Thesis for the Degree of Master of Science in Physics

Renormalization and Supersymmetry Breaking Phenomenology

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Abstract

In this thesis we address some issues of current interest in particle physics and quantum field theory (QFT).

First we give an introduction to renormalized perturbation theory and loop computations in QFT. Quantum chromodynamics (QCD) is used as an example and it is explicitly renormalized to first order, with all counterterms computed. The β-function is derived and used to show that QCD is asymptotically free.

Secondly we give a short introduction to supersymmetry (SUSY): the algebra, superfields and SUSY breaking. We present a simple model (the O'Raifeartaigh model) and show how to deal with the case of strong SUSY breaking in a manifestly supersymmetry invariant way.

Finally we compute the tree level cross section for production of a hidden vector boson present in a specific model of SUSY breaking (semi-direct gauge mediation). Unfortunately, the resulting cross section is too small to give a signal at the LHC. We also compute the decay rate of the vector boson and show that it is actually a candidate for dark matter.

Acknowledgments

My greatest gratitude goes to my supervisor Gabriele Ferretti for all his ideas and guidance through the maze that is particle physics. No subject was considered too remote and no question went unanswered, more often than not they developed into a discussion deserving its own lecture.

My second greatest thanks goes to my good friend Hampus Linander for his perpetual company when getting coffee, and his superhuman ability in asking that 'But why...'-question that I could never answer. It always sent me straight back to my desk, questing for better understanding.

I would also like to thank past and present members at the Institution of Fundamental Physics for all help, interesting discussions, 'fredagsfika', 'lunchlöpning' and, most of all, for providing a very pleasant and welcoming atmosphere where a confused master student can conduct his (or her) studies.

Finally, I would like to thank my family: Inger, Lasse, Jon and Sivert, without whose support (though generally more concerned with how I was doing than the specifics of what I did) I would never have dared take the step into what is considered one of the most difficult areas in existence.

"Nothing travels faster than the speed of light with the possible exception of bad news, which obeys its own special laws."

– Douglas Adams (chapter one of 'Mostly Harmless')

Contents

1

Introduction

During my fourth year of study I took a course in quantum field theory (QFT). I found it extremely complicated and to my great regret (and horror!) I failed, even though I knew that it is the basic groundwork for all particle physics, in which I have a great interest. After having thought about how to face my fear of the subject for the better part of a year, I finally realized that the only way I was ever going to learn, was by putting myself in a position where I had no other choice. That said, I obviously had to do my master thesis on a subject connected to quantum fields! In retrospect I have to say that even though the logic seems flawed, I have not regretted the decision at all.

The goal of a particle physicist is to describe the basic constituents of matter and how it works, because from there virtually all physics can be derived. Ideas about the smallest parts of nature have been around for centuries, but in 1789 the French chemist A. Lavoisier (who was later executed in 1794 during the French revolution, to which Lagrange responded: 'It took them only an instant to cut off his head, but France may not produce another such head in a century.') defined an element as a basic substance that could not be broken down, and this was termed an atom. Around a hundred years later (1897) it was shown that the atom was not the smallest constituent, when the negatively charged electron was discovered by J.J Thompson.

In 1909 an experiment led by E. Rutherford discovered the atomic nucleus. Two years later he suggested a model where the atom was built out of a nucleus, where most of the mass is concentrated, and a number of electrons circling around it (at a distance of roughly $10^{-10}\,\text{m} = 1$ Å). The nucleus was furthermore shown to be built out of the positively charged protons (found by Rutherford in 1915) and neutrons (found by J. Chadwick in 1932) and these were later shown to have roughly the size 10^{-15} m.

In the 60's M. Gell-Mann and G. Zweig proposed on theoretical grounds that all hadrons, including protons and neutrons, are in turn composite par-

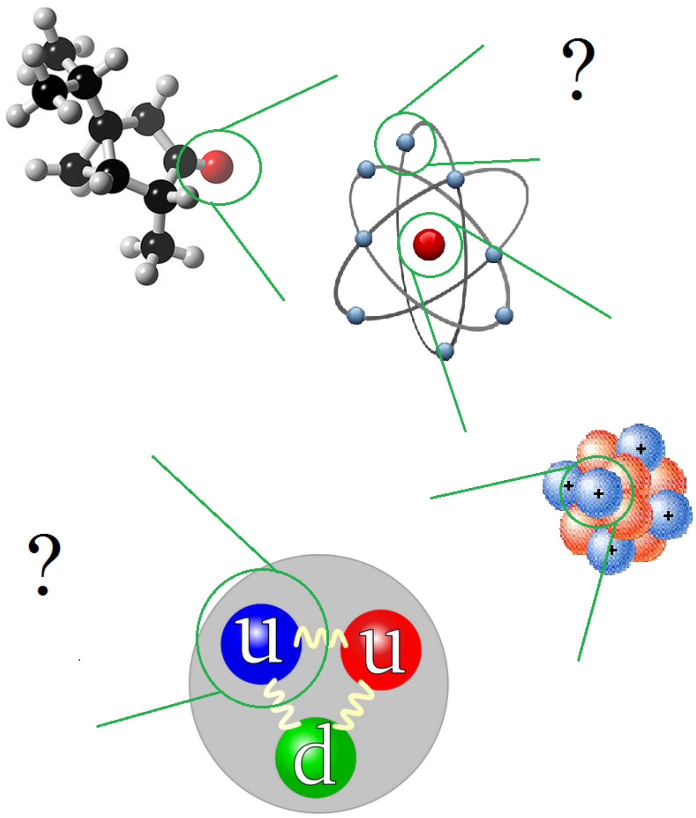


Figure 1.1: The molecule in the top left is built from hydrogen, carbon and oxygen atoms, and each of these consist of a nucleus with electrons around it. The nucleus is built out of protons and neutrons, which are in turn composed of three quarks each. It is unknown whether quarks and electrons have any inner structure and as far as we know they behave as points (they could, for example, be strings). The size of the molecule (Thujone, $C_{10}H_{16}O$) is typically 10^{-9} m, the atom 10^{-10} m and the nucleus 10^{-15} m.

ticles consisting of quarks[1]. This proposal was verified in 1968 and today it is still the accepted model. At the time of writing six different types of quarks have been found. An illustration of the situation is shown in figure 1.1[2].

Furthermore, the electron have two heavier cousins: the muon and the tau. Along with the neutrinos, which interact very weakly with matter, they are

[1]The baryons, to which protons and neutrons belong, consist of three quarks whereas the mesons are built out of two: a quark and an anti-quark.

[2]The picture is composed from the following images: the molecule: `http://czechabsinthe.files.wordpress.com/2007/04/molecule.jpg`; the atom: `http://stuffthathappens.com/blog/wp-content/uploads/2007/11/atom.png`; the nucleus `http://www.theo-phys.uni-essen.de/tp/ags/guhr_dir/media/nucleus3.jpg`; the proton `http://upload.wikimedia.org/wikipedia/commons/9/92/Quark_structure_proton.svg`.

gathered into the family called leptons. Together with the quarks, these two different types of particles are the basic constituents of matter, as far as we know today.

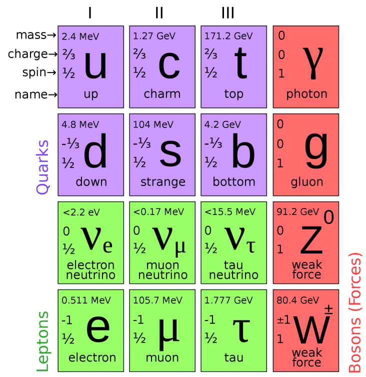


Figure 1.2:

In addition the particles may interact with each other in four different ways: by gravity, by electromagnetism, by the weak force or by the strong force. The first two need no introduction, but the last two may be more unfamiliar because they have very short range and do not appear in ordinary life. The weak force is, for example, responsible for β-decay. It is the reason why there are no free neutrons: they decay into a proton, an electron and an anti-neutrino each. The strong force is a short range interaction responsible for binding the quarks inside a hadron and also, as a consequence, for keeping the protons and neutrons together inside the atomic nucleus[3].

The quarks, leptons and all forces, except gravity, are built into the Standard model of particle physics (along with the Higgs particle, which has not been discovered experimentally yet), all shown in figure 1.2[4]. It is a model based on symmetry and it was developed in the 70's and 80's, and its final parts were experimentally verified in the 90's. It is extremely successful in that its predictions match experimental measurements to an astounding accuracy.

However, it cannot be the theory of everything that particle physicists are looking for. The most obvious thing is that it does not include gravity, but that will not be the main subject here. Another puzzle is that, within the standard model, it is impossible to explain why the mass of the Higgs particle is so light. It could be much bigger and there is no reason why it is not so. A model that fixes this and then some, see [1] or [2], is supersymmetry (SUSY).

Supersymmetry is a remarkable symmetry in that it is a transformation between two completely different classes of particles, it transforms bosons into fermions and vice versa. The two behave very differently, obeying different statistics: two fermions in a system cannot have the same quantum numbers but bosons can, and all particles belong to one class or the other. Supersymmetry has the effect that the standard model is extended so that each particle get a supersymmetric partner, essentially doubling the number of particles. The resulting model is called the minimal supersymmetric standard model (MSSM), but at the time of writing there has been no experimental sign of the new particles. It is an open question whether the symmetry exists or

[3] Protons are positively charged so in all nuclei there are enormous electromagnetic forces that want to rip them apart. Alpha decay and spontaneous fission are examples of such instabilities.

[4] The picture was taken from `http://upload.wikimedia.org/wikipedia/commons/0/00/Standard_Model_of_Elementary_Particles.svg`.

not, but it is widely considered to be the most promising model for physics beyond the standard model. Fortunately, with the start of the Large Hadron Collider (LHC) in 2010, we are on the brink of discovering whether it really does describe reality or not.

The main subject here will be supersymmetry: how to break it and the phenomenology of the breaking. When writing a thesis there is usually one specific question that must be answered, but it is difficult to state one single question for this problem. Instead the purpose has been to acquire enough knowledge and skill to be able to understand the latest research material on the subject.

As for the report itself, there is some thought behind what has been included and what has been left out. The first section deals with renormalization and loop computations in QFT, necessary for essentially any advanced phenomenological application. The next chapter develops the basic working tools of supersymmetry in superspace. The methods of both sections are then used when working with the theory presented in section 4. In the final section we apply (a more complicated version of) the theory of the previous chapter to examine the phenomenology.

1.1 Method

The method is a combination of a literature study (the most important books and papers used in each section are listed in their respective introduction), a series of discussions and some theoretical research. In sections 2 and 3 the tools were books, papers and discussions with my supervisor. In sections 2.3 and 2.4 we review a specific example, originally worked out in the 70's by D. Gross, H. Politzer and F. Wilczek.

The final two chapters contain some new theoretical results that have not been previously published. The work has been carried out in collaboration with my supervisor, whom deserves the credit for all ideas. The computations have been done by both of us separately and then compared.

2

Asymptotic Freedom in QCD

Tree level quantum field theory computations often give good predictions, however if one wants to compute higher order corrections, to get even better results, trouble arises. Often the corrections seem to be infinite, a prediction that does not go well with reality because experimentalists measure finite results. The solution to this problem offers a great deal of new physical insight. In principle, to be able to go on into more advanced matters in quantum field theory, such as supersymmetry, one has to learn about loops and renormalization.

The outline for the section is the following: the first part will introduce the problem and offer some qualitative explanation to why it appears. The following two sections treat it and show how computations can be performed, using quantum chromodynamics (QCD, the theory of quarks and gluons) as an example. This is far from the easiest theory to work with but it is satisfying, because as the computations become more and more complicated so does the physics. In the last part of the section, the concepts introduced so far are used and treated in a way that offers a more natural interpretation. The goal is to show that QCD is asymptotically free, i.e can be treated perturbatively at high energies. This is a computation that was done already in 1973 by H. Politzer, D. Gross and F. Wilczek and won them the Noble prize in 2004. Good introductions to the subject are [3] and [4].

2.1 The Problem with Divergences

The first impression you get from renormalization is usually how ugly it is. You hear that the theory actually predicts infinities when you compute cross sections and that you, somehow, are able to shove them away into some corner and then leave them there to rot. The experimentalist do not measure infinite cross sections and so, you think, the theoreticians just try to lie the infinities away to make the theory fit reality.

This is as far from the truth as you can get. When you start working with it and understand more and more you soon realize, at least that is what I did, how subtle and beautiful it really is. But it is not all about understanding, it takes some getting used to also. Even though you are comfortable with it, there will be times when you are in the middle of a computation and you suddenly start thinking 'what am I really doing?'.

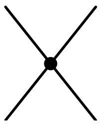
To see where the problem arises we will use ϕ^4-theory as a basic example. It is defined by the familiar Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{2.1}$$

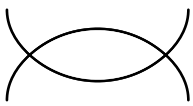
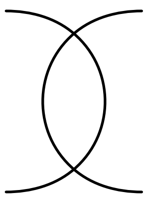
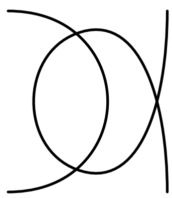
Even with this basic example it is impossible to compute the propagators exactly. This is nothing strange to physicists and the way to handle it is the same as in non-relativistic quantum mechanics - we use perturbation theory. Let us compute the four-point interaction: there are two incoming particles with four-momenta p_1^μ and p_2^μ that scatter into the two outgoing particles with q_1^μ and q_2^μ. The probability of scattering is gotten by time evolving both states and computing the overlap

$$\lim_{T\to\infty} \langle p_1, p_2 |\, e^{-2iHT} \,| q_1, q_2\rangle = i\mathcal{M}\delta^4(p_1 + p_2 - q_1 - q_2) \tag{2.2}$$

With H the Hamiltonian. The first order perturbation is the tree level process

$\mathcal{M}_{tree} =$  $= -i\lambda$

The scattering amplitude is $|\mathcal{M}|^2_{tree} = \lambda^2$ and it is very much finite. If we want to get a more precise result we have to go on and compute the second order perturbation. This will be of order λ^2 and if $\lambda < 1$, which it must be if perturbation theory is to work, the second order contribution should probably be a lesser correction.

$\mathcal{M}_2 =$  $+$  $+$ 

We only have to compute the first diagram, the s-channel process[1].

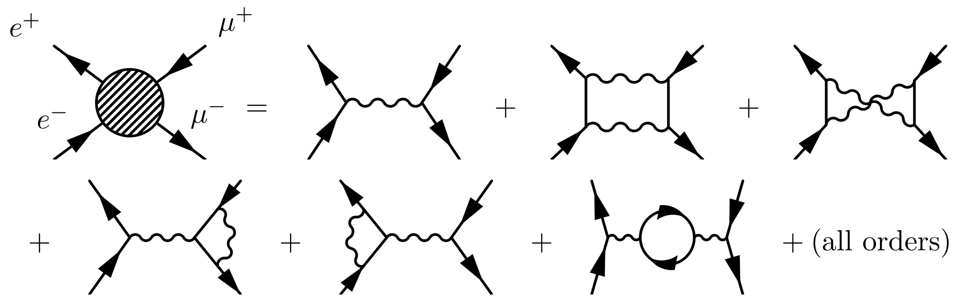
[1]Technically speaking it is not really a s-channel because the outgoing propagators cannot form a bound state and thus there is no pole. I call it s-channel because the diagrammatic form is similar to such processes.

$$\mathcal{M}_2 = -\lambda^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\varepsilon} \frac{1}{(k + p_1 + p_2)^2 - m^2 + i\varepsilon} \tag{2.3}$$

But this is suspicious: there are four powers of momenta both in the numerator and the denominator so the integral could diverge. We are interested in large momenta so assume $k^2 >> p_1^2$, p_2^2. Let $k^0 \to ik^0$ to get an Euclidean metric

$$\begin{aligned}\mathcal{M}_2 \sim -\lambda^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 + m^2)^2} &= -\lambda^2 \frac{\Omega_3}{(2\pi)^4} \int dk \frac{k^3}{(k^2 + m^2)^2} \\ &\sim \lim_{k \to \infty} -\frac{\lambda^2}{2} \frac{\Omega_3}{(2\pi)^4} \log \frac{k^2 + m^2}{m^2} \end{aligned} \tag{2.4}$$

This is the famous logarithmic divergence of quantum field theory! Does this mean that the whole theory is complete nonsense? To answer that we will need to discuss experimental measurements and QED may be more suitable for that purpose, as it is easier to use physical intuition. Therefore consider the scattering of two electrons into to muons. The full scattering process is given by



The logarithmic divergences appear here as well, the three last diagrams on the right-hand side have such behaviour while the top two loop diagrams are finite. Despite the fact that the theory suggests divergences, experiments measure the left-hand side and they get finite results. An estimate of the cross section using only the tree level result gives the correct order of magnitude for the scattering (see [5]). In other words the theory behave as we would expect if we didn't know about the divergences. Let us ponder what physically may happen inside the loop

When computing the scattering amplitude the particle circling inside the loop was allowed to have any momentum, in other words, arbitrarily high energy. In the most extreme case the particle would have energy of the order of the Planck mass and we would have to start worrying about black holes. Quantum field theory cannot handle that because it knows nothing about quantum gravity. At that energy, and probably long before that[2], the theory

[2]For the specific case of QED, it breaks down already around 100 GeV when electroweak effects become important.

breaks down. Conversely the appearance of infinities tells us that quantum field theory is a good model, because when there are effects it cannot handle it is intelligent enough to give a warning sign. It would be a lot worse if everything came out finite but did not agree with experiments. The true underlying theory should be free of infinities (note that string theory is) but at least there is a reason why they pop up in these computations.

The divergences discussed here are called ultraviolet divergences. There is another type of divergence, called infrared, that comes about because for any process containing a massless propagator, there will be a pole as its momentum approaches zero. Although it is an interesting subject it can be solved independently of renormalization. It will not be treated here but a good treatment can be found in [6] and [7].

A final comment: note that what was done so far was computing loops, this is not renormalization! Renormalization is the process where the divergences are built into the theory by reinterpreting the fields, masses and coupling constants as actually containing them from the start and making a distinction between the measurable physical quantities and theoretical bare quantities. QCD will be used next as an example to really step up the difficulty.

2.2 Renormalizing a Non-Abelian Gauge Theory

In the previous section the appearance of the divergences was explained. Here the plan of attack is to interpret them and make them a natural part of the theory, this is what is called renormalization. There are several ways to do it and the one used here is called renormalized perturbation theory. It is suitable for computing physical quantities such as cross sections, but there are more sophisticated methods that offer greater physical insight. In the end, all methods must of course give the same result for an observable quantity.

In section 2.3 we will compute loops and to do so the integrals must be regulated to preserve gauge invariance. For this matter we will have to work in d-dimensions ($d-1$ space dimension), only taking the limit $d \to 4$ in the very end. Therefore everything in this section is also done for a d-dimensional spacetime.

As example a $SU(3)$ gauge theory will be used. The Lagrangian, called the Yang-Mills Lagrangian, is

$$\mathcal{L} = -\frac{1}{4} F^a_{0\mu\nu} F^{\mu\nu a}_0 + \bar{\psi}_0 (i\partial\!\!\!/ - m_0)\psi_0 + g_0 \bar{\psi}_0 \gamma^\mu T^a \psi_0 A^a_{0\mu} \tag{2.5}$$

With $F^a_{0\mu\nu}$ the field strength of the gluon field

$$F^a_{0\mu\nu} = \partial_\mu A^a_{0\nu} - \partial_\nu A^a_{0\mu} + g_0 f^{abc} A^b_{0\mu} A^c_{0\nu} \tag{2.6}$$

The index 0 on the fields will be useful to keep this Lagrangian separate from the renormalized one. Equation (2.5) does not look very much different from the Lagrangian of QED except for the generators T^a (which are actually

1/2 times the Gell-Mann matrices) but some care is needed. In addition to the usual hidden spinor indices ψ has an additional hidden index, since it belongs to the fundamental representation of $SU(3)$ ($\bar{\psi}$ is in the anti-fundamental). This is contracted with the hidden indices on T^a in the same way the spinor indices are contracted with γ^μ. When all non-linear terms are written out there are a lot of new interactions

$$\mathcal{L}_0 = \bar{\psi}_0(i\partial\!\!\!/ - m_0)\psi_0 - \frac{1}{4}\left(\partial_\mu A^a_{0\nu} - \partial_\nu A^a_{0\mu}\right)(\partial^\mu A_0^{\nu a} - \partial^\nu A_0^{\mu a}) + g_0\bar{\psi}_0\gamma^\mu T^a\psi_0 A^a_{0\mu} - g_0 f^{abc}(\partial_\mu A^a_{0\lambda})A_0^{\mu b}A_0^{\lambda c} - \frac{1}{4}g_0^2(f^{eab}A^a_{0\mu}A^b_{0\nu})(f^{ecd}A_0^{\mu c}A_0^{\nu d}) \quad (2.7)$$

From now on this will be refereed to as the bare Lagrangian and the fields with index 0 will be the bare fields. Actually, additional terms will have to be added later on to account for gauge invariance, but they will not affect the discussion here. This is further commented in appendix B.

The first step in the renormalization process is a rescaling of the bare fields

$$A^a_{0\mu} = \sqrt{Z_A}A^a_\mu, \qquad \psi_0 = \sqrt{Z_\psi}\psi \quad (2.8)$$

This is the first place the infamous divergences appear in loop calculations. What we have changed is the normalization of the fields and it is divergent as $d \to 4$ but as long as d is arbitrarily small there is no cause for alarm. Putting this back into the Lagrangian yields

$$\begin{aligned}\mathcal{L} = {} & Z_\psi\bar{\psi}(i\partial\!\!\!/ - m_0)\psi - \frac{1}{4}Z_A\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a}) \\ & + g_0 Z_\psi\sqrt{Z_A}\bar{\psi}\gamma^\mu T^a\psi A^a_\mu - g_0 Z_A^{3/2}f^{abc}(\partial_\mu A^a_\lambda)A^{\mu b}A^{\lambda c} \\ & - \frac{1}{4}g_0^2 Z_A^2(f^{eab}A^a_\mu A^b_\nu)(f^{ecd}A^{\mu c}A^{\nu d})\end{aligned} \quad (2.9)$$

Unfortunately the coupling constant acquired a unit when going to d-dimensions. In practice this is not a problem but it would be nice to keep it dimensionless. The dimensions of the fields in an arbitrary dimension can be worked out by looking at the kinetic terms and any of the interactions. They are, in units of energy,

$$[\mathcal{L}] = E^d, \quad [\psi] = E^{(d-1)/2}, \quad A_\mu = E^{d/2-1}, \quad [g_0] = E^{(4-d)/2}$$

To get a dimensionless coupling constant, substitute $\sqrt{Z_A}Z_\psi g_0 = Z_g\mu^\varepsilon g$, where $[g] = E^0$, $[\mu] = E^1$ and $2\varepsilon = 4 - d$. The downside here is that in any computation where previously only g_0 appeared we will have to drag around an extra factor of μ^ε. The same thing can be done to the other interactions.

$$\begin{aligned} Z_A^{3/2}g_0 = Z_3\mu^\varepsilon g \quad &\Rightarrow \quad Z_3 = Z_A\frac{Z_g}{Z_\psi} \\ Z_A^2 g_0 = Z_4\mu^\varepsilon g \quad &\Rightarrow \quad Z_4 = Z_A^{3/2}\frac{Z_g}{Z_\psi}\end{aligned} \quad (2.10)$$

The indices 3 and 4 may seem arbitrary, but they have been chosen because the interactions describe three- and four-point vertices. These are not independent, because it is the coupling constant that is redefined (not the vertices themselves) and gauge invariance guarantees that it is the same for all interactions.

The Lagrangian in equation (2.9) does not look canonically normalized. To get it in a form we are used to, that is, in a form where it is more apparent what the Feynman rules are, the wavefunction renormalizations must be removed in some way. Rescaling again would achieve nothing except getting back to the bare version. Instead we will do something that at first may seem foolish: we rewrite the Zs as $Z = 1 + \delta$, effectively breaking up the Lagrangian in a canonically normalized part and additional interaction terms. The new terms are called counterterms and are defined by

$$\begin{array}{lll} \delta_A = Z_A - 1, & \delta_\psi = Z_\psi - 1, & \delta_m = Z_\psi m_0 - m \\ \delta_g = Z_g - 1, & \delta_3 = Z_3 - 1, & \delta_4 = Z_4 - 4 \end{array} \tag{2.11}$$

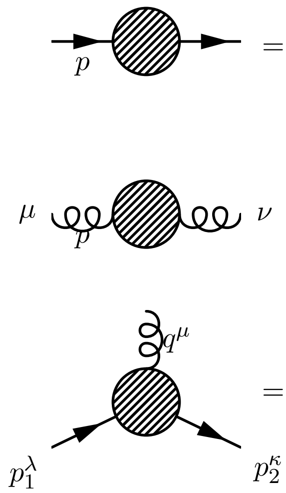
With them we rewrite the Lagrangian to its final form

$$\begin{aligned} \mathcal{L} = {} & \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)^2 + \mu^\varepsilon g \bar{\psi}\gamma^\mu T^a \psi A_\mu^a \\ & - \mu^\varepsilon g f^{abc}(\partial_\mu A_\lambda^a) A^{\mu b} A^{\lambda c} - \frac{1}{4}\mu^{2\varepsilon} g^2 (f^{eab} A_\mu^a A_\nu^b)(f^{ecd} A^{\mu c} A^{\nu d}) + \mathcal{L}_{CT} \end{aligned} \tag{2.12}$$

$$\begin{aligned} \mathcal{L}_{CT} = {} & \bar{\psi}(i\delta_\psi \partial\!\!\!/ - \delta_m)\psi - \frac{1}{4}\delta_A\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)^2 + \mu^\varepsilon g \delta_g \bar{\psi}\gamma^\mu T^a \psi A_\mu^a \\ & - \mu^\varepsilon g \delta_3 f^{abc}(\partial_\mu A_\lambda^a) A^{\mu b} A^{\lambda c} - \frac{1}{4}\mu^{2\varepsilon} g^2 \delta_4 (f^{eab} A_\mu^a A_\nu^b)(f^{ecd} A^{\mu c} A^{\nu d}) \end{aligned} \tag{2.13}$$

This is the renormalized Lagrangian. The price we pay to reinterpret the divergences is that the theory has to be defined at some energy scale, otherwise the counterterms cannot be specified. This scale is arbitrary and the renormalization parameter will affect the theoretical results. In practice it means that the general behaviour of a process can be predicted, but no numbers can be given unless we first go to an experimentalist and ask how big a (for example) cross section is at a specific energy. For example: no calculation in QED will ever tell us how big the fine-structure constant is, but once it is known from experimental measurements, there is no end to the potential applications of QED.

Setting this scale is next on the agenda. By fixing the propagators and vertices at the scale μ, the counterterms and thus the entire theory can be specified. Remember that QCD is non-perturbative at low energies so it would not be clever to choose the renormalization scale as the mass of one of the light quarks. Instead μ must be chosen to at least a few GeV. The theory is defined by the diagrams in figure 2.1, with the conditions



$$= i\frac{\not{p}+m}{p^2-m^2+i\varepsilon} - i\Sigma(\not{p})$$

$$= -i\frac{\eta_{\mu\nu}\delta^{ab}}{p^2+i\varepsilon} + i\Pi^{ab\mu\nu}(p^2)$$

$$= ig\Gamma^\mu(p_1,p_2)D^a$$

Figure 2.1: The QCD renormalization diagrams

$$\begin{array}{ll} \frac{d}{d\not{p}}\Sigma(\not{p}) = 0, & p^2 = \mu^2 \\ \Sigma(\not{p}) = 0, & p^2 = \mu^2 \\ \Pi^{ab\mu\nu}(p^2) = 0, & p^2 = \mu^2 \\ \Gamma^\mu = ig\gamma^\mu T^a, & p_1^2 = p_2^2 = \mu^2 \end{array} \tag{2.14}$$

At this point we can solemnly state that QCD has been renormalized! The divergences have been reinterpreted into the definitions of the fields and the coupling constant. The three- and four-point gluon interactions could be specified as well, but those counterterms will be given in terms of the others by equations (2.10) and (2.11). In practice this is not the end of the road. The loop diagrams still need to be computed to get the counterterms, if the theory is to be used beyond tree level.

2.3 Computing Loops and Counterterms

The aim of this section is to compute the divergent parts of the counterterms to one loop. The first thing we need is the Feynman rules for the relevant terms in $\mathcal{L}_{CT}$ (see e.g [3]), shown in figure 2.2.

For a theory that is as complicated as QCD computing loops is rather hard. The only diagram that will be treated here is the first order loop correction to the quark propagator. The rest of the diagrams that come into the computation of the counterterms are found in appendix C.

To first order, the diagrams contributing to the quark propagator are shown in figure 2.3. Write the rhs according to figure 2.1

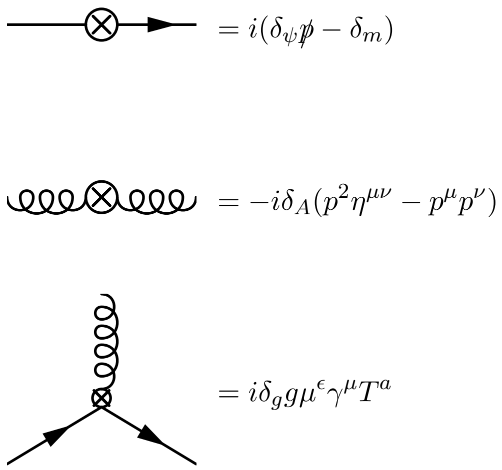


Figure 2.2: Feynman rules for the counterterms. Note how similar the structure of these are to the tree level Feynman rules. This is essential when isolating the divergence.

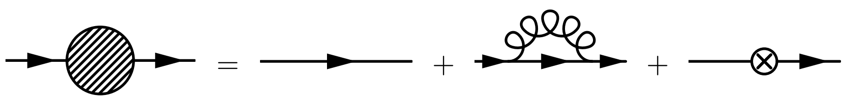
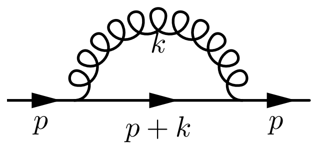


Figure 2.3: All diagrams that contribute to the quark propagator to one loop.

$$-i\Sigma(p^2) = \mathcal{M}_q(p^2) + i\delta_\psi \not{p} - i\delta_m \tag{2.15}$$

Already at this point we can see how the counterterm works. Once computed it will cancel the divergence coming from the propagator corrections. If the correction is a part of bigger diagram, the counterterm has to be included as well. The rest of the section goes into calculating this thing.



Here we only consider the amputated diagram so there are no external propagators. If we count powers of momenta we immediately see that it seems to diverge linearly. Direct application of the Feynman rules gives the expression

$$\mathcal{M}_q = \int \frac{d^dk}{(2\pi)^d} (ig\mu^\varepsilon T^a\gamma^\mu) \frac{\not{p} + \not{k} + m}{(p+k)^2 - m^2 + i\epsilon} (ig\mu^\varepsilon T^b \gamma^\nu) \frac{-i\eta_{\mu\nu}\delta^{ab}}{k^2 + i\epsilon} \tag{2.16}$$

The point of dimensional regularization is that for sufficiently small d the integral will converge. This means that we have greater mathematical freedom, such as variable substitution. The denominator can be put on a more convenient form by a method due to Feynman. The factor $1/AB$ can be written as

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{(xA + (1-x)B)^2} \tag{2.17}$$

In which case the denominator takes the form

$$\begin{aligned} D^2 &= \left(x\left((k+p)^2 - m^2 + i\epsilon\right) + (1-x)\left(k^2 + i\varepsilon\right)\right)^2 \\ &= \left(k^2 + 2xkp + xp^2 - xm^2 + i\epsilon\right)^2 \end{aligned} \tag{2.18}$$

If the integral converges we can make whatever coordinate transformations we want, as long as it is invertible. If the choice $\ell^\mu = k^\mu + xp^\mu$ is made the denominator becomes

$$D = \ell^2 - \Delta, \qquad \Delta = xm^2 - x(1-x)p^2 - i\epsilon \tag{2.19}$$

For space-like momentum Δ will be positive, whereas if p^μ is time-like some caution is needed and we cannot take $\epsilon = 0$ as usual. The numerator can be simplified with some Dirac algebra

$$N = \gamma^\mu \left(\not{\ell} + (1-x)\not{p} + m \right) \gamma_\mu = (2-d)(\not{\ell} + (1-x)\not{p}) + dm \tag{2.20}$$

Next consider the Lie algebra factor. The gauge group is $SU(3)$ but it is more convenient to keep it completely general, to keep the integration constants separate from the Lie algebra constants. The factor $T^a T^a$ is the Casimir operator and commutes with all generators of a simple Lie algebra

$$[T^b, T^a T^a] = f^{bac} T^c T^a + f^{bac} T^a T^c = 0 \tag{2.21}$$

By Shur's lemma, it should be proportional to the identity matrix times some constant $C_2(r)$, that may be different for each representation. The final expression is

$$\mathcal{M}_q = -g^2 \mu^{2\varepsilon} C_2(r) \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{(2-d)(\not{\ell} + (1-x)\not{p}) + dm}{(\ell^2 - \Delta)^2} \tag{2.22}$$

To evaluate it we make a Wick rotation to get an Euclidean integration measure, by doing the following substitution

$$\ell^0 = i\ell_w^0, \quad \boldsymbol{\ell} = \boldsymbol{\ell}_w \quad \Rightarrow \quad \det(d\ell/d\ell_w) = i \tag{2.23}$$

With an Euclidean metric it is obvious that any expression with an odd power of ℓ in the numerator disappears, due to spherical symmetry, because the denominator is even and positive for all ℓ. What is left to evaluate is

$$-ig^2 \mu^{2\varepsilon} C_2(r) \int_0^1 dx \left[((2-d)(1-x)\not{p} + dm) \int \frac{d^d\ell_w}{(2\pi)^d} \frac{1}{(\ell_w^2 + \Delta)^2} \right] \tag{2.24}$$

The easiest way to compute it is to use hyperspherical coordinates. The integrand is independent of all angles and they integrate to give the surface area of the $d-1$ dimensional sphere. The rest could be looked up in a table or in Mathematica but my secret passion for integrals demands that I do it properly. To be more general consider the denominator to the n-th power.

$$\begin{aligned} \int \frac{d^d\ell_w}{(2\pi)^d} \frac{1}{(\ell_w^2 + \Delta)^n} &= \frac{\Omega_{d-1}}{(2\pi)^d} \int d\ell_w \frac{\ell_w^{d-1}}{(\ell_w^2 + \Delta)^n} \\ &= \frac{\Omega_{d-1}}{2(2\pi)^d} \int d(\ell_w^2) \frac{(\ell_w^2)^{d/2-1}}{(\ell_w^2 + \Delta)^n} \end{aligned} \tag{2.25}$$

A nice way to evaluate it is to make the substitution $x = \frac{\Delta}{\ell^2 + \Delta}$ and then compare it to the Beta function.

$$\int d(\ell_w^2)\, \frac{(\ell_w^2)^{d/2-1}}{(\ell_w^2+\Delta)^n} \to \Delta^{d/2-n} \int_0^1 dx\, (1-x)^{d/2-1} x^{n-d/2-1}$$
$$= \Delta^{d/2-n} \frac{\Gamma(n-d/2)\Gamma(d/2)}{\Gamma(n)} \quad (2.26)$$

Now set $n = 2$ to get back to the original problem. Using $\Omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ and putting everything together, equation (2.24) takes the form

$$\mathcal{M}_q = -ig^2\mu^{2\varepsilon} C_2(r) \frac{\Gamma(2-d/2)}{(4\pi)^{d/2}} \int_0^1 dx\, \frac{(2-d)(1-x)\not{p} + dm}{\Delta^{2-\frac{d}{2}}} \quad (2.27)$$

The divergence manifests itself in the Gamma function as it is singular when $d \to 4$. Write $2 - d/2 = \varepsilon$ to make it explicit. The Gamma function has the nice property that $\zeta\Gamma(\zeta) = \Gamma(\zeta+1)$, i.e we can write $\Gamma(\varepsilon) = \frac{\Gamma(\varepsilon+1)}{\varepsilon}$ and the numerator is finite when ε goes to zero. To identify the different parts write $(\mu^2/\Delta)^\varepsilon = \exp(\varepsilon \log(\mu^2/\Delta))$ and expand in ε. The final expression is

$$\mathcal{M}_q = -i\frac{g^2 C_2(r)}{16\pi^2} \int_0^1 dx\, \Big((4-\varepsilon)m - (2-\varepsilon)(1-x)\not{p}\Big)$$
$$\times \frac{1}{\varepsilon}\left(1 + \varepsilon \log\left(\frac{\mu^2}{\Delta}\right) + O(\varepsilon^2)\right) \quad (2.28)$$

Note that the $\frac{1}{\varepsilon}$ stands for a logarithmic divergence. To fulfill the specified condition in equation (2.14) the quark and mass counterterms have to take the value $i\delta_\psi = -\frac{d}{d\not{p}}(\mathcal{M}_q)$ and $i(\not{p}\delta_\psi - \delta_m) = -\mathcal{M}_q$, both conditions evaluated at $p^2 = \mu^2$.

As can be seen from the above, the finite part of any counterterm can be changed by a different choice of renormalization scale. This means that the constant is arbitrary and we do not have to be very specific about it in the counterterm. Therefore we can define them as

$$\delta_\psi = -\frac{g^2 C_2(r)}{16\pi^2} \int_0^1 dx\, \frac{2(1-x)}{\varepsilon} \left(1 + \varepsilon \log\left(\frac{\mu^2}{\Delta}\bigg|_{p^2=\mu^2}\right) + \mathcal{O}(\varepsilon^2)\right) \quad (2.29)$$

$$\delta_m = -\frac{g^2 C_2(r)}{16\pi^2} \int_0^1 dx\, \frac{4m}{\varepsilon} \left(1 + \varepsilon \log\left(\frac{\mu^2}{\Delta}\bigg|_{p^2=\mu^2}\right) + \mathcal{O}(\varepsilon^2)\right) \quad (2.30)$$

This is called the minimal subtraction renormalization scheme. For our purpose the mass counterterm can be forgotten from now on, because it is unimportant for the β-function. The wavefunction renormalization is vital, however, and the divergent part of it is

$$\lim_{\varepsilon\to 0}\varepsilon\delta_\psi = -\frac{g^2 C_2(r)}{16\pi^2} \tag{2.31}$$

Some comments are needed before the section is concluded. Using dimensional regularization here was, strictly speaking, not completely necessary. The integrals could have been regulated just as well with the Pauli-Villars method, meaning that another heavy propagating fermion is included in the Lagrangian that cancels the divergence for very high momenta but does not effect the low energy result. Dimensional regularization was used anyway because it is both more versatile and not that much more complicated, and therefore there is no real point in doing anything else. For the gluon counterterm the Pauli-Villars regulation does NOT work properly.

The rest of the counterterm computations have been placed in appendix C because they are much the same as the one already done, albeit with some complications. These are technical issues however and lead to fun calculations but it is not much fun to read.

2.4 The β-function of QCD

It has been pointed out several times that the renormalization scale is arbitrary. It may be worrisome at first that the theory seems not to give unique answers, but the physics must of course be independent of the choice of μ. Therefore no measurable quantity can depend on the renormalization scale and we may use that to our advantage to derive consistency equations. The derivation of the β-function is due to 't Hooft in [8].

Consider the n-point function both in terms of renormalized and bare quantities

$$\begin{aligned} G^{(n)}(g,m,\mu;x_i^\nu) &= \langle\Omega|\,\psi(x_1^\nu)\ldots\psi(x_n^\nu)|\,\Omega\rangle \\ &= Z_\psi^{-n/2}\langle\Omega|\,\psi_0(x_1^\nu)\ldots\psi_0(x_n^\nu)|\,\Omega\rangle = Z_\psi^{-n/2}G_0^{(n)}(g_0,m_0,\varepsilon;x_i^\nu) \end{aligned} \tag{2.32}$$

The lhs is the renormalized n-point function which is finite and independent of ε, whereas the correlation function on the rhs is its bare counterpart and independent of the renormalization scale but divergent. If the equation is to be consistent the wavefunction renormalization must be a function of g_0, μ and ε such that the dependence on variables are the same on both sides of the equality, i.e $Z_\psi = Z_\psi(g_0\mu^{-\varepsilon},\varepsilon)$. The combination $g_0\mu^{-\varepsilon}$ must appear together because Z_ψ is dimensionless and can therefore only depend on dimensionless parameters. The spacetime dependence is of no importance here and the label will be removed from the equations. Note that the wavefunction renormalization can always be chosen independent of the (bare) mass[3].

[3] This is not obvious, but it is shown in e.g [9]. The statement is true as long as the energy is greater than the mass of all particles in the theory.

Hold the bare parameters g_0 and m_0 fixed. The renormalized counterparts must then be functions of μ, at least implicitly, to account for the difference between renormalized and bare parameters. This reasoning makes clear the dependence on μ and allows us to differentiate both sides of equation (2.32) with respect to it.

$$\frac{d}{d\mu}G^{(n)}(g,m,\mu) = \left[\frac{\partial g}{\partial \mu}\frac{\partial}{\partial g} + \frac{\partial m}{\partial \mu}\frac{\partial}{\partial m} + \frac{\partial}{\partial \mu}\right] G^{(n)} \tag{2.33}$$

$$\begin{aligned}\frac{d}{d\mu}\left[Z_\psi(g_0\mu^\varepsilon,\varepsilon)^{-n/2}G_0^{(n)}(g_0,m_0,\varepsilon)\right] &= -\frac{n}{2}Z_\psi^{-n/2-1}G_0^{(n)}\frac{d}{d\mu}Z_\psi \\ &= -\frac{n}{2}G^{(n)}\frac{d}{d\mu}\log Z_\psi \end{aligned}\tag{2.34}$$

Traditionally this is written as

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + m\gamma_m(g)\frac{\partial}{\partial m} + n\gamma(g)\right] G^{(n)}(g,m,\mu) = 0 \tag{2.35}$$

Where

$$\beta(g) = \lim_{\varepsilon\to 0}\mu\frac{\partial}{\partial\mu}g(g_0\mu^{-\varepsilon},\varepsilon) \tag{2.36}$$

$$\gamma_m(g) = \lim_{\varepsilon\to 0} -\mu\frac{\partial}{\partial\mu}\log Z_m(g_0\mu^{-\varepsilon},\varepsilon) \tag{2.37}$$

$$\gamma(g) = \lim_{\varepsilon\to 0}\frac{1}{2}\mu\frac{\partial}{\partial\mu}\log Z_\psi(g_0\mu^{-\varepsilon},\varepsilon) \tag{2.38}$$

Finally the β-function has been properly defined. Even though the methods have been purely mathematical its physical meaning is clear: it shows how the coupling constant shifts with the renormalization scale. Since it is a dimensionless parameter it must be independent of μ, simply because there is nothing that can kill off its unit.

In the γ_m equation the Z_m is defined as $m_0 = Z_m m$, likewise we define $g_0 = Z_G g\mu^\varepsilon$. In the notation of section 2.2 we have $Z_G = Z_G(g,\varepsilon) = Z_g/\sqrt{Z_A}Z_\psi$. Using this for the rhs of the β-function, we have

$$\beta(g) = \lim_{\varepsilon\to 0}\mu\frac{\partial}{\partial\mu}\left(\frac{g_0}{\mu^\varepsilon Z_G}\right) = \lim_{\varepsilon\to 0} -\varepsilon g - \mu g Z_G^{-1}\frac{\partial Z_G}{\partial\mu} \tag{2.39}$$

But Z_G is always computed perturbatively as a function of ε and g

$$\mu\frac{\partial Z_G}{\partial\mu} = \mu\frac{\partial g}{\partial\mu}\frac{\partial Z_G}{\partial g} = \beta(g)\frac{\partial}{\partial g}Z_G$$

Inserting into equation (2.39) results in the Callan-Symanzik equation

$$\left[\varepsilon g + g\beta(g)\frac{\partial}{\partial g} + \beta(g)\right] Z_G(g) = 0 \tag{2.40}$$

This is a nice result but we can do even better when considering how Z_G and β depend on ε, which has to go to zero in the end. If the limit in equation (2.39) exists we can evaluate β as a Taylor expansion in ε, likewise Z_G can be expressed as a Laurent series

$$\begin{aligned}\beta(g) &= \beta^{(0)}(g) + \varepsilon\beta^{(1)}(g) + \varepsilon^2\beta^{(2)}(g) + \dots \\ Z_G(g) &= 1 + \frac{Z_G^{(1)}}{\varepsilon} + \frac{Z_G^{(2)}}{\varepsilon^2} + \dots\end{aligned}$$

For the equality in equation (2.40) to hold, the coefficient of every ε^k must be zero. For $\varepsilon^{n\geq 2}$ we get the condition

$$\beta^{(n)} + \beta^{(n+1)}\left(Z_G^{(1)} + g\frac{\partial}{\partial g}Z_G^{(1)}\right) + \beta^{(n)}\left(Z_G^{(2)} + g\frac{\partial}{\partial g}Z_G^{(2)}\right) + \dots = 0 \quad (2.41)$$

The solution is easy to see if the equations are written on matrix form

$$\begin{pmatrix} 1 & (1+g\partial_g)Z^{(1)} & (1+g\partial_g)Z^{(2)} & \dots \\ 0 & 1 & (1+g\partial_g)Z^{(1)} & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix}\beta^{(2)} \\ \beta^{(3)} \\ \beta^{(4)} \\ \vdots\end{pmatrix} = 0 \quad (2.42)$$

The matrix is triangular so the only solution is the trivial one with every $\beta^{(n)} = 0$. Using this to write the rest of the equations yields

$$\begin{aligned}\varepsilon^1 :&\quad \beta^{(1)} + g = 0 \\ \varepsilon^0 :&\quad \beta^{(0)} + \beta^{(1)}Z_G^{(1)} + gZ_G^{(1)} + g\beta^{(1)}\partial_g Z_G^{(1)} = 0 \\ \varepsilon^{-k} :&\quad \beta^{(0)}Z_G^{(k)} + \left(\beta^{(1)} + gZ_G^{(k)}\right) + g\beta^{(0)}\partial_g Z_G^{(k)} + g\beta^{(1)}Z_G^{(k)} = 0\end{aligned}$$

With the solution

$$\begin{aligned}&\beta^{(1)} = -g \\ &\beta = g^2\partial_g Z_G^{(1)} \\ &\beta\left(Z_G^{(k)} + g\partial_g Z_G^{(k)}\right) = g^2\partial_g Z_G^{(k+1)} \qquad k \geq 1\end{aligned} \quad (2.43)$$

Where we have dropped the index (0). The last equation is for consistency, it is the middle one that is interesting. It says that the β-function ONLY depends on the simple pole Z_G. This is a huge simplification as we do not have to worry about the finite contributions or higher poles when computing counterterms (unless we want to do cross sections). Furthermore if we are only interested in the high momentum limit we may work with a massless theory, because the momentum can be taken sufficiently high to make the mass a very minor correction. The same methods may be used to derive equations for γ_m and γ but they will not be used.

The next step is to solve the Callan-Symanzik equation once and for all. To do so we will work with the two-point correlation function. Due to dimensional arguments it must be possible to write it as

$$G^{(2)} = \frac{i}{p^2} f(p^2/\mu^2) \tag{2.44}$$

For some function f. By using the chain rule we can swap the μ-derivative for a p

$$\mu \frac{\partial}{\partial \mu} G^{(2)} = -\left(p \frac{\partial}{\partial p} + 2\right) G^{(2)} \tag{2.45}$$

And the Callan-Symanzik equation takes the form

$$\left[p \frac{\partial}{\partial p} - \beta(g) \frac{\partial}{\partial g} + 2 - 2\gamma(g)\right] G^{(2)}(g, p, \mu) = 0 \tag{2.46}$$

The advantage compared to the previous form of the C-S equation is that now there is no implicit dependence left. A very nice way to solve it was worked out by Sidney Coleman (see [10]) who compared it to a fluid, full of bacteria, running through a pipe. Interpret p as time, β as velocity, γ as growth rate and $G^{(2)}$ as a density. Do the following substitutions

$$\begin{gathered} \log(p/\mu) = t \\ g = x \\ -\beta(g) = v(x) \\ 2\gamma(g) - 2 = \rho(x) \\ G^{(2)}(p, g) = D(t, x) \end{gathered}$$

The C-S equation becomes

$$\left[\frac{\partial}{\partial t} + v(x) \frac{\partial}{\partial x} - \rho(x)\right] D(t, x) = 0 \tag{2.47}$$

The bacterial density is the unknown function $D(t, x)$, with $\rho(x)$ being the rate of growth, x the position in the pipe and $v(x)$ the speed of the water. If we take the Lagrangian viewpoint and travel with the water the equation is simple, as the velocity term disappears. The solution in this case is the same as for a stationary situation, i.e some initial density $D_i(x)$ integrated with the growth rate along the path $x(t)$. The problem reduces to finding the trajectory of a fluid element in the running water to get back to the Euclidean picture.

$$\frac{d}{dt} \bar{x}(t, x) = -v(\bar{x}), \quad \bar{x}(0, x) = x \tag{2.48}$$

Combining the two will give the full solution

$$D(t,x) = D_i(\bar{x}(t,x)) \exp\left(\int_0^t dt'\, \rho(\bar{x}(t',x))\right)$$
$$= D_i(\bar{x}(t,x)) \exp\left(\int_{\bar{x}(t)}^x dx'\, \frac{\rho(\bar{x}')}{v(x')}\right) \quad (2.49)$$

Curiously, the trajectory equations are at least as interesting as the full solution. In the original variables equation (2.48) is

$$\frac{d}{d\log(p/\mu)}\bar{g}(p,g) = \beta(\bar{g}), \quad \bar{g}(\mu,g) = g \quad (2.50)$$

These are called the renormalization group equations. Just like $\bar{x}$ is a coordinate that runs with the fluid, $\bar{g}$ must be interpreted as a running coupling constant whose rate of change is the β-function. Finally we have ended up with something that can be measured directly. Just for reference we write down the solution for $G^{(2)}$ also

$$G^{(2)}(p,g) = \frac{i}{p^2} H_i(\bar{g}(p,g)) \exp\left(\int_{p'=\mu}^{p'=p} d\log(p'/\mu)\, 2[1-\gamma(\bar{g}(p',g))]\right) \quad (2.51)$$

For some unknown function H_i. In practice, H_i is evaluated in terms of the coupling constant and then coefficients in front of g are matched between lhs and rhs. The appearance of an unknown function tells us that the C-S equation does not contain all physics on its own.

Depending on the value of β three very different things can happen.

For $\beta > 0$ the coupling constant will grow with momentum, thus at some point the theory will become non-perturbative and strongly interacting. This happens in QED, but the point where $\bar{e}$ is greater than one is way above the planck mass. QED breaks down already at around 100 GeV.

For $\beta < 0$ the opposite happens, meaning that as momentum increases there is some point where perturbation theory becomes valid. This is what is called asymptotic freedom. It is great news for physicists as it is extremely difficult to compute anything when non-perturbative methods are required.

For $\beta = 0$ the coupling constant does not change with μ. It means that the divergent terms that come into Z_G cancel each other. Note however that there may still be divergences, for example in the wavefunction renormalizations, and these will still have to be renormalized to specify the theory.

Finally we are at a point where the initial goal is in sight. For QCD the β-function is

$$\beta(g) = g^2 \frac{\partial}{\partial g} Z_G^{(1)} = g^2 \frac{\partial}{\partial g} \frac{1+\delta_g}{\sqrt{1+\delta_A}(1+\delta_\psi)} = g^2 \frac{\partial}{\partial g}\left(1 + \delta_g - \frac{1}{2}\delta_A - \delta_\psi\right) \quad (2.52)$$

The expressions for the divergent parts of δ_ψ, δ_A and δ_g are found in equations (2.31), (C.19) and (C.28) respectively

$$\beta(g) = \frac{2g^3}{16\pi^2}\left[-(C_2(r)+C_2(G)) - \frac{1}{2}\left(\frac{5}{3}C_2(G) - \frac{4}{3}n_f C_2(r)\right) + C_2(r)\right]$$
$$= \frac{g^3}{16\pi^2}\left(\frac{4}{3}n_f C(r) - \frac{11}{3}C_2(G)\right) \quad (2.53)$$

For a sufficiently small number of quark species this will be negative. The underlying reason is the fact that the gauge group is non-abelian. To see it, take QED as an example of an abelian group. The photon propagator is not charged, because the adjoint representation is trivial for $U(1)$, so all $C_2(G)$ disappear. The consequence is that the divergent parts of δ_ψ and δ_g cancel each other and only the negative part of δ_A remains nonzero, meaning that the end result is positive.

For $SU(3)$ the coefficients are (see e.g [11]) $C(r) = 1/2$, for the fundamental representation, and $C_2(G) = 3$ so $\beta = -\frac{g^3}{16\pi^2}(11 - 2n_f/3) = -\frac{b_0 g^3}{16\pi^2}$. Equation (2.50) solves to

$$\bar{g}^2 = \frac{g^2}{1 - 2(g^2 b_0/16\pi^2)\log(p/\mu)} \quad (2.54)$$

Usually it is written with $\alpha_s = g^2/16\pi^2$ instead

$$\bar{\alpha}_s = \frac{\alpha_s}{1 - (b_0\alpha_s/2\pi)\log(p/\mu)} \quad (2.55)$$

Experiments have so far discovered six quark species: u, d, c, s, t and b, which means that $\beta = -\frac{7g^2}{16\pi^2}$ and QCD asymptotically free[4]. This is what we set out to prove and in the process learn something about renormalization and a non-abelian quantum field theory.

[4]It is asymptotically free up to energies around the top quark mass, but above that we do not know. If there are other flavours with bigger masses, β may still be positive for high enough energy and that is for experiments, such as the LHC, to discover.

3

Some Introductory Supersymmetry

The outline for this section is to introduce the main concepts of supersymmetry. We do this by diving straight into the SUSY algebra and from there go on to define superfields and superspace. Usually one would rather go about doing things in component fields first, to get more comfortable, and for such treatment see [1] or [2].

3.1 Basic SUSY

In this section we will introduce the very basics of supersymmetry and establish the conventions. It will not be a lengthy introduction and some derivation will be omitted. For more information see [12], [13], [14] and [15].

As was noted in the introduction, supersymmetry is a symmetry between bosons and fermions. It is a very special symmetry in that it entails an enlargement of the Poincaré algebra, with the generators ${Q_\alpha}^A$ and $\overline{Q}_{\dot{\alpha}B} = ({Q_\alpha}^B)^\dagger$. These will necessarily be fermionic operators, because they are supposed to take an integer spin state to a half-integer spin state and vice versa, therefore a spinorial index is present. As fermionic entities they must obey anticommutation relations, which will preliminary be written as

$$\begin{aligned}
&\{{Q_\alpha}^A, \overline{Q}_{\dot{\alpha}B}\} = \Sigma_{\alpha\dot{\alpha}} {Z_B}^A \\
&\{{Q_\alpha}^A, {Q_\beta}^B\} = \Xi_{\alpha\beta} X^{AB}, \qquad \{\overline{Q}_{\dot{\alpha}A}, \overline{Q}_{\dot{\beta}B}\} = X^\dagger_{AB} \Xi^\dagger_{\dot{\alpha}\dot{\beta}}
\end{aligned}$$

As humble as they may look, this is the basic reason why people[1] are excited. The reason is that by having symmetry generators that obey anticommutation relations, the restrictions from the Coleman-Mandula theorem

[1] To be read as 'physicists'.

are avoided and they may be used to build a model with a non-trivial S-matrix, in other words, a model with interactions.

The roman letter index denotes the number of SUSY generators and for $d = 4$ the maximal number is four[2]. In phenomenological applications one deals only with one generator, because if several are present chiral fermions cannot be constructed. This means we will stick to $N = 1$ SUSY.

To construct the supersymmetry extended Poincaré algebra, consider the commutator with the momentum operator and a SUSY generator. Bosons and fermions behave the same under (infinitesimal) translations, so the commutator will act on a state to give

$$\begin{aligned}\varepsilon^{\mu}[Q_{\alpha}, P_{\mu}]|\,boson, x^{\mu}\rangle \\ = Q_{\alpha}|\,boson, x^{\mu} + \varepsilon^{\mu}\rangle - \varepsilon^{\mu} P_{\mu}|\,fermion, x^{\mu}\rangle = 0\end{aligned} \tag{3.1}$$

Likewise for $\overline{Q}_{\dot{\alpha}}$. Thus we may conclude that the generators commute with the momentum operator. The same argument can be made with the angular momentum operator, but the conclusion here is that Q_{α} and $\overline{Q}_{\dot{\alpha}}$ do not commute with $\mathbf{J}$, because fermions and bosons behave differently under rotations. This is not strange when considering the generator of Lorentz transformations $M_{\mu\nu}$. Since Q_{α} is a spinorial operator, it should transform as a spinor and we can immediately write down the commutator

$$[Q_{\alpha}, M_{\mu\nu}] = (\sigma_{\mu\nu})_{\alpha}{}^{\beta} Q_{\beta}, \quad [\overline{Q}^{\dot{\alpha}}, M_{\mu\nu}] = (\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \overline{Q}^{\dot{\beta}} \tag{3.2}$$

With this established, $\{Q_{\alpha}, \overline{Q}_{\dot{\alpha}}\}$ must transform as $(\frac{1}{2}, \frac{1}{2})$, i.e a vector. The only vector present in the Poincaré algebra is the momentum operator, hence $\{Q_{\alpha}, \overline{Q}_{\dot{\alpha}}\}$ must be proportional to that

$$\{Q_{\alpha}, \overline{Q}_{\dot{\alpha}}\} = 2\sigma^{\mu}_{\alpha\dot{\alpha}} P_{\mu} \tag{3.3}$$

The $\sigma^{\mu}_{\alpha\dot{\alpha}}$ is there to soak up the indices and the factor 2 is just a convention. The part of the Superpoincaré algebra containing the SUSY generators is, in its full glory,

$$\{Q_{\alpha}, \overline{Q}_{\dot{\beta}}\} = 2\sigma^{\mu}_{\alpha\dot{\alpha}} P_{\mu} \tag{3.4}$$

$$\{Q_{\alpha}, Q_{\beta}\} = \{\overline{Q}_{\dot{\alpha}}, \overline{Q}_{\dot{\beta}}\} = 0 \tag{3.5}$$

$$[P_{\mu}, Q_{\alpha}] = [P_{\mu}, \overline{Q}_{\dot{\alpha}}] = 0 \tag{3.6}$$

$$[Q_{\alpha}, M^{\mu\nu}] = (\sigma^{\mu\nu})_{\alpha}{}^{\beta} Q_{\beta} \tag{3.7}$$

$$[\overline{Q}^{\dot{\alpha}}, M^{\mu\nu}] = (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \overline{Q}^{\dot{\beta}} \tag{3.8}$$

[2] This is true for global SUSY. In supergravity, when supersymmetry is made local, the maximum number of generators is eight. It will not be treated here.

Note in particular that all generators will commute with the operator P^2, meaning that both the bosonic and the fermionic state will have the same mass. This will be the main subject of discussion in the last section.

Next step will be to find out whether the dimension of the representation is useful to describe nature. To do so, we will use the spin projection operator $J^i = \frac{1}{2}\epsilon^{ijk}M^{jk}$. Its behaviour with the SUSY generators is

$$[Q_\alpha, J^i] = \frac{1}{2}\sigma^i_{\alpha\dot{\alpha}}\sigma^{0\dot{\alpha}\beta}Q_\beta, \quad [\overline{Q}^{\dot{\alpha}}, J^i] = \frac{1}{2}\bar{\sigma}^{i\dot{\alpha}\alpha}\sigma^0_{\alpha\dot{\beta}}\overline{Q}^{\dot{\beta}} \tag{3.9}$$

The interpretation here is that Q_1 lowers the spin by $\frac{1}{2}$ and Q_2 raises it by $\frac{1}{2}$. This can be seen by acting on a state $|\,p, j, m\rangle$ (p is the momentum, j is the spin and m spin projection along z-axis)

$$(J^3 Q_\alpha - Q_\alpha J^3)|\,p, j, m\rangle = \frac{1}{2}\sigma^3_{\alpha\dot{\alpha}}\sigma^{\dot{\alpha}\beta}Q_\beta|\,p, j, m\rangle \quad \Rightarrow$$
$$\begin{cases} J^3 Q_1|\,p, j, m\rangle = \left(m - \frac{1}{2}\right) Q_1|\,p, j, m\rangle \\ J^3 Q_2|\,p, j, m\rangle = \left(m + \frac{1}{2}\right) Q_2|\,p, j, m\rangle \end{cases} \tag{3.10}$$

The same goes for $\overline{Q}^{\dot{\alpha}}$. Hence, the space can be constructed if one starts with the state of lowest spin projection $|\,p, j, -j\rangle$. For now, let it be a massless state with four-momentum $p_\mu = E(1, 0, 0, 1)$. Only Q_2 and $\bar{Q}^{\dot{2}} = \bar{Q}_{\dot{1}}$ can give a non-zero result when acting on this state, but the last one must vanish by the SUSY algebra because

$$\begin{aligned} \langle p, j, -j|\,Q_1\overline{Q}_{\dot{1}}|\,p, j, -j\rangle &= \langle p, j, -j|\,\{\overline{Q}_{\dot{1}}, Q_1\}|\,p, j, -j\rangle \\ &= 2\sigma^\mu_{1\dot{1}}\langle p, j, -j|\,P_\mu|\,p, j, -j\rangle = 0 \end{aligned} \tag{3.11}$$

Thus only Q_2 has a nonzero impact and $Q_2|\,p, j, -j\rangle \sim |\,p, j, \frac{1}{2} - j\rangle$. Acting again with Q_2 gives zero because of anticommutivity, as does acting with Q_1. The other possibilities are

$$\begin{aligned} \overline{Q}_{\dot{1}}Q_2|\,p, j, -j\rangle &= \left(\{Q_2, \overline{Q}_{\dot{1}}\} - Q_2\overline{Q}_{\dot{1}}\right)|\,p, j, -j\rangle = \\ &\qquad \left(2\sigma^\mu_{2\dot{1}}P_\mu - Q_2\overline{Q}_{\dot{1}}\right)|\,p, j, -j\rangle = 0 \\ \overline{Q}_{\dot{2}}Q_2|\,p, j, -j\rangle &= \left(\{\overline{Q}_{\dot{2}}, Q_2\} - Q_2\overline{Q}_{\dot{2}}\right)|\,p, j, -j\rangle = \\ &\qquad (2E - Q_2\overline{Q}_{\dot{2}})|\,p, j, -j\rangle \sim |\,p, j, -j\rangle \end{aligned}$$

The end result is that there are only two states in a (massless) supermultiplet. If we take CPT-invariance into consideration, the states with spin projection j and $j - \frac{1}{2}$ must also be included to account for the antiparticles.

The particle types needed to describe nature (disregarding gravitation) are scalars, fermions and gauge bosons. The quarks and leptons are fermions and to describe those we need a supermultiplet of spin projection $(0, \frac{1}{2})$ and its conjugate $(-\frac{1}{2}, 0)$, called the chiral multiplet. The gauge sector have spin-1 bosons so here we must use a $(\frac{1}{2}, 1)$ and $(-1, -\frac{1}{2})$, called the vector multiplet.

A massive multiplet is different. The four-momentum is most conveniently chosen as $p_\mu = (m, \mathbf{0})$, and using this for the lhs in equation (3.11) gives a non-zero result, but the rest is analogous. The massive chiral multiplet is the same as the massless, but the vector multiplet is more complicated and looks like $(-1, -\frac{1}{2}, -\frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, 1)$. This can be gotten from a massless vector multiplet that eats a chiral one through a Higgs mechanism.

The next thing we want to do is construct the transformation induced by the generators onto the fields. Consider the simplest supersymmetric model: a massless chiral multiplet. Here we will use a coordinate basis with the ground state $|\,\Omega\rangle$ of spin 0 such that $|\,x\rangle = \phi(x)|\,\Omega\rangle$, with ϕ a complex scalar field. Impose the constraint $[\phi, \overline{Q}_{\dot{\alpha}}] = 0$ for simplicity.

Some very useful relations are the graded Jacobi identities (see [15]). Let B_i be bosonic operators and F_i fermionic operators, then the Jacobi identity can be generalized to

$$[[B_1, B_2], B_3] + [[B_2, B_3], B_1] + [[B_3, B_1], B_2] = 0 \tag{3.12}$$

$$[[F_1, B_2], B_3] + [[B_2, B_3], F_1] + [[B_3, F_1], B_2] = 0 \tag{3.13}$$

$$[\{F_1, F_2\}, B_3] - \{[F_2, B_3], F_1\} + \{[B_3, F_1], F_2\} = 0 \tag{3.14}$$

$$[\{F_1, F_2\}, F_3] + [\{F_2, F_3\}, F_1] + [\{F_3, F_1\}, F_2] = 0 \tag{3.15}$$

If we use the third identity with $F_1 = Q_\alpha$, $F_2 = \overline{Q}_{\dot{\alpha}}$ and $B_3 = \phi$, the impact of Q_α on ϕ can be worked out by using the SUSY algebra

$$\{[\phi, Q_\alpha], \overline{Q}_{\dot{\alpha}}\} + \{[\phi, \overline{Q}_{\dot{\alpha}}], Q_\alpha\} = [\phi, \{Q_\alpha, \overline{Q}_{\dot{\alpha}}\}] = 2\sigma^\mu_{\alpha\dot{\alpha}}[\phi, P_\mu] \tag{3.16}$$

The rhs has the familiar form of an infinitesimal translation. When P_μ is represented on the field by a differential operator, the relations takes the form

$$[\phi, \{Q_\alpha, \overline{Q}_{\dot{\alpha}}\}] = 2i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\phi \tag{3.17}$$

Now define the fields $\psi_\alpha(x)$, $F_{\alpha\beta}(x)$ and $X_{\alpha\dot{\beta}}(x)$ as

$$[\phi, Q_\alpha] = i\sqrt{2}\psi_\alpha, \quad \{\psi_\alpha, Q_\beta\} = -i\sqrt{2}bF_{\alpha\beta}, \quad \{\psi_\alpha, \overline{Q}_{\dot{\beta}}\} = X_{\alpha\dot{\beta}} \tag{3.18}$$

When playing around with the SUSY algebra and the Jacobi identities, one can work out what the generators do to these fields as well. Equation (3.17) becomes

$$2i\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu\phi = i\sqrt{2}\{\psi_\alpha, \overline{Q}_{\dot{\beta}}\} = i\sqrt{2}X_{\alpha\dot{\beta}} \tag{3.19}$$

With ϕ, Q_α, Q_β and equation (3.14) instead, it is

$$0 = [\phi, \{Q_\alpha, Q_\beta\}] = i\sqrt{2}(\{\psi_\alpha, Q_\beta\} + \{\psi_\beta, Q_\alpha\}) = -2(F_{\alpha\beta} + F_{\beta\alpha}) \tag{3.20}$$

The equation implies that $F_{\alpha\beta}(x) = \epsilon_{\alpha\beta}F(x)$, for some complex scalar field $F(x)$. This must be used to define new fields in the same way ϕ was in equation (3.18).

$$[F, Q_\alpha] = \lambda_\alpha, \quad [F, \overline{Q}_{\dot{\alpha}}] = \overline{\chi}_{\dot{\alpha}} \tag{3.21}$$

Now we can check what the generators do to ψ_α. Using the last Jacobi identity with ψ_α, Q_β and Q_γ yields

$$0 = [\psi_\alpha, \{Q_\beta, Q_\gamma\}] = -i\sqrt{2}(\epsilon_{\alpha\beta}\lambda_\gamma + \epsilon_{\alpha\gamma}\epsilon_\beta) \quad \Rightarrow \quad \lambda_\alpha = 0 \tag{3.22}$$

With $\overline{Q}_{\dot{\beta}}$ instead of Q_γ, the equation reads

$$2i\sigma^\mu_{\beta\dot{\beta}}\partial_\mu\psi_\alpha = [\psi_\alpha, \{Q_\beta, \overline{Q}_{\dot{\beta}}\}] = -i\sqrt{2}\epsilon_{\alpha\beta}\overline{\chi}_{\dot{\beta}} + 2i\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu\psi_\beta \quad \Rightarrow$$
$$\chi_{\dot{\beta}} = -\sqrt{2}\partial_\mu\psi^\alpha\sigma^\mu_{\alpha\dot{\beta}} \tag{3.23}$$

Already the fields defined in equation (3.21) have been expressed in the original definitions and the rest of the commutators are superfluous, but need to be checked for consistency. Three of them are trivial

$$[\psi_\alpha, \{\overline{Q}_{\dot{\alpha}}, \overline{Q}_{\dot{\beta}}\}] = [F, \{Q_\alpha, Q_\beta\}] = 0, \qquad [F, \{Q_\alpha, \overline{Q}_{\dot{\beta}}\}] = 2i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu F$$

The final commutator is

$$[F, \{\overline{Q}_{\dot{\alpha}}, \overline{Q}_{\dot{\beta}}\}] = i8\sqrt{2}\bar{\sigma}^{\mu\nu}_{\dot{\alpha}\dot{\beta}}\partial_\mu\partial_\nu\phi = 0 \tag{3.24}$$

The rhs is zero because $\bar{\sigma}^{\mu\nu}$ is antisymmetric under $\mu \leftrightarrow \nu$. To define a SUSY transformation on the fields, introduce the Grassman numbers ξ^α and $\bar{\xi}_{\dot{\alpha}}$. An infinitesimal transformation is written in the usual way as

$$(\delta_\xi + \delta_{\bar{\xi}})\Phi = -i[\Phi, \xi Q + \overline{\xi Q}] \tag{3.25}$$

The transformations on the fields are thus

$$(\delta_\xi + \delta_{\bar{\xi}})\phi = \sqrt{2}\xi\psi \tag{3.26}$$
$$(\delta_\xi + \delta_{\bar{\xi}})\psi_\alpha = i\sqrt{2}\sigma^\mu_{\alpha\dot{\alpha}}\bar{\xi}^{\dot{\alpha}}\partial_\mu\phi + \sqrt{2}\xi_\alpha F \tag{3.27}$$
$$(\delta_\xi + \delta_{\bar{\xi}})F = -i\sqrt{2}\partial_\mu\psi\sigma^\mu\bar{\xi} \tag{3.28}$$

The constants, signs and factors of i can be placed, more or less, in whatever place you feel is best. Here they have been chosen to get the same transformations as in [14].

3.2 The Superfield Formalism

Unfortunately, when dealing with supersymmetry even the simplest computation is rather lengthy, with lots of opportunities to make mistakes. When computing something more complicated, such as proving that the gauge sector of the MSSM is invariant under a SUSY-transformation, there is a vast number of terms to consider and it is hard to get an overview. For that matter, a new formalism was introduced using the concept of superspace and superfields. There are a lot of good reviews on this and here mostly [13] and [12] have been used, with conventions according to [14]. Any of these offer more information than what is presented below.

The basic idea is to construct fields that behave in such a way that the SUSY-transformations can be represented as differential operators on some space, in analogy to how the momentum operator can be represented as a derivative in spacetime. In order to do this some formalism is needed. The SUSY generators obey anticommutation relations and there is little chance to get the normal spacetime coordinates to behave in such a way. The thing to do is extend space with new coordinates that are Grassman numbers, and anticommute naturally. Choose four such coordinates, θ^α and $\bar{\theta}_{\dot{\alpha}}$ where $\alpha = 1, 2$

$$\{\theta^\alpha, \theta^\beta\} = \{\bar{\theta}_{\dot{\alpha}}, \bar{\theta}_{\dot{\beta}}\} = \{\theta^\alpha, \bar{\theta}_{\dot{\alpha}}\} = 0 \tag{3.29}$$

Spacetime extended in this way, with four fermionic dimensions, will from now on be called superspace. Any function of such variables will be very simple, because the Taylor expansion cancels after second order as $\theta_\alpha\theta_\beta\theta_\gamma = 0$. The basic rules for differentiation and integration can be found in appendix D.

A general superfield is an arbitrary function $F = F(x, \theta, \overline{\theta})$ and after expanding in θ and $\bar{\theta}$, it can be written on the form

$$\begin{aligned} F(x,\theta,\overline{\theta}) = f(x) + \theta\psi(x) + \overline{\theta\chi}(x) + \theta\theta\, m(x) + \overline{\theta\theta}\, n(x) + \theta\sigma^\mu\bar{\theta}v_\mu(x) \\ + \theta\theta\,\overline{\theta\lambda}(x) + \overline{\theta\theta}\,\theta\rho(x) + \theta\theta\,\overline{\theta\theta}\, d(x) \end{aligned} \tag{3.30}$$

The component fields get their properties based on their relation to θ and $\bar{\theta}$. The fields f, m, n and d must be scalars, while ψ_α, ρ_α, $\bar{\chi}^{\dot{\alpha}}$ and $\bar{\lambda}^{\dot{\alpha}}$ are two-component Weyl spinors, and v_μ must be a vector.

To represent the generators as differential operators, postulate that ξQ generates a linear translation by ξ^α in θ^α, plus some other translation in x^μ. For the superfield F, this means

$$(1 + \xi Q)F(x, \theta, \bar{\theta}) = F(x + \delta x, \theta + \xi, \bar{\theta}) \tag{3.31}$$

$$(1 + \overline{\xi Q})F(x, \theta, \bar{\theta}) = F(x + \delta x^\dagger, \theta, \bar{\theta} + \bar{\xi}) \tag{3.32}$$

To satisfy it, one possible representation is

$$Q_\alpha = \partial_\alpha - i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu \tag{3.33}$$

$$\overline{Q}^{\dot{\alpha}} = \overline{\partial}^{\dot{\alpha}} + i\bar{\sigma}^{\mu\dot{\alpha}\alpha}\theta_\alpha\partial_\mu \tag{3.34}$$

To make sense, the above expressions must also satisfy the algebra in equation (3.4), and it is easy to check that it does. A SUSY transformation on a superfield is written as (compare to the component transformations in equation (3.26))

$$(\delta_\xi + \delta_{\bar{\xi}})F = (\xi Q + \overline{\xi Q})F \tag{3.35}$$

The individual component transformations are identified by their dependence on θ and $\bar{\theta}$. For example: the scalar term after a transformation is $\sqrt{2}(\xi\psi + \overline{\xi\chi})$, so this must be the transformation law of the (complex) scalar field f.

We can furthermore define two differential operators that anticommute with the generators and amongst themselves

$$D_\alpha = \partial_\alpha + i(\sigma^\mu\bar{\theta})_\alpha\partial_\mu \tag{3.36}$$

$$\overline{D}_{\dot{\alpha}} = -\overline{\partial}_{\dot{\alpha}} - i(\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu \tag{3.37}$$

$$\{D_\alpha, D_\beta\} = \{\overline{D}_{\dot{\alpha}}, D_{\dot{\beta}}\} = 0, \quad \{D_\alpha, \overline{D}_{\dot{\beta}}\} = -2i\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu \tag{3.38}$$

$$\{D_\alpha, Q_\beta\} = \{D_\alpha, \overline{Q}_{\dot{\beta}}\} = \{\overline{D}_{\dot{\alpha}}, Q_\beta\} = \{\overline{D}_{\dot{\alpha}}, \overline{Q}_{\dot{\beta}}\} = 0 \tag{3.39}$$

These will be called covariant derivatives because $Q_\alpha(D_\beta\Phi) = -D_\beta(Q_\alpha\Phi)$. A general superfield has too many components to be of much use, but the covariant derivatives can be used to impose constraints. Define a chiral (antichiral for the conjugate) superfield by

$$\overline{D}_{\dot{\alpha}}\Phi(x,\theta,\bar{\theta}) = 0 \tag{3.40}$$

$$D_\alpha\overline{\Phi}(x,\theta,\bar{\theta}) = 0 \tag{3.41}$$

As usual $\overline{\Phi} = \Phi^\dagger$. The solution to equation (3.40) would be very simple if the field depended only on θ and the spacetime coordinate $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$, because $\overline{D}_{\dot{\alpha}}y^\mu = 0$. The covariant derivative of the field would also be zero in that case because of the chain rule. The solution can therefore be written

$$\Phi(y,\theta) = A(y) + \sqrt{2}\theta\psi(y) + \theta\theta\, F(y) \tag{3.42}$$

$$\overline{\Phi}(y^\dagger,\bar{\theta}) = A^*(y^\dagger) + \sqrt{2}\,\overline{\theta\psi}(y^\dagger) + \overline{\theta\theta}\, F^*(y^\dagger) \tag{3.43}$$

With $y^{\dagger\mu} = x^\mu - i\theta\sigma^\mu\bar{\theta}$. The full component expansion is gotten when inserting the expression for y and expanding again

$$\Phi(x,\theta,\bar{\theta}) = A(x) + i\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}A(x) + \frac{1}{4}\theta\theta\,\overline{\theta\theta}\,\partial^{\mu}\partial_{\mu}A(x) + \sqrt{2}\theta\psi(x) + \frac{i}{\sqrt{2}}\theta\theta\,\partial_{\mu}\psi(x)\sigma^{\mu}\bar{\theta} + \theta\theta\,F(x) \tag{3.44}$$

$$\overline{\Phi}(x,\theta,\bar{\theta}) = A^{*}(x) - i\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}A^{*}(x) + \frac{1}{4}\theta\theta\,\overline{\theta\theta}\,\partial^{\mu}\partial_{\mu}A^{*}(x) + \sqrt{2}\overline{\theta}\overline{\psi}(x) + \frac{i}{\sqrt{2}}\overline{\theta\theta}\,\theta\sigma^{\mu}\partial_{\mu}\overline{\psi}(x) + \overline{\theta\theta}\,F^{*}(x) \tag{3.45}$$

The highest component in Φ is the auxiliary field F and the rest are space-time derivatives. Therefore F must transform into a total derivative under a SUSY transformation

$$(\delta_{\xi} + \delta_{\bar{\xi}})F = i\sqrt{2}\partial_{\mu}\psi\sigma^{\mu}\overline{\xi} \tag{3.46}$$

This will be useful when constructing an action later on. Doing things by components necessarily gets messy, simply because of the number of terms that have to be included for a consistent theory. The point of having superfields is that they contain essentially the same information, but in a much more compact notation. A very convenient definition is

$$\Phi|_{\theta=\bar{\theta}=0} = A(x) \tag{3.47}$$

$$D_{\alpha}\Phi|_{\theta=\bar{\theta}=0} = \psi_{\alpha}(x) \tag{3.48}$$

$$D^{2}\Phi\Big|_{\theta=\bar{\theta}=0} = F(x) \tag{3.49}$$

The normalization of the fields in these two points of view is not the same[3], but it makes no difference which definition is chosen if the conventions are followed.

Another superfield that needs to be mentioned is the vector superfield, defined by the reality condition

$$V(x,\theta,\bar{\theta}) = V^{\dagger}(x,\theta,\bar{\theta}) \tag{3.50}$$

The expansion in component fields is

$$V(x,\theta,\bar{\theta}) = C(x) + i\theta\chi(x) - i\overline{\theta\chi}(x) + \frac{i}{2}\theta\theta\,(M(x) + iN(x)) - \frac{i}{2}\overline{\theta\theta}\,(M(x) - iN(x)) - \theta\sigma^{\mu}\bar{\theta}v_{\mu}(x) + i\theta\theta\,\bar{\theta}\left(\overline{\lambda}(x) + \frac{i}{2}\bar{\sigma}^{\mu}\partial_{\mu}\chi(x)\right) - i\overline{\theta\theta}\,\theta\left(\lambda(x) + \frac{i}{2}\sigma^{\mu}\partial_{\mu}\overline{\chi}(x)\right) + \frac{1}{2}\theta\theta\,\overline{\theta\theta}\,\left(D(x) + \frac{1}{2}\partial^{\mu}\partial_{\mu}C(x)\right) \tag{3.51}$$

[3] To converge with the notation in [14], the rhs of equation (3.48) needs to be multiplied by $\frac{1}{\sqrt{2}}$ and similarly in equation (3.49) a factor $\frac{1}{4}$.

With C, D, M, N all real. Note that under a SUSY transformation the D-field will transform as a total derivative

$$(\delta_\xi + \delta_{\bar{\xi}})D = -\frac{1}{2}\left[\xi\sigma^\mu\partial_\mu\overline{\lambda} + \partial_\mu\lambda\sigma^\mu\overline{\xi} - \frac{i}{2}\partial_\mu\partial_\nu\chi\sigma^\nu\bar{\sigma}^\mu\xi - \frac{i}{2}\partial_\mu\partial_\nu\overline{\chi}\bar{\sigma}^\nu\sigma^\mu\overline{\xi}\right] \tag{3.52}$$

The number of component fields can be reduced considerably by noting that the combination $\Phi + \bar{\Phi}$ is a vector superfield. This can be seen as a gauge transformation (because v_μ transforms as $v'_\mu \to v_\mu + i\partial_\mu\Lambda$) and all components except λ_α, D and v_μ can be set to zero. Therefore V can be divided into parts: $V = V_{WZ} + \Phi + \bar{\Phi}$, where

$$V_{WZ} = -\theta\sigma^\mu\bar{\theta}v_\mu + i\theta\theta\,\overline{\theta\lambda}(x) - i\overline{\theta\theta}\,\theta\lambda(x) + \frac{1}{2}\theta\theta\,\overline{\theta\theta}\,D(x) \tag{3.53}$$

This is called Wess-Zumino gauge. The field, however, does not respect SUSY transformations since it has too few components to give the correct relations. Let us use it to define two particular chiral superfields, the left- and right-handed spinor superfields

$$W_\alpha = -\frac{1}{4}\overline{DD}D_\alpha V_{WZ}, \quad \overline{W}_{\dot{\alpha}} = -\frac{1}{4}DD\overline{D}_{\dot{\alpha}}V_{WZ} \tag{3.54}$$

W_α is naturally chiral ($\overline{W}_{\dot{\alpha}}$ anti-chiral), since $D_\alpha D_\beta D_\gamma = 0$ because of anticommutivity, and furthermore

$$\overline{DD}D_\alpha V = \overline{DD}D_\alpha(V_{WZ} + \Phi + \overline{\Phi}) = \overline{DD}D_\alpha V_{WZ} \tag{3.55}$$

The component expansion is given, in functions of the bosonic coordinates y and $y^\dagger$, by

$$\begin{aligned} W_\alpha &= -i\lambda_\alpha(y) + \theta_\alpha D(y) - \frac{i}{2}(\sigma^\mu\bar{\sigma}^\nu\theta)_\alpha(\partial_\mu v_\nu(y) - \partial_\nu v_\mu(y)) \\ &\quad + \theta\theta\,(\sigma^\mu\partial_\mu\bar{\lambda}(y))_\alpha \end{aligned} \tag{3.56}$$

$$\begin{aligned} \overline{W}_{\dot{\alpha}} &= i\bar{\lambda}_{\dot{\alpha}}(y^\dagger) + \bar{\theta}_{\dot{\alpha}}D(y^\dagger) + \frac{i}{2}(\bar{\sigma}^\mu\sigma^\nu\bar{\theta})_{\dot{\alpha}}(\partial_\mu v_\nu(y^\dagger) - \partial_\nu v_\mu(y^\dagger)) \\ &\quad + \epsilon_{\dot{\alpha}\dot{\beta}}\overline{\theta\theta}\,\bar{\sigma}^{\mu\dot{\beta}\alpha}\partial_\mu\lambda_\alpha(y^\dagger) \end{aligned} \tag{3.57}$$

There is also a reality constraint $D^\alpha W_\alpha = \overline{D}_{\dot{\alpha}}\overline{W}^{\dot{\alpha}}$. These fields respect SUSY transformations and include a vector, which means that they may be used to describe gauge fields.

The next question is how to use the new tools to construct Lagrangians and actions. For an arbitrary number of chiral superfields the most general, supersymmetrically invariant action is

$$S = \int d^4x\left[\int d^4\theta\,K(\Phi_i, \overline{\Phi}_{\bar{j}}) + \left(\int d^2\theta\,W(\Phi_i) + c.c\right)\right] \tag{3.58}$$

The function K is the called the Kähler potential and it is a vector superfield. The superpotential $W(\Phi_i)$ is a chiral superfield and it must be an analytic function, so it cannot depend on $\overline{\Phi}_{\bar{j}}$. The SUSY invariance is easy to prove by using the equations (3.46) and (3.52). It is not renormalizable, in general, and imposing this condition results in the Wess-Zumino model

$$\mathcal{L}_{WZ} = \int d^4\theta\, \overline{\Phi}_{\bar{i}}\Phi_i + \left(\int d^2\theta\, \lambda_i\Phi_i + \frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3}g_{ijk}\Phi_i\Phi_j\Phi_k\right) \tag{3.59}$$

When dealing with non-abelian fields equation (3.58) must be generalized, for more information see [12]. Another possibility is to construct an action with the spinor superfields. The prescribed combination is

$$S = \int d^4x \frac{1}{4}\left[\int d^2\theta\, W^\alpha W_\alpha + \int d^2\bar{\theta}\, \overline{W}_{\dot{\alpha}}\overline{W}^{\dot{\alpha}}\right] \tag{3.60}$$

This is works even for a non-abelian gauge group. An interesting application of this action is found in section 5.

The final thing that will be mentioned is the classical scalar potential. It is the basic working tool when analyzing SUSY breaking. Consider the action in equation (3.58) and swap Berezin integration for differentiation

$$\begin{aligned}
\int d^4\theta\, K(\Phi_i, \overline{\Phi}_{\bar{i}}) &= D^2\,\overline{D}^2\, K(\Phi_i, \overline{\Phi}_{\bar{j}})\Big|_{\theta=\bar{\theta}=0} \\
\int d^2\theta\, W(\Phi_i) &= D^2\, W(\Phi_i)\Big|_{\theta=\bar{\theta}=0} \\
\int d^2\bar{\theta}\, \overline{W}(\overline{\Phi}_{\bar{i}}) &= \overline{D}^2\, \overline{W}(\overline{\Phi}_{\bar{i}})\Big|_{\theta=\bar{\theta}=0}
\end{aligned}$$

Hitting K repeatedly yields lots of terms, but only those depending on the auxiliary field are interesting for the classical potential. The other terms are couplings that do not come into the potential. The notation is

$$\partial_i = \frac{\partial}{\partial\Phi_i}, \quad \bar{\partial}_{\bar{i}} = \frac{\partial}{\partial\overline{\Phi}_{\bar{i}}}$$

The above relations are equivalent to derivatives with respect to the component scalar field ϕ, because $K(\Phi, \overline{\Phi})|_{\theta=\bar{\theta}=0} = K(\phi, \phi^\dagger)$, as we've taken it to be independent of covariant derivatives. For the scalar potential, the interesting part of the Lagrangian is

$$\begin{aligned}
\mathcal{L}_{auxiliary} = \partial_i\bar{\partial}_{\bar{j}}K\, F^i F^{\bar{j}*} + \partial_i\bar{\partial}_{\bar{j}}\bar{\partial}_{\bar{k}}K\, F^i\bar{\psi}^{\bar{j}}\bar{\psi}^{\bar{k}} + \partial_i\partial_j\bar{\partial}_{\bar{k}}K\, \psi^i\psi^j F^{\bar{k}*} \\
+ \partial_i W\, F^i + \bar{\partial}_{\bar{i}}\overline{W}\, F^{\bar{i}*}
\end{aligned} \tag{3.61}$$

The term $\partial_i\bar{\partial}_{\bar{j}}K = g_{i\bar{j}}$ is called the Kähler metric and plays a big role in supergravity. The fermion terms will not be a part of the classical potential and we can neglect them. The auxiliary field can be solved for by the normal procedure of functional derivatives. The classical potential is defined as

$$V = -g_{i\bar{j}} F^{\bar{j}*} F^i = g^{i\bar{j}} \partial_i W \bar{\partial}_{\bar{j}} \overline{W} \tag{3.62}$$

With this final piece of the puzzle we can at last begin to use this new formalism.

4

A SUSY Breaking Sector

The first thing a phenomenologist would want to do with a brand new symmetry, is to build an extended version of the standard model and this has been done, see [1] or [2]. The Lagrangian of the MSSM was first written down using component fields and the result is rather long, but with the superfield formulation it is much more compact.

It is certainly encouraging to know that it is possible to construct a supersymmetric extension of the standard model, but there are a great many things that need a more detailed analysis. Here we will be interested in supersymmetry breaking, in other words, explaining how and why the standard model particles and their supersymmetric partners have different masses. For a deeper review see [16] and [17], while more phenomenological arguments can be found in [1] or [18].

4.1 A SUSY Breaking Model

In section 3.1 it was briefly commented that, because the SUSY generators commute with P^2, the masses of both the particle and its supersymmetric partner are the same. This is trivial to show

$$P^2|\,\phi\rangle = m^2|\,\phi\rangle \;\Rightarrow\; P^2\left(Q_\alpha|\,\phi\rangle\right) = Q_\alpha P^2|\,\phi\rangle = m^2\left(Q_\alpha|\,\phi\rangle\right) \tag{4.1}$$

From a phenomenological point of view this is obviously wrong. In that case supersymmetry would have been discovered long ago and the superpartners would have been found at the same time the standard model particles were. This was not the case and that implies that supersymmetry is not an exact symmetry, but must somehow be broken. Theoretically we would like it to be spontaneously broken, which means that when the fields get their vacuum expectation values (vev) the Lagrangian is still invariant but the ground state is not. For SUSY this happens if the minimum of the scalar potential is non-zero.

However, the complicated structure of the MSSM makes it more or less impossible to break in such a way. One way to avoid this difficulty is to let the breaking take place in a different gauge sector than the $SU(3) \times SU(2) \times U(1)$ of the standard model, and then let the breaking be mediated by some other field.

An example of a possible SUSY breaking sector is the O'Raifeartaigh model (see [19]), defined by the Lagrangian

$$\begin{aligned}
&\mathcal{L} = \int d^4\theta \, K(\mathbf{X}, \overline{\mathbf{X}}, \Phi_1, \overline{\Phi}_{\bar{1}}, \Phi_2, \overline{\Phi}_{\bar{2}}) + \left(\int d^2\theta \, W(\mathbf{X}, \Phi_1, \Phi_2) + c.c \right) \\
&K = \mathbf{X}\overline{\mathbf{X}} + \Phi_1 \overline{\Phi}_{\bar{1}} + \Phi_2 \overline{\Phi}_{\bar{2}}, \quad W = \frac{h}{2} \mathbf{X} \Phi_1^2 + m \Phi_1 \Phi_2 + f \mathbf{X} \\
&\mathbf{X} = X + \sqrt{2}\theta\psi_X + \theta\theta \, F, \quad \Phi_i = \phi_i + \sqrt{2}\theta\psi_i + \theta\theta \, F_i
\end{aligned} \tag{4.2}$$

Note that the superfield $\mathbf{X}$ is written in bold and its scalar component X in normal font. This is only a toy model and it does not describe nature (the universe would be a boring place if it did), but it is a good place to test new ideas. The classical potential is given by formula (3.62) and takes the following form

$$V = |hX\phi_1 + m\phi_2|^2 + \left| \frac{h}{2}\phi_1^2 + f \right|^2 + |m\phi_1|^2 \tag{4.3}$$

All three constants can be made real by rotating each field by a phase, but to be slightly more general they will be considered complex. The fields will have their vevs at the minimum of the potential

$$\begin{aligned}
\frac{\partial V}{\partial X^*} &= h^* \phi_1^* (hX\phi_1 + m\phi_2) \\
\frac{\partial V}{\partial \phi_2^*} &= m^* (hX\phi_1 + m\phi_2) \\
\frac{\partial V}{\partial \phi_1^*} &= h^* X^* (hX\phi_1 + m\phi_2) + h^* \phi_1^* \left(\frac{h}{2}\phi_1^2 + f \right) + |m|^2 \phi_1
\end{aligned}$$

Choosing $\langle \phi_2 \rangle = \frac{h\langle X \rangle \langle \phi_1 \rangle}{m}$ sets the first and second equation to zero, but the third one must be analyzed more thoroughly. Writing $\phi_1(x) = Re^{i\theta}$ yields

$$\begin{aligned}
\text{Re}: \quad & |m|^2 R\cos\theta + \frac{1}{2}|h|^2 R^3 \cos\theta + fh^* R\cos\theta = 0 \\
\text{Im}: \quad & |m|^2 R\sin\theta + \frac{1}{2} R^3 \sin\theta - fh^* R\sin\theta = 0
\end{aligned}$$

Both equations are satisfied if $\langle \phi_1 \rangle = 0$, but there may be a second solution for

$$\theta = \pm\frac{\pi}{2}, \quad R^2 = \frac{2}{|h|^2}(fh^* - |m|^2)$$

This is only possible if the dimensionless parameter $\xi = \frac{fh^*}{|m|^2} > 1$, in which case there will be two different branches of SUSY breaking vacua. Here we will only be interested in the case $\xi < 1$, and the potential is minimal when

$$\langle\phi_1\rangle = \langle\phi_2\rangle = 0, \quad \langle X\rangle \,\text{arbitrary} \quad \Rightarrow \quad V_{min} = |f|^2 \tag{4.4}$$

There is no constraint for $\langle X\rangle$ and SUSY will be broken regardless of its value. The degeneracy is lifted when quantum effects are considered (first order loop corrections) and that is the reason why we refer to X as the pseudomodulus.

Next up is to compute the spectrum of the theory, i.e the masses of the particles. To find the scalar masses, let the fields take their vev plus some small quantum fluctuation $\phi_i(x) \to \langle\phi_i\rangle + \varphi_i(x)$. Expand the classical potential to second order in φ_i, and gather the couplings into a matrix. The potential is given in terms of the superfields, so the easiest way to get the matrix is to compute the Hessian and arrange it to be Hermitian

$$M_b^2 = \begin{pmatrix} \partial_{\varphi_1}\partial_{\overline{\varphi}_1}V & \partial_{\varphi_1}\partial_{\varphi_1}V & \partial_{\varphi_1}\partial_{\overline{\varphi}_2}V & \partial_{\varphi_1}\partial_{\varphi_2}V \\ \partial_{\overline{\varphi}_1}\partial_{\overline{\varphi}_1}V & \partial_{\overline{\varphi}_1}\partial_{\varphi_1}V & \partial_{\overline{\varphi}_1}\partial_{\overline{\varphi}_2}V & \partial_{\overline{\varphi}_1}\partial_{\overline{\varphi}_2}V \\ \partial_{\varphi_2}\partial_{\overline{\varphi}_1}V & \partial_{\varphi_1}\partial_{\varphi_1}V & \partial_{\varphi_2}\partial_{\overline{\varphi}_2}V & \partial_{\varphi_2}\partial_{\overline{\varphi}_2}V \\ \partial_{\overline{\varphi}_2}\partial_{\overline{\varphi}_1}V & \partial_{\overline{\varphi}_2}\partial_{\varphi_1}V & \partial_{\overline{\varphi}_2}\partial_{\overline{\varphi}_2}V & \partial_{\overline{\varphi}_2}\partial_{\varphi_2}V \end{pmatrix}$$

To get the masses the matrix must be diagonalized. For convenience, define a second dimensionless parameter $x = h\langle X\rangle/m^*$. The eigenvalues are

$$m^2_{b_1,b_2} = \frac{|m|^2}{2}\left(2 + |x|^2 + |\xi| \pm \sqrt{4|x|^2 + |x|^4 + |\xi|^2 + 2|\xi||x|^2}\right) \tag{4.5}$$

$$m^2_{b_3,b_4} = \frac{|m|^2}{2}\left(2 + |x|^2 - |\xi| \pm \sqrt{4|x|^2 + |x|^4 + |\xi|^2 - 2|\xi||x|^2}\right) \tag{4.6}$$

$$m^2_{X,\bar{X}} = 0$$

There are two bosonic masses for each field, corresponding to the two degrees of freedom from one complex scalar field. The fermion masses are easier to compute as the mass matrix can be gotten straight from the superpotential, without having to go all the way around the classical potential.

$$M_f = \begin{pmatrix} \partial^2_{\Phi_1}W(\Phi_i) & \partial_{\Phi_1}\partial_{\Phi_2}W(\Phi_i) \\ \partial_{\Phi_1}\partial_{\Phi_2}W(\Phi_i) & \partial^2_{\Phi_2}W(\Phi_i) \end{pmatrix} \tag{4.7}$$

Diagonalizing gives the following eigenvalues

$$m^2_{f_1,f_2} = \frac{|m|^2}{2}\left(|x|^2 + 2 \pm \sqrt{|x|^4 + 4|x|^2}\right) \tag{4.8}$$

$$m^2_{\psi_X} = 0$$

The massless fermion of the $\mathbf{X}$ field is the Goldstino and it will always appear when SUSY is broken, in the same way a Goldstone boson does when a global symmetry is broken.

In order to compute the quantum correction it is convenient to define the full mass matrix $\mathcal{M}^2$

$$\mathcal{M}^2 = \begin{pmatrix} M_b^2 & 0 \\ 0 & M_f^2 \end{pmatrix} \tag{4.9}$$

Also, define the supertrace as $\mathrm{Str}(\mathcal{M}^2) = \mathrm{tr}\, M_b^2 - 2\mathrm{tr}\, M_f^2$. Note that there are an equal number of fermionic and bosonic degrees of freedom, which means that $\mathrm{Str}(1) = n_b - n_f = 0$.

4.2 An Effective Field Theory

Our mission here is to find an effective potential from the O'Raifeartaigh model in the previous chapter. First introduce the energy scale Λ: above it there may be some UV-complete theory and below there is an O'Raifeartaigh model. Furthermore, let Φ_1 and Φ_2 be heavy superfields, i.e $m_{\phi_1} \sim m_{\phi_2} >> m_X$. If we are interested only in energy lower than m_ϕ, we can integrate out the two heavy fields to get an effective theory for the remaining light field.

To get an effective theory one normally computes an effective Kähler potential. If there are terms in the superpotential that depend only on the light field, they are kept and provide a (effective) superpotential below the cut-off m_ϕ. In the specific case of the O'Raifeartaigh-model, the term $f\mathbf{X}$ will be unaffected and provide a Polony model. The effective Kähler potential is computed through loop calculations (see [20]), and can be specified to whatever order one prefers.

Usually this is good enough. However, if the SUSY breaking is strong something else is required, because the auxiliary field appears at most quadratically in the effective Kähler potential and by equation (3.62) we see that the auxiliary field encodes the SUSY breaking. Therefore, if higher order terms in F are neglected their information will be lost. If the breaking is weak on the other hand, the higher order terms will be very small and little information is lost when neglecting them.

To treat the problem of strong SUSY breaking we will have to generalize the Kähler potential. We denote its generalization by H and let it depend on $\mathbf{X}$, $D_\alpha \mathbf{X} D^\alpha \mathbf{X}$ and $D^2\mathbf{X}$. For simplicity, let $\mathbf{X}$ be constant in spacetime so that $\partial_\mu \mathbf{X} = 0$ (called a spurion field). Recall that $\mathbf{X}$ is a chiral superfield and thus H must be on the form

$$H = \mathbf{X}^a (D^\alpha \mathbf{X} D_\alpha \mathbf{X})^b (D^2 \mathbf{X})^c \overline{\mathbf{X}}^e (\overline{\mathbf{D}}_{\dot{\alpha}} \overline{\mathbf{X}} \overline{D}^{\dot{\alpha}} \overline{\mathbf{X}})^f (\overline{D}^2 \overline{\mathbf{X}})^g \tag{4.10}$$

With only $b, f = 0, 1$ because of anticommutivity. The Lagrangian is

$$\mathcal{L} = \int d^4\,\theta H(\mathbf{X}, \overline{\mathbf{X}}, D_\alpha \mathbf{X}, \overline{D}_{\dot\alpha}\overline{\mathbf{X}}, D^2\mathbf{X}, \overline{D}^2\overline{\mathbf{X}}) + \left(\int d^2\theta\, f\mathbf{X} + \text{c.c}\right) \tag{4.11}$$

The $D^\alpha \mathbf{X} D_\alpha \mathbf{X}$-term (likewise for the conjugate) can be removed because integration by parts yields

$$\begin{aligned} &D^\alpha \left(\mathbf{X}^{a+1} D_\alpha \mathbf{X} (D^2\mathbf{X})^c \overline{\mathbf{X}}^e (\overline{D}_{\dot\alpha}\overline{\mathbf{X}}\overline{D}^{\dot\alpha}\overline{\mathbf{X}})^f (\overline{D}^2\overline{\mathbf{X}})^g\right) \\ &= (a+1)\mathbf{X}^a (D^\alpha \mathbf{X} D_\alpha \mathbf{X})^b (D^2\mathbf{X})^c \overline{\mathbf{X}}^e (\overline{D}_{\dot\alpha}\overline{\mathbf{X}}\overline{D}^{\dot\alpha}\overline{\mathbf{X}})^f (\overline{D}^2\overline{\mathbf{X}})^g \\ &+ \mathbf{X}^{a+1} (D^2\mathbf{X})^{c+1} \overline{\mathbf{X}}^e (\overline{D}_{\dot\alpha}\overline{\mathbf{X}}\overline{D}^{\dot\alpha}\overline{\mathbf{X}})^f (\overline{D}^2\overline{\mathbf{X}})^g \end{aligned} \tag{4.12}$$

The lhs is zero when it is hit by D^2. This will always be the case since we will deal with H only through the Lagrangian. Thus the generalized Kähler potential can be chosen with $b = f = 0$ in equation (4.10).

The bosonic part of the action is gotten as usual but with some extra difficulty

$$S = \int d^4x\ [\partial_X \partial_{\bar{X}} H F F^* + fX + f^* X^*] \tag{4.13}$$

Here H is a function not only of X but also of F, so when solving for the auxiliary field the equation is more complicated than usual

$$F(1 + F^* \partial_{F^*}) \partial_X \partial_{\overline{X}} H + f^* = 0 \tag{4.14}$$

Likewise for the conjugate. Putting this result back into the Lagrangian gives the classical (scalar) potential on the form

$$V = |f|^2 \frac{(1 - F\partial_F - F^*\partial_{F^*})\partial_X \partial_{\overline{X}} H}{|1 + F\partial_F \partial_X \partial_{\overline{X}} H|^2} \tag{4.15}$$

There are also some new couplings for the Goldstino. When computing $D^2 \overline{D}^2 H$ by brute force the new terms that appear are

$$\mathcal{L}_{new} = -\frac{F}{2}\bar{\psi}^2 \partial_X \partial_{\overline{X}} \partial_{\overline{X}} H - \frac{F^*}{2}\psi^2 \partial_X \partial_X \partial_{\overline{X}} H + \frac{1}{4}\psi^2 \bar{\psi}^2 \partial_X \partial_X \partial_{\overline{X}} \partial_{\overline{X}} H \tag{4.16}$$

For now, let us compute H to one loop for the specific case of the O'Raifeartaigh model. One way to do it is by a path integral, see [21] for a review. Consider a Lagrangian (in Euclidean space)

$$\mathcal{L}_{heavy} = \frac{1}{2}(\partial\phi)^2 + \frac{m}{2}\phi^2, \quad m = m(X) \tag{4.17}$$

Where the mass is a function of the light field X. The generating functional is

$$\int \mathcal{D}\phi \exp\left(-\frac{1}{2}\int d^4x \left((\partial\phi)^2 + m\phi^2\right)\right) = \frac{1}{\sqrt{\det\left(\frac{-\partial^2+m^2)}{\Lambda^2}\right.}} \tag{4.18}$$

But this can be thought of as a potential for the X-field, because it is the only thing left after integration. Thus we can write

$$\int \mathcal{D}\phi \exp\left(-\frac{1}{2}\int d^4x \left((\partial\phi)^2 - m\phi^2\right)\right) = \exp\left(\int d^4x\, V(X)\right) \tag{4.19}$$

The rhs of these two equations can be identified, but the resulting expressions must be massaged to a sensible form. Take the logarithm of both to get

$$\int d^4x\, V(X) = -\frac{1}{2}\log\left[\det\left(\frac{-\partial^2+m^2}{\Lambda^2}\right)\right] \tag{4.20}$$

The rhs needs some interpretation. By diagonalizing the matrix the determinant can be reduced to the trace of the operator. This in turn must be interpreted as

$$\mathrm{tr}\,\mathcal{O} = \mathrm{tr}\left(\{\langle x|\,\mathcal{O}|\,y\rangle\}\right) = \int d^4x\,\langle x|\,\mathcal{O}|\,x\rangle \tag{4.21}$$

Equation (4.20) is thus

$$\int d^4x\, V(X) = -\frac{1}{2}\int d^4x \int \frac{d^4p}{(2\pi)^4}\log\left(\frac{p^2+m^2}{\Lambda^2}\right) \tag{4.22}$$

Where we have gone over to a momentum basis to get rid of the differential operator. The integration over space may diverge, but it is just the volume of space itself and should not cause too much worry. The integrands can be identified, with the result

$$\begin{aligned} V(X) &= -\frac{1}{2}\int_{|p|\leq\Lambda}\frac{d^4p}{(2\pi)^2}\log\left(\frac{p^2+m^2}{\Lambda^2}\right) \\ &= \frac{1}{64\pi^2}\left[\Lambda^4\left(\frac{1}{2}-\log\left(\frac{\Lambda^2+m^2}{\Lambda^2}\right)\right)+\Lambda^2m^2+m^4\log\left(\frac{m^2}{\Lambda^2+m^2}\right)\right] \end{aligned} \tag{4.23}$$

In the limit $\Lambda >> m$ it simplifies to

$$V(X) = \frac{1}{64\pi^2}\left[\frac{\Lambda^4}{2}+\Lambda^2m^2+m^4\log\left(\frac{m^2}{\Lambda^2}\right)\right] \tag{4.24}$$

For several heavy fields, the mass $m(X)$ will instead be a X-dependent matrix and it should be traced over in the final expression. It is best written using the supertrace. In the end the potential becomes

$$V(X) = \frac{1}{64\pi^2}\left[\frac{\Lambda^4}{2}\mathrm{Str}(1) + \Lambda^2\mathrm{Str}(\mathcal{M}^2) + \mathrm{Str}\left(\mathcal{M}^4 \log\frac{\mathcal{M}^2}{\Lambda^2}\right)\right] \tag{4.25}$$

As was already noted, $\mathrm{Str}\,1 = \mathrm{Str}\,\mathcal{M}^2 = 0$ for the O'Raifeartaigh model[1]. Note that to one loop only the quadratic terms in the Lagrangian are relevant.

For the O'Raifeartaigh model, the quadratic terms are the masses in equations (4.5)-(4.8), but now the light field is NOT on shell and the dimensionless constants $x = \frac{hX}{m}$ and $\xi = \frac{hF}{m^2}$ depend on the components themselves, not on their vev as before. With these masses, the expression for the scalar potential can be computed through equation (4.25) and put equal to the generalized Kähler potential by way of equation (4.15), in which case H is specified to one loop order as

$$\begin{aligned}\partial_X\partial_{\overline{X}}H &= \frac{1}{64\pi^2F^2}\mathrm{Str}\,\mathcal{M}^4\log\frac{\mathcal{M}^2}{\Lambda^2}\\ &= \frac{1}{64\pi^2F^2}\left[\mathrm{Str}\,\mathcal{M}^4\log\frac{|m|^2}{\Lambda^2} + \mathrm{Str}\,\mathcal{M}^4\log\frac{\mathcal{M}^2}{|m|^2}\right]\end{aligned} \tag{4.26}$$

This is a nice result and it calls for some discussion. The point is that we have constructed a manifestly supersymmetric, effective theory for **X**, even in the regime of strong breaking. Previously one was confined to leave **X** on shell when this was the case, and proceed by computing the Coleman-Weinberg potential, but then the theory is no longer manifestly invariant under a SUSY transformation. Therefore the method presented here is superior.

To make certain that what we have done is correct, we can compare the result in equation (4.26) to the method with an effective Kähler potential mentioned in the beginning. To do so we have to assume that the breaking is small, i.e $fh^*/|m|^2 << 1$, otherwise the effective Kähler potential is not the full answer. From [17] the expression to one loop is

$$K_{eff} = -\frac{1}{32\pi^2}\mathrm{tr}\left(\hat{\mathcal{M}}^\dagger\hat{\mathcal{M}}\log\frac{\hat{\mathcal{M}}^\dagger\hat{\mathcal{M}}}{\Lambda^2}\right) \tag{4.27}$$

The mass matrix $\hat{\mathcal{M}} = \hat{\mathcal{M}}(\mathbf{X})$ is gotten straight from the superpotential by looking at the quadratic terms

$$\hat{\mathcal{M}}: \qquad W(\mathbf{X},\Phi_i) = \lambda_i(\mathbf{X})\Phi^i + \hat{\mathcal{M}}(\mathbf{X})\Phi_i\Phi_j + \ldots \tag{4.28}$$

For the O'Raifeartaigh model it is

[1] This is true not only in the specific case of the O'Raifeartaigh model, but in general for globally supersymmetric theories. Once again the situation is more complicated in supergravity and $\mathrm{Str}\,\mathcal{M}^2 = 2(n-1)m_{3/2}^2 - 2\langle R_{i\bar{j}}G^iG^{\bar{j}}\rangle m_{3/2}^2$, where n is the number of chiral fields, $R_{i\bar{j}}$ the Ricci tensor and $G = K + \log W + \log\overline{W}$.

$$\hat{\mathcal{M}}(\mathbf{X}) = \begin{pmatrix} f\mathbf{X} & m \\ m & 0 \end{pmatrix} \tag{4.29}$$

The classical potential is computed in the usual way by equation (3.62)

$$V = \frac{\partial_X W \bar{\partial}_{\bar{X}} \overline{W}}{\partial_X \bar{\partial}_{\bar{X}} K_{eff}} = |f|^2 (1 - \partial_X \bar{\partial}_{\bar{X}} K_{eff} + \mathcal{O}(h^4)) \tag{4.30}$$

This matches the potential in (4.26). We could also consider our method but for a meta-stable model, with a high-energy superpotential on the form

$$W(\Phi_1, \Phi_2, \mathbf{X}) = \frac{h}{2}\mathbf{X}\Phi_1^2 + m\Phi_1\Phi_2 + f\mathbf{X} + \frac{\epsilon m}{2}\mathbf{X}^2 \tag{4.31}$$

This is somewhat more complicated than a normal O'Raifeartaigh model. The classical potential has a supersymmetric vacuum for $\langle X \rangle = -\frac{f}{\epsilon m}$, but it goes to infinity as ϵ goes to zero. Therefore the original vacuum should be approximate and SUSY broken[2]. The end result is the same and the extra complication adds no extra spice to the problem. There is no need for further details.

There is a different method presented in [22] which deals with the regime of strong SUSY breaking. This method also deals with the cases where the pseudomodulus is integrated out or where there is no pseudomodulus at all, but these topics will not be disucssed.

In this method the pseudomodulus was taken to be a constrained superfield, which means

$$\mathbf{X}^2 = 0 \tag{4.32}$$

In terms of components, this equation has the solution that the scalar component is

$$X = -\frac{\psi_X^2}{2F} \tag{4.33}$$

It looks simple but works remarkably well, however it prohibits all couplings between the Goldstino and the pseudomodulus since $X\psi_X \sim (\psi_X)^3 = 0$. Therefore the terms in equation (4.16) will never appear. The conclusion is that when one is interested specifically in how the Goldstino couples to the pseudomodulus, using a constrained superfield is not an option.

A subject that has been avoided here is the discussion of R-symmetry, a $U(1)$-symmetry, which can be used in spontaneous SUSY-breaking. In the

[2] A master thesis at Chalmers is required to discuss sustainable development and this is my addition to that discussion. If the universe is in a meta-stable state it could, at any time, tunnel through from the approximate, SUSY breaking vacuum to the true supersymmetric vacuum at infinity, and reality would cease to exist. This certainly wouldn't be in humanity's best interest, in which case it is of highest importance, for a sustainable society, to further investigate the issue.

O'Raifeartaigh model the field would transform as $R(\mathbf{X}) = R(\Phi_2) = 2$ and $R(\Phi_1) = 0$. For a detailed discussion on how this ties to spontaneous SUSY breaking see [23].

Although the O'Raifeartaigh model was used to specify the properties of H in equation (4.26), the derivation is more general. The developed method is certainly applicable to much more complicated models and it would be very interesting to apply it to something that describes reality better. Hopefully that would lead to more interesting conclusions and deeper a physical insight.

5

An Experimental Sign of SUSY Breaking

In this section we will investigate if there is a chance of experimental detection at the LHC of a specific SUSY breaking model. We will use a semi-direct gauge mediation model (see [24], [25] and [26]), which means that the field that mediates the SUSY breaking from the hidden sector to the messengers is not the usual spurion chiral superfield but some other gauge field (here taken to be $U(1)$). The messenger fields are charged under both the SM gauge group and the hidden one. These fields mediate the breaking between the hidden sector and the MSSM.

Gauge mediation models with messengers have a very nice feature. Without them, coupling the MSSM to the hidden sector directly forces one to include too many extra charged fields. Consider equation (2.53). In order to break SUSY dynamically a lot of new fields need to be added and each of these provide a correction, a possible particle that can circle the loop in figure C.1, that needs to be considered when computing the counterterms. This bumps up the n_f high enough to make the β-function too large thus introducing Landau poles (i.e the running coupling constant diverges) at energies below the GUT scale.

The advantage of having messengers is that they screen the MSSM from the SUSY breaking fields, and the only correction that needs to be considered is the one where the messenger itself is circling the loop. In this way Landau poles can be avoided.

In so called "minimal gauge mediation" the messengers acquire a mass via coupling to a spurion chiral superfield, whereas in semi-direct gauge mediation they have a tree level supersymmetric mass and get non supersymmetric mass corrections from couplings with the hidden gauge field.

We will assume that the messengers are very heavy, heavier than the CM energy in the LHC (14 TeV). Therefore we may use an effective Lagrangian

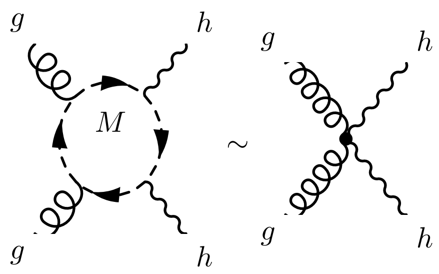


Figure 5.1: The Feynman diagram for the scattering process $gg \to hh$. The loop diagram to the left reduces to the simpler tree level process when the messengers are integrated out.

where they have been integrated out. Since the gauge group of the hidden sector is chosen to be $U(1)$, the effective theory will be the similar to the Euler-Heisenberg Lagrangian of S-QED (for a derivation see [27]).

5.1 Producing the Hidden Particles

The effective action is $S_{eff} = S_0 + S_{int}$, where S_0 are the kinetic terms (compare to equation (3.60))

$$S_0 = \frac{1}{4}\int d^4x \left[\left(\int d^2\theta\, W^2 + c.c\right) + \frac{1}{4}\left(\int d^2\theta\, W^{a\,2} + c.c\right)\right] \tag{5.1}$$

The first part is the hidden sector kinetic energy and the second one, with the color index, is the corresponding one for $SU(3)_c$. The interacting part comes from the Euler-Heisenberg Lagrangian

$$S_{int} = \frac{g_s^2 g_h^2}{192\pi^2 M^4}\int d^4x \int d^2\theta d^2\bar{\theta}\left(W^{a\,2}\overline{W}^2 + W^2\overline{W}^{a\,2} + 4W^{a\alpha}W_\alpha \overline{W}^a_{\dot{\alpha}}\overline{W}^{\dot{\alpha}}\right) \tag{5.2}$$

Here we have taken the effective coupling constant to be $g_s g_h/M^2$, where g_s is the strong coupling constant, g_h is the interaction strength in the hidden sector and M is the mass of the messenger. The Feynman diagram for the scattering process is shown in figure 5.1.

The components of the spinor superfield W_α are given by equation (3.56), but here we will neglect the D-field because we are only interested in the vector component. Furthermore, use the gluons' equations of motion

$$\begin{array}{l} \partial^\mu F_{\mu\nu} = 0 \\ \partial_{[\rho}F_{\mu\nu]} = 0 \end{array} \qquad \Rightarrow \qquad \frac{1}{4}\sigma^{\mu\nu}{}_\alpha{}^\beta\theta_\beta\theta\sigma^\rho\bar{\theta}\partial_\rho F_{\mu\nu} = 0$$

Since we are interested in producing the hidden particles at the LHC, we can neglect the gluinos[1] in the incoming state and also the fermionic component in the outgoing state. The latter is a choice: we choose to look at

[1]This is a well-founded approximation because we want to investigate a cross section and the LHC is, more or less, a gluon collider. If there is some other universe where people are using gluino colliders, they would do it the other way around and neglect the gluons instead.

the boson rather than its fermionic component. In the end the result will be similar no matter which component we look at. The only parts of W_α that cannot be neglected are

$$W_\alpha = -\frac{i}{4}\sigma^{\mu\nu}{}_\alpha{}^\beta\theta_\beta F_{\mu\nu}, \quad \text{with} \quad \sigma^{\mu\nu}{}_{\alpha\beta}F_{\mu\nu} = f_{\alpha\beta} \tag{5.3}$$

$$\overline{W}_{\dot\alpha} = \frac{i}{4}\bar\sigma^{\mu\nu}{}_{\dot\alpha\dot\beta}\bar\theta^{\dot\beta}F_{\mu\nu}, \quad \text{with} \quad \bar\sigma^{\mu\nu}{}_{\dot\alpha\dot\beta}F_{\mu\nu} = \bar f_{\dot\alpha\dot\beta} \tag{5.4}$$

The action can be written in a more transparent form using the component fields instead of the superfields

$$S_{int} = \frac{1}{48}\frac{\alpha_s\alpha_h}{M^4}\int d^4x\left(f^2\bar f^{a\,2} + f^{a\,2}\bar f^2 + 4ff^a\bar f\bar f^{a\,2}\right) \tag{5.5}$$

Note that this is not renormalizable, because the effective coupling constant $\frac{g_s g_h}{M^2}$ has a dimension, it goes like $[E]^{-2}$. As long as we restrict the computation to tree level amplitudes and $\hat s << M^2$ (i.e the messengers circling the loop in figure 5.1 are off shell) everything is ok.

Denote the hidden field strength by $G_{\mu\nu}$. The matrix element for the scattering process is written simply as

$$\mathcal{M} = -i\frac{1}{48}\frac{\alpha_s\alpha_h}{M^4} \times \langle p_1,\varepsilon_1;p_2,\varepsilon_2|\, T^{\mu_1\nu_1\ldots\mu_4\nu_4}F_{\mu_1\nu_1}F_{\mu_2\nu_2}G_{\mu_3\nu_3}G_{\mu_4\nu_4}|\, k_1,\lambda_1;k_2,\lambda_2\rangle \tag{5.6}$$

Here p_1, p_2, ε_1 and ε_2 are the momenta and polarization vectors of the gluons and k_1, k_2, λ_1 and λ_2 the corresponding ones for the hidden particles. The tensor $T^{\mu_1\nu_1\ldots\mu_4\nu_4}$ is gotten from the action

$$T^{\mu_1\nu_1\ldots\mu_4\nu_4} = \mathrm{tr}\left(\sigma^{\mu_1\nu_1}\sigma^{\mu_2\nu_2}\right)\mathrm{tr}\left(\bar\sigma^{\mu_3\nu_3}\bar\sigma^{\mu_4\nu_4}\right) + \mathrm{tr}\left(\sigma^{\mu_3\nu_3}\sigma^{\mu_4\nu_4}\right) \times \mathrm{tr}\left(\bar\sigma^{\mu_1\nu_1}\bar\sigma^{\mu_2\nu_2}\right) + 4\mathrm{tr}\left(\sigma^{\mu_1\nu_1}\sigma^{\mu_3\nu_3}\right)\mathrm{tr}\left(\bar\sigma^{\mu_2\nu_2}\bar\sigma^{\mu_4\nu_4}\right) \tag{5.7}$$

When contracting the fields with the external states it becomes

$$\mathcal{M} = -i\frac{\alpha_s\alpha_h}{3M^4}V^{\mu_1\nu_1\ldots\mu_4\nu_4}p_{1\,\mu_1}p_{2\,\mu_2}k_{1\,\mu_3}k_{2\,\mu_4}\varepsilon_{1\,\nu_1}\varepsilon_{2\,\nu_2}\lambda_{1\,\nu_3}\lambda_{2\,\nu_4} \tag{5.8}$$

$$V^{\mu_1\nu_1\ldots\mu_4\nu_4} = T^{\mu_1\nu_1\ldots\mu_4\nu_4} + (1\leftrightarrow 2) + (3\leftrightarrow 4) + \begin{pmatrix} 1\leftrightarrow 2 \\ 3\leftrightarrow 4 \end{pmatrix} \tag{5.9}$$

The traces can be simplified by some neat identities, see [28] for reference. There is no real point in proceeding further by hand, but rather use some software[2] to compute it. The polarization sums were calculated using the standard substitution

[2]Here the computations were performed using the Mathematica package Ricci, see `http://www.math.washington.edu/~lee/Ricci/` for more information.

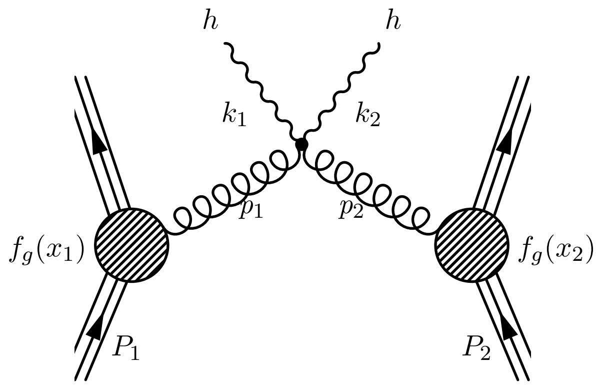


Figure 5.2: This is the true collision in the lab frame, involving the gluons as partons of the colliding protons. Note that the momenta of the gluons are $p_1 = x_1 P_1$ and $p_2 = x_2 P_2$. In the LHC the protons are approximately massless and their four momenta can be written $P_1 = \frac{\sqrt{s}}{2}(1, \hat{z})$ and $P_2 = \frac{\sqrt{s}}{2}(1, -\hat{z})$.

$$\sum_{\lambda=1,2} \varepsilon_\mu^\lambda \varepsilon_\nu^\lambda = -\eta_{\mu\nu} \tag{5.10}$$

It has been checked that the Ward identities are still satisfied when using this. The final expression for the scattering amplitude is written in a simple form using the Mandelstam variables

$$\begin{aligned} &\hat{s} = (p_1 + p_2)^2, \quad \hat{t} = (p_1 - k_1)^2, \quad \hat{u} = (p_1 - k_2)^2 \\ &\hat{s} + \hat{t} + \hat{u} = m_1^2 + m_2^2 + m_3^2 + m_4^2 \end{aligned}$$

Here we have gluons (massless) and the hidden particles (of mass m) so $\hat{s} + \hat{t} + \hat{u} = 2m^2$. The averaged matrix element is

$$|\overline{\mathcal{M}}|^2 = \frac{1}{4}\frac{1}{64}\left(\frac{\alpha_s \alpha_h}{3M^4}\right)^2 \left(2m^4\hat{s}^2 + 2\hat{s}^2(-2m^2 + \hat{s})^2 + 2(m^2 - \hat{t})^4 + 2(m^2 - \hat{u})^4\right) \tag{5.11}$$

To obtain the full cross section there are some extra difficulties that need to be taken into account. In the above derivation of the scattering amplitude we have only considered the colliding gluons, but the LHC collides protons! The two processes are not the same, but by using Feynman's parton model of hadrons they can be related to each other. Introduce the parton distribution functions $f_i(x)$ (see [29], [30]), and let it denote the probability of finding a quark (or gluon) of flavor i with a fraction x of the proton's momentum. The situation is shown in figure 5.2.

The gluon cross section can be written in the usual form (see e.g [31] or [32])

$$d\sigma_{ij} = \frac{1}{2\hat{s}}|\mathcal{M}|^2 dLips_2 \tag{5.12}$$

$$dLips_n = \left(\prod_i^n \frac{d^3\mathbf{k}_i}{(2\pi)^3}\frac{1}{2E_i}\right)\delta^4(p_{in} - \sum_i^n k_i) \tag{5.13}$$

Using Lorentz invariance, the two integrals coming from $dLips_2$ can be computed in the center of mass frame of the gluons[3], and the differential cross section written

$$\frac{d\sigma_{ij}}{d\hat{t}} = \frac{|\overline{\mathcal{M}}|^2}{16\pi\hat{s}^2}, \quad t_\pm = m^2 - \frac{\hat{s}}{2}\left(1 \pm \sqrt{1 - \frac{4m^2}{\hat{s}}}\right) \tag{5.14}$$

Introduce another factor 1/2 (because the outgoing particles are identical) and integrate between t_- and t_+ to get the full cross section for the gluon collision

$$\sigma_{ij} = \frac{1}{128}\left(\frac{\alpha_s\alpha_h}{3}\right)^2 \frac{\hat{s}^3}{40\pi M^8}\sqrt{1 - \frac{4m^2}{\hat{s}}}\left(27\frac{m^4}{\hat{s}^2} - 26\frac{m^2}{\hat{s}} + 7\right) \tag{5.15}$$

The cross section for the protons is gotten when summing up all possible parton configurations. With the previously introduced PDFs[4] this is simply done by including them and integrating over the fraction of momenta carried by the gluons

$$\sigma = \int_0^1 dx_1 \int_0^1 dx_2 \sum_{i,j} f_i(x_1)f_j(x_2)\sigma_{ij} \tag{5.16}$$

Here $\sum f_i(x_1)f_j(x_2) = f_g(x_1)f_g(x_2)$. The integration is done numerically and the process is plotted versus the hidden particle mass in figure 5.3, with $M = 1$ TeV$/c^2$, $\alpha_h = \alpha_s = 0.069$ and $s = 14$ TeV. Equation (5.15) can be used to get σ for any value of M or α_h.

Unfortunately the cross section is very small. A good landmark for comparison is to use an integrated luminosity of one inverse femtobarn to estimate the number of events, which is simply the integrated luminosity times the cross section. For this process we would have approximately 0.07 events, i.e none at all.

Furthermore, the messenger mass was taken rather low: it is expected to be much bigger and since the equation (5.15) goes like M^{-8}, a small increase

[3] This is a very nice simplification, otherwise one would have to compute the integrals in the lab frame. In this case it is the same as the CM frame of the protons, where $\mathbf{p}_1 \neq \mathbf{p}_2$ and there would be x_1 and x_2 everywhere. In the end, this way of doing things must of course give the same answer and it has been checked that it does.

[4] Numerical values for the PDFs are based on measurements form the Hera experiment and it was taken from `http://hepdata.cedar.ac.uk/pdf/pdf3.html`.

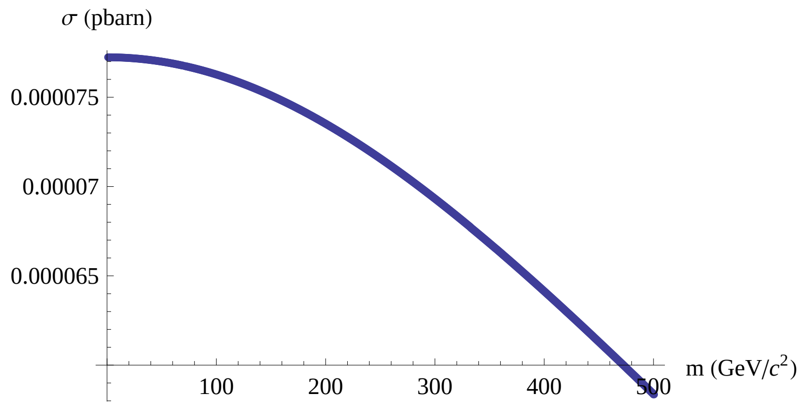


Figure 5.3: The figure shows the total cross section $\sigma_{gg\to hh}$ as a function of the mass of the hidden particle, for $M = 1$ TeV/c^2.

in M means a huge decrease in the already very small cross section. The conclusion is that there is very little chance of producing the hidden particles at the LHC, and no chance of getting statistically reliable measurements.

5.2 Decay of the Hidden Particle

The decay rate of the previously mentioned hidden particles may still provide some interesting results, for cosmological purposes. With the action given in equation (5.1), one hidden particle can be coupled to three photons (or any gauge field) through the messenger field. Note that both fields have a $U(1)$-charge, but they are of different origin. The same procedure that gave equation (5.2) can be used here, but as already mentioned we want to couple one hidden particle to three photons

$$S_{int} = -\frac{3e^3 g_h}{192\pi^2 M^4}\int d^4x \int d^2\theta d^2\bar{\theta}\, \left(W^{(e)\,2}\bar{W}^{(h)}_{\dot{\alpha}}\overline{W}^{(e)\dot{\alpha}} + W^{(h)\alpha}W^{(e)}_{\alpha}\bar{W}^{(e)\,2}\right) \tag{5.17}$$

There are no gauge indices here so instead we have introduced the label (e) for the photon and (h) for the hidden field. The combinatorics and coupling constants are different, but the rest is similar. The tensor structure corresponding to equation (5.7) is

$$\tilde{T}^{\mu_1\nu_1\ldots\mu_4\nu_4} = \mathrm{tr}\left(\sigma^{\mu_1\nu_1}\sigma^{\mu_2\nu_2}\right)\mathrm{tr}\left(\bar{\sigma}^{\mu_3\nu_3}\bar{\sigma}^{\mu_4\nu_4}\right) + \mathrm{tr}\left(\sigma^{\mu_3\nu_3}\sigma^{\mu_4\nu_4}\right) \times \mathrm{tr}\left(\bar{\sigma}^{\mu_1\nu_1}\bar{\sigma}^{\mu_2\nu_2}\right) \tag{5.18}$$

The matrix element is

$$\mathcal{M} = \frac{e^3 g_h}{(4\pi)^2 M^4} B^{\mu_1\nu_1\ldots\mu_4\nu_4} p_{\mu_1}\lambda_{\nu_1} k_{1\,\mu_2}\varepsilon_{1\,\nu_2} k_{2\,\mu_3}\varepsilon_{2\,\nu_3} k_{3\,\mu_4}\varepsilon_{3\,\nu_4} \tag{5.19}$$

$$B^{\mu_1\nu_1\ldots\mu_4\nu_4} = \tilde{T}^{\mu_1\nu_1\ldots\mu_4\nu_4} + \text{(all permutations of 234)} \tag{5.20}$$

$$\tag{5.21}$$

This is a $1 \to 3$ process and is thus best presented using the variables

$$s = (p_1 - k_1)^2, \quad t = (p_1 - k_2)^2, \quad u = (p_1 - k_3)^2 \tag{5.22}$$

$$s + t + u = m^2 + m_1^2 + m_2^2 + m_3^2 \tag{5.23}$$

Here the decay products are photons so $m_1 = m_2 = m_3 = 0$. The same procedure for the sum over polarization is used here, but the hidden particle can have three different polarizations

$$|\overline{\mathcal{M}}|^2 = \frac{1}{3}\sum_{\text{polarizations}} |\mathcal{M}|^2 = \frac{\alpha_e^3\alpha_h}{6M^8}\Big(s^4 + t^4 + u^4 - 2m^2(s^3 + t^3 + u^3) + m^4(s^2 + t^2 + u^2)\Big) = \frac{\alpha_e^3\alpha_h}{6M^8} g(s,t,u) \tag{5.24}$$

The expression for the differential decay rate can be found in e.g [3]

$$d\Gamma(1 \to 3) = \frac{1}{2m}|\overline{\mathcal{M}}|^2 dLips_3 \tag{5.25}$$

The kinematic integrals in the massless case turn out to be

$$dLips_3 = \frac{1}{3!}\frac{1}{16(2\pi)^3}\frac{ds\,dt}{m^2}, \qquad s,t \in [0, m^2] \tag{5.26}$$

The extra factor of $\frac{1}{3!}$ is there to avoid counting identical configurations several times, because the three decay products are identical particles. The decay rate can be computed in closed form

$$\Gamma = \frac{\alpha_e^3\alpha_h}{1152(2\pi)^3 m^3 M^8}\int ds\,dt\, g(s,t,m^2 - s - t) = \frac{\alpha_e^3\alpha_h}{30720\pi^3}\frac{m^9}{M^8} \tag{5.27}$$

This can be plotted but it is more informative to plot its inverse $\tau = 1/\Gamma$, with τ the mean life time, shown in figure 5.4.

We see that the decay rate can vary over many orders of magnitude. Note that if the hidden particle exist, lots of them would have been created at the Big Bang. With a more likely value for the messenger mass, say $M = 10$ TeV, the mean life time of the hidden particle can, depending on its mass, be greater than the age of the universe (roughly 10^{19} s). This means that most of it would still be around, providing a possible candidate for dark matter.

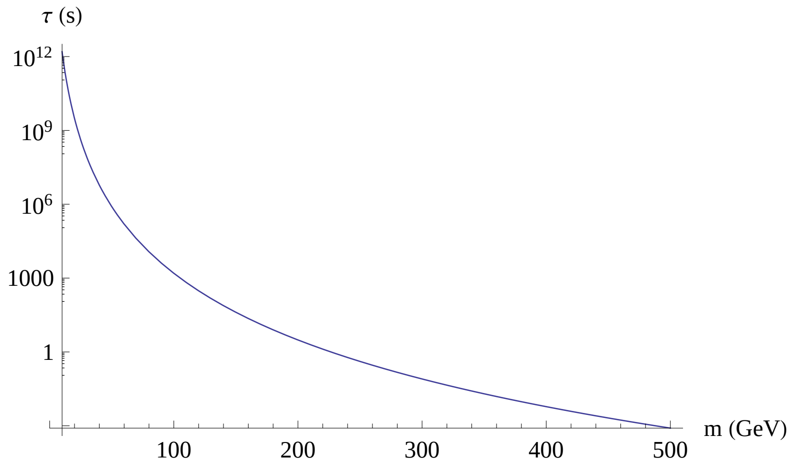


Figure 5.4: Logarithmic plot of the mean life time of the process $h \to \gamma\gamma\gamma$, with $M = 1$ TeV$/c^2$ and $\alpha_h = \alpha_s$. There are several other decay channels (e.g $h \to ggg$ or $h \to g\tilde{g}\tilde{g}$) and τ is therefore smaller, but the order of magnitude should be correct.

A

Conventions

Unless otherwise stated we work in god given units, i.e

$$\hbar = c = 1$$

The Minkowski metric

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \Rightarrow \quad p^2 = m^2 \quad \text{(on shell)}$$

The usual notation is used for coordinates, momentum etc

$$x^\mu = (t, \mathbf{x}), \quad p^\mu = (E, \mathbf{p}), \quad \partial_\mu = (\partial/\partial t, \boldsymbol{\nabla})$$

The sigma matrices

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma^\mu_{\alpha\dot{\alpha}} = (\sigma^0, \boldsymbol{\sigma}), \quad \bar{\sigma}^{\mu\dot{\alpha}\alpha} = (\sigma^0, -\boldsymbol{\sigma}), \quad [\sigma^i, \sigma^j] = 2\epsilon^{ijk}\sigma^k, \quad \{\sigma^i, \sigma^j\} = 2\delta^{ij}$$

Especially important are

$$\sigma^{\mu\nu}{}_\alpha{}^\beta = \frac{1}{4}\left(\sigma^\mu_{\alpha\dot{\alpha}}\bar{\sigