frac{v^3+v_c^3}{v_\alpha^3+v_c^3}\right)^{D_\alpha\tau_s\lambda_j/3} \quad (3.48)$$

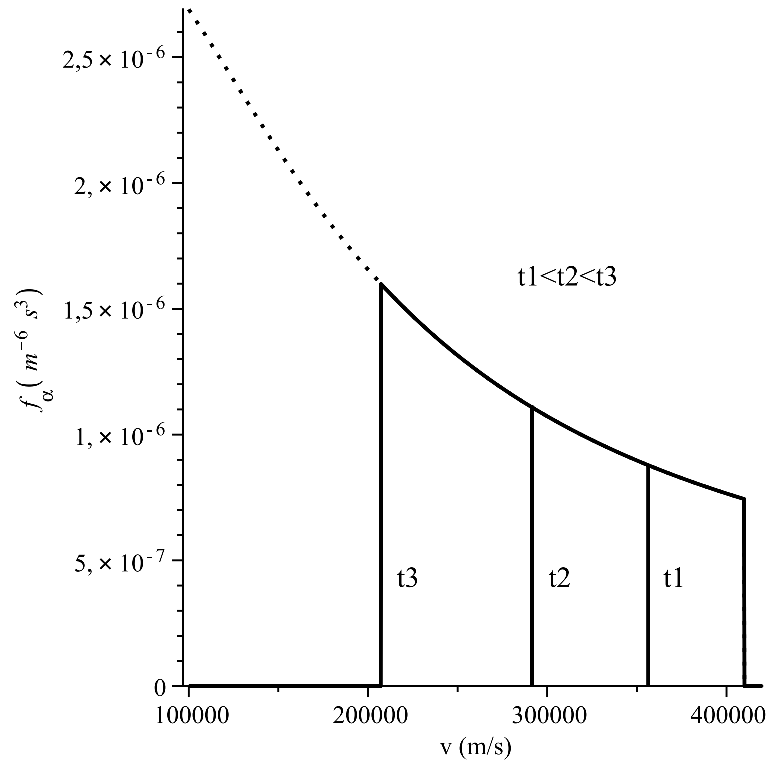

Figure 3.1: Qualitative graphic of the evolution in time of the alpha particle distribution function. All the particles are born with the same velocity, but the friction with the background plasma slows down their energy.

where λ_j are determined by $J_0(r_0\sqrt{\lambda_j}) = 0$. The time-dependancy of the distribution function (3.48) consists of a velocity front at

$$v_f(t) = \sqrt[3]{(v_\alpha^3 + v_c^3)\exp\left[-3\int_0^t \frac{dt'}{\tau_s(t)}\right] - v_c^3} \tag{3.49}$$

Sliding along the steady-state solution towards thermal energies, cf. Figure 3.1 . Assuming for example, $\tau_s = 370\,ms$ and $v_\alpha/v_c = 3.2$ gives that $v = 0$ after $t = 430\,ms$.

4

Analysis of the Inversion Condition

As it has been pointed out in the introduction, the occurence of the velocity-space thermonuclear instabilities is that the alpha particles velocity distribution function is inverted in some part of the velocity space, i.e. $\partial f_\alpha/\partial v > 0$. In this situation free energy is available to drive instabilities. Starting from the Fokker-Planck equation describing the slowing-down of energetic alpha particles in the absence of spatial diffusion, the criterion for determining whether the inversion of the alpha particle function occurs has been derived in Refs.[30,31] and independently [32]. In a simplified form this criterion may be written as

$$\frac{d\ln S_0}{dT_i}\frac{dTi}{dt} + \frac{d}{dt}\ln(nT_e^{3/2}) > \frac{3}{\tau_s} \tag{4.1}$$

which can directly be derived from Eq.(3.45). In physical terms, Eq.(4.1) states that if the source S_0, (or the electron temperature T_e) is increased too rapidly, inversion occurs. Since changes in S_0 are often predominantly due to changes in the ion temperature T_i, Eq.(4.1) can be used to obtain a limit on the rate at which the temperature must be increased to become inverted. It is convenient to demonstrate the above condition by assuming $T_e = T_i = T$ and introduce the critical heating rate, $\dot{T}_{cr}$, determined by $\partial f_\alpha/\partial v = 0$. The dependence of $\dot{T}_{cr}$ on T for differnt values of the parameter $\eta = (\dot{n}/n)/(\dot{T}/T)$ is shown in Figure 4.1 taking $n = 10^{14}cm^{-3}$ and expecting $\dot{T} < 10KeV/s$ in the case of plasma heating by neutral injection (NBI) or RF waves, it can be concluded from Figure 4.1 that the planned heating of large tokamaks will not generated inverted alpha particle distribution functions (and hence cannot drive thermonuclear velocity space instabilities).

Let us now examine the possibility of the occurance of such instabilities in the presence of spatial diffusion losses of alpha particles. In order to determine the inversion condition

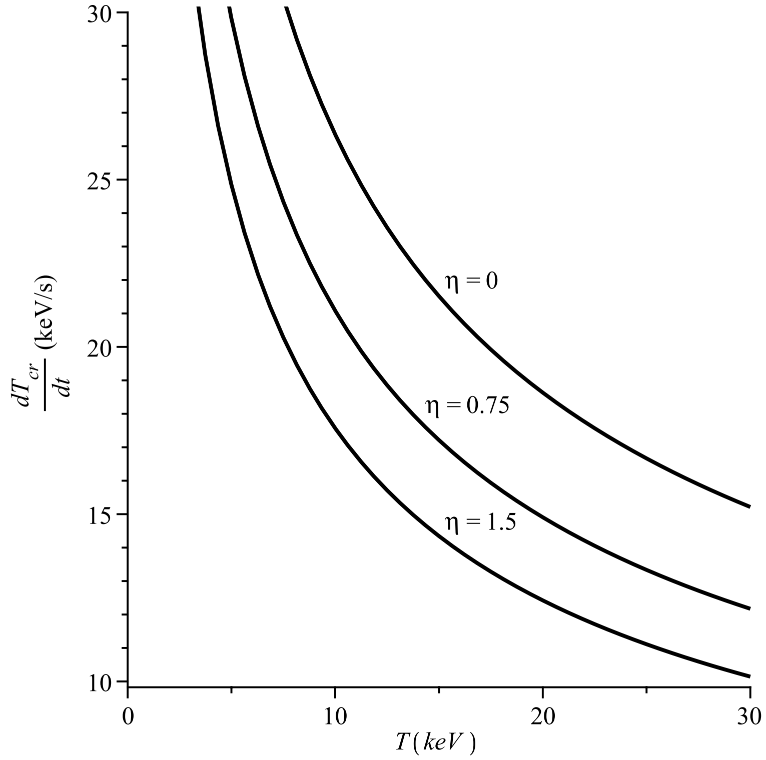

Figure 4.1: Critical plasma heating as a function of temperature

for the distribution function, f_α, we use the obtained solution (3.45), to differentiate f_α with respect to v and obtain

$$\frac{\partial f_\alpha}{\partial v} = \frac{1}{4\pi r^{k/2}} \sum_{j=1}^{\infty} J_{\frac{k}{2-k}}(\gamma_j(r)) \exp\left[\lambda_j \tau_s(t) \int_{v_\alpha}^{v} d(v') \frac{v'^2}{v'^3 + v_c^3}\, dv'\right]$$
$$\left[\frac{1}{\tau_s}\frac{\partial \tau_s}{\partial \tau^*} + \lambda_j d(v) + \lambda_j \frac{\partial \tau_s}{\partial \tau^*} \int_{v_\alpha}^{v} d(v') \frac{v'^2}{v'^3 + v_c^3}\, dv' + \frac{\int_0^{r_0} \frac{\partial S_0}{\partial \tau^*} r^{\frac{2-k}{2}} J_{\frac{k}{2-k}}(\gamma_j(r))\, dr}{\int_0^{r_0} S_0(r,\tau^*) r^{1-k} J^2_{\frac{k}{2-k}}(\gamma_j(r))\, dr}\right]$$
$$\frac{\int_0^{r_0} S_0(r,\tau^*) r^{\frac{2-k}{2}} J_{\frac{k}{2-k}}(\gamma_j(r))\, dr}{\int_0^{r_0} r^{1-k} J^2_{\frac{k}{2-k}}(\gamma_j(r))\, dr} \frac{\tau_s(\tau^*) v^2}{(v^3 + v_c^3)^2} \quad (4.2)$$

The inversion condition $\frac{\partial f_\alpha}{\partial v} > 0$ is fulfilled when

$$\frac{\partial}{\partial t} \ln\left[\int_0^{r_0} S_0(r,\tau^*) r^{1-k} J^2_{\frac{k}{2-k}}(\gamma_1(r))\, dr\right] + \frac{\partial}{\partial t} \ln \tau_s > \frac{3}{\tau_s} -$$
$$- \lambda_1 d(v) \left[1 - \frac{\partial \tau_s}{\partial t} \frac{1}{d(v)} \int_{v}^{v_\alpha} d(v') \frac{v'^2}{v'^3 + v_c^3}\, dv'\right] \quad (4.3)$$

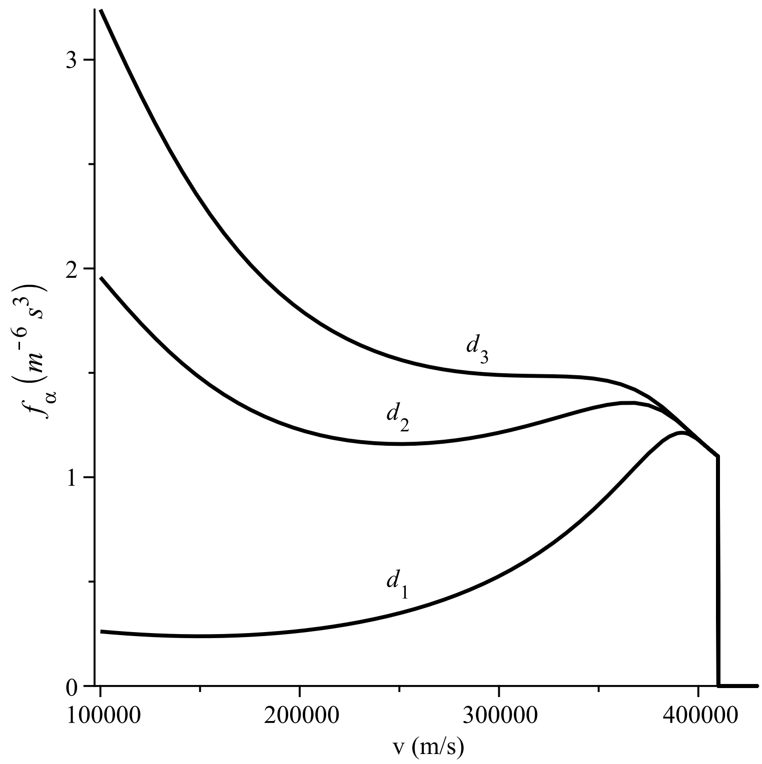

Figure 4.2: Representation of the velocity distribution function for ITER-like parameters ($\tau_s = 0.37s, r_0 = 2.2m, E_\alpha = 3.5MeV, v_\alpha/v_c = 3.2, r = 0.1m, S_0 = 2.65\,10^{18}s^{-1}, k = 0$). Considering that $d_3 < d_2 < d_1$ note that the inversion occurs for higher velocities as we decrease the value of d

The above equation represents the generalized condition for the instability due to the time varying particle distribution function. In the equilibrium, when S_0 and τ_s are both constant in time the condition (4.3) simply reduces to

$$\lambda_1 d(v) > \frac{3}{\tau_s} \tag{4.4}$$

where $\lambda_1 = \min \lambda_j$. Thus, the steady state alpha particle distribution function will be inverted in velocity space due to spatial diffusion if the criterion is satisfied. Of course, any finite heating rate will only increase the degree of inversion. It also follows from Eq.(4.4) that in case of a velocity dependance diffusion coeficient, i.e. $d(v) = d_0 v^p$, the inversion occurs only for

$$v > \sqrt[p]{\frac{3}{\lambda_m d_0 \tau_s}} \tag{4.5}$$

and the distribution function looks like in Figure 4.2 or Figure 4.3 depending on which parameter we are changing. The analysis shows that it is necessary to know explicitly

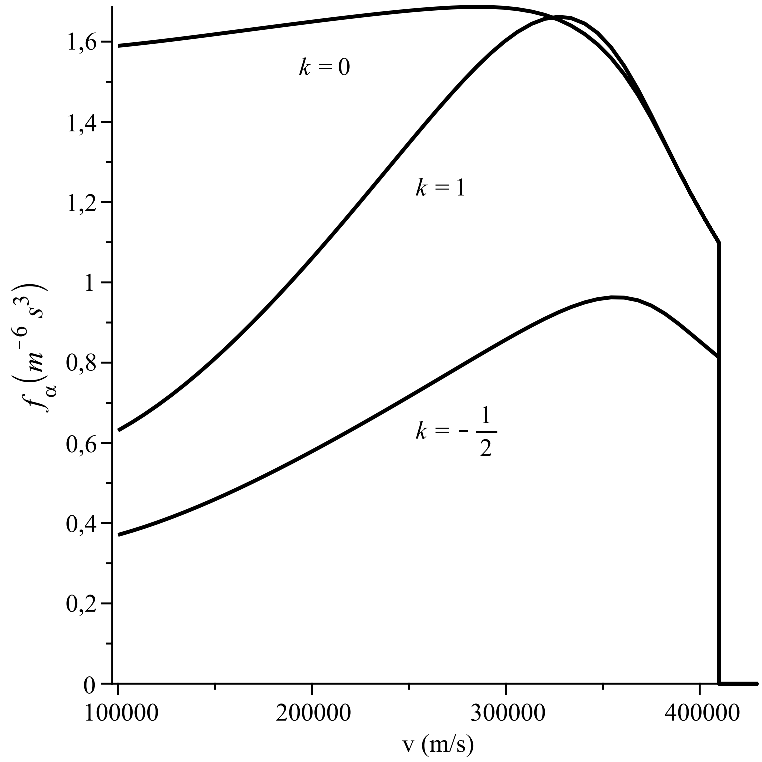

Figure 4.3: Representation of the velocity distribution function for ITER-like parameters ($\tau_s = 0.37s, r_0 = 2.2m, E_\alpha = 3.5MeV, v_\alpha/v_c = 3.2, r = 0.1m, S_0 = 2.65\,10^{18}s^{-1}, d_0 = 7$) for different values of k

the diffusion coefficient in order to determine the inversion conditions. This rely on the knowledge of the physical mechanism behind the considered alpha particle diffusion process. For example, in the case of a constant diffusion coefficient ($k = 0, p = 0$), we obtain $\lambda_m = 5.78/r_0^2$ and the condition for inversion becomes

$$D_\alpha > 0.52\frac{r_0^2}{\tau_s} \tag{4.6}$$

Assuming ITER-like parameters: $r_0 = 2.2, T = 15\,keV, n = 2 \cdot 10^{14}\,cm^{-3}$, we have $\tau_s = 370\,ms$ and the necessary value of the diffusion coefficient to get inversion is $D_\alpha > 6.8\,m^2/s$. Similarly, in the case when $D_\alpha = d_0 r^k (k = 1, p = 0)$, we find $\lambda_m = 3.67/r_0$ and the condition for inversion becomes

$$d_0 > 0.82\frac{r_0}{\tau_s} \tag{4.7}$$

which for the above parameters yields $d_0 > 4.9\,m/s$. Another way of expressing the inversion criterion in the case of a velocity independent diffusion coefficient is to introduce an effective loss time, τ_L, in the Fokker-Planck equation. This can be done by applying a formal averaging procedure, which reduces Eq.(3.1) to a velocity and time dependent problem for the averaged alpha particle distribution function. In order to average Eq.(3.2), an assumption about the profile of f_α has to be made. A convenient model compatible with the boundary conditions is $f_\alpha = f_{\alpha 0}(1-r^2/r_0^2)$. Defining a formal average procedure as

$$\bar{A} \equiv \frac{2}{r_0^2}\int_0^{r_0} rA\,dr \tag{4.8}$$

and applying it to Eq.(3.1) yields

$$\frac{\partial \bar{f}_\alpha}{\partial t} = \frac{\chi}{\bar{\tau}_s}\frac{1}{v^2}\frac{\partial}{\partial v}[(v^3+v_c^3)\bar{f}_\alpha] - \frac{\bar{f}_\alpha}{\tau_L} + \frac{\bar{S_0(r)}}{4\pi v_\alpha^2}\delta(v-v_\alpha) \tag{4.9}$$

where we have assumed the profiles $T(r) = T_0(1-r^2/r_0^2)^\delta$ and $n(r) = n_0(1-r^2/r_0^2)^\gamma$. Thus

$$\chi = \frac{2}{\left(2+\gamma-\frac{3}{2}\delta\right)\left(1+\frac{3}{2}\delta-\gamma\right)} \tag{4.10}$$

and the diffusion loss time is

$$\tau_L = \frac{1}{8d_0 r_0^{k-2}} \tag{4.11}$$

Assuming that for $T > 10keV$

$$<\sigma v> \simeq 1.5\cdot 10^{16} T_i^{3/2}\,(m/s) \tag{4.12}$$

When T is over $10\,keV$, the averaged value of S_0 can be written as

$$\bar{S_0} \simeq \frac{1.5\cdot 10^{-16}(1+\gamma)^2(1+\delta)^{3/2}\bar{n}^2\bar{T}^{3/2}}{\left(1+\frac{3}{2}\delta+2\gamma\right)4} \tag{4.13}$$

The solution of Eq.(4.9) is given by

$$\bar{f}_\alpha(v,t) = \frac{\bar{S_0}(\tau^*)\bar{\tau}_s(\tau^*)}{\chi 4\pi(v_\alpha^3+v_c^3)}\exp\left[-\int_{\tau^*}^t dt'\left(\frac{1}{\tau_L}-\frac{3\chi}{\bar{\tau}_s(t')}\right)\right] \tag{4.14}$$

where τ^* is determined by

$$\int_\tau^{\tau^*}\frac{\chi}{\tau_s}\,dt' = \frac{1}{3}\ln\left(\frac{v^3+v_c^3}{v_\alpha^3+v_c^3}\right) \tag{4.15}$$

The inversion condition obtained directly from Eqs.(4.14) and (4.15) is

$$\frac{1}{\bar{S_0}}\frac{\partial \bar{S_0}}{\partial t} + \frac{1}{\bar{\tau}_s}\frac{\partial \bar{\tau}_s}{\partial t} > \frac{3\chi}{\bar{\tau}_s} - \frac{1}{\tau_L} \tag{4.16}$$

Note that this condition is equivalent with the result obtained in Refs. [30,31,32], where τ_L has been introduced in a heuristic way by replacing the effect of the diffusion operator in Eq.(3.1) by the diffusion loss time, i.e.

$$\frac{d_0}{r}\frac{\partial}{\partial r}\left(r^{k+1}\frac{\partial f_\alpha}{\partial r}\right) \to -\frac{f_\alpha}{\tau_L} \tag{4.17}$$

This shows that the steady state distribution is always inverted if

$$\frac{1}{\tau_L} > 3\frac{\chi}{\tau_s} \tag{4.18}$$

Assuming for example $\delta = \gamma = 1/4$ and $k = 0$, the condition (4.18) together with Eqs.(4.10) and (4.11) give

$$D_\alpha > 0.4\frac{r_0^2}{\tau_{s0}} \tag{4.19}$$

where $\tau_{s0} = \tau_s(r = 0)$. This is in good agreement with the corresponding criterion (4.4). For finite heating rates we can assume that the time dependences of the plasma temperature and density are

$$T = t^\nu; \quad n_e = t^\mu \tag{4.20}$$

Then the general condition for inversion (4.16) predicts that inversion will be possible for heating times satisfying

$$t < \left(\frac{3\nu + \mu}{3\chi/\bar{\tau}_s} - \frac{1}{6\tau_L}\right)^{-1} \tag{4.21}$$

For example for $\nu = 2$, $\mu = 0$ and $\delta = \gamma = 1/4$ the velocity inversion of f_α is possible for

$$t < \left(\frac{15}{8\tau_{s0}} - \frac{1}{6\tau_L}\right)^{-1} \tag{4.22}$$

Note that large diffusion loss times, represent small values of spatial diffusion. Then, Eq.(4.22) expresses qualitately that the smaller the spatial diffusion is, the smaller the heating time must be in order to satisfy the inversion condition.

5

Conclusions

The derived inversion condition, shows that the change in the slope of the velocity distribution function for high energy alpha particle, and consequently the posible emergence of wave instabilities, can be achieved either by fast variations of temperature, the presence of anomalous spatial diffusion or both effects at the same time.

In previous research regarding also the inversion criterion, where the effect of anomalous spatial diffusion was not included, the conclusions were that the inversion of the velocity distribution function is only achievable through fast variations of temperature that are completely out of the scope of the present plasma heating systems. Therefore, the posible wave instabilities excited by the studied mechanism are not a factor to take into account.

However, the conclusion of the present master thesis is that the posibility of inversion in the velocity distribution function cannot be rejected because, besides the effects of fast variations of temperature, the presence of spatial diffusions has to be considered. Even in the steady-state case, large enough values of D_α can cause the inversion.

Finally, it has been left for further research, the derivation of an explicit function for D_α, allowing the comparison of the values of D_α that satisfy the inverison condition with the experimental values, and the quantitatively assessment of the implications of the anomalous spatial diffusion in the system.

Bibliography

[1] A. B. Mikhajlovskii, in Problems of Plasma Physics (Edited by M. A. Leontivoch), Vol.9, Moscow (1979).

[2] D. Pfirsh, in Theory of Magnetically Confined Plasmas, Pergamon Press, Oxford and New York (1979).

[3] Y. I. Kolesnichenko, Nuclear Fusion 20 (1980) 727.

[4] A. B. Mikhajlovskii, in Physics of Plasma in Thermonuclear Regimes (Edited by B. Coppi and W. Sadowskij), U. S. Department of energy (1981)

[5] M. Lisak, Physica Scripta 29 (1984) 87.

[6] L. V. Korablev, Sov. Phys. - JETP 26 (1968) 922.

[7] T. D. Kaladze, A. B. Mikhajlovskii, Sov. J. Plasma Phys. 1 (1975) 128 Physica Scripta 29 (1984) 87.

[8] T. D. Kaladze, D. G. Lominadze, A. B. Mikhajlovskii, A. B. Pokhotelov, Nuclear Fusion 16 (1976) 465.

[9] T. D. Kaladze, A. B. Mikhajlovskii, Nuclear Fusion 17 (1977) 729.

[10] D. G. Lominadze, A. B. Mikhajlovskii, Sov. J. Plasma Physics 1 (1975) 291.

[11] T. D. Kaladze,Sov. J. Plasma Physics 7 (1981) 451.

[12] A. B. Mikhajlovskii,T. D. Kaladze, D. G. Lominadze, L. V. Tsamalashvili, Sov. J. Plasma Physics 5 (1979) 173.

[13] W. Sutton, D. J. Sigmar, G. H. Miley, Fusion Techn. 7 (1985) 374.

[14] T. D. Kaladze,Physica Scripta T16 (1987) 27.

[15] V. S. Belikov, Y. I. Kolesnichenko, V. I. Oraevskij, Sov. Phys. JETP 39 (1974) 828.

[16] C. O. Beasley, A. B. Mikhajlovskii,D. G. Lominadze, Sov. J. Plasma Physics 2 (1976) 95.

[17] T. D. Kaladze, A. B. Mikhajlovskii, Nuclear Fusion 17 (1977) 411.

[18] V. S. Belikov, Y. I. Kolesnichenko, A. B. Mikhajlovskii, V. A. Yavorskii, Sov. J. Plasma Physics 3 (1977) 146.

[19] D. J Sigmar, H. C. Chan, Nuclear Fusion 18 (1978) 1569.

[20] A. B. Mikhajlovskii, Sov. Physics - JETP 41 (1975) 890.

[21] A. B. Mikhajlovskii, A. L. Frenkel, Sov. J. Plasma Phys. 3 (1977) 677.

[22] M. N. Rosenbluth, P. H. Rutherford, Phys. Rev. Lett. 34 (1975) 1428.

[23] K. T. Tsang, D. J. Sigmar, J. C. Whitson, Phys. Fluids 24 (1981), 1508.

[24] Y. M. Li, S. M. Mahajan, D. W. Ross, Phys. Fluids 30 (1987) 1466.

[25] M. Bornatici, F. Engelmann, Quasilinear Transport of Alpha Particles in a Tokamak Plasma, FOM Institute for Plasma Physics, Rijnhuizen, Nieuwegein, the Netherlands, Res. Rep. PP. 79/04 (1979).

[26] V. S. Belikov, Y. I. Kolesnichenko, Nuclear Fusion 20 (1980) 1153.

[27] V. S. Belikov, Y. I. Kolesnichenko, A. D. Fursa, in Plasma Physics and Controlled Nuclear Fusion Research (Proc. 7th Int. Conf. Innsbruck, 1978) Vol. 1, IAEA, Vienna (1979) 561

[28] D. Anderson, M. Lisak, Phys. Fluids 27 (1984) 925.

[29] M. Lisak, D. Anderson, H. Hamnén, Phys. Fluids 26 (1983) 3308.

[30] Y. I. Kolesnichenko, Sov. J. Plasma Phys. 6 (1980) 531.

[31] V. S. Belikov, Y. I. Kolesnichenko, V. A. Yavorskij, Nucl. Fusion 21 (1981) 1311.

[32] J. G. Cordey, R. J. Goldston, D. R. Mikkelsen, Nucl. Fusion 2 (1981) 581.

[33] D. V. Sivukhin in Reviews of Plasma Physics, edited by M. Leontovich, Vol.4 (1963).

[34] John Wesson, Tokamaks, International series of monographs on Physics 18, Oxford Science Publications (2004).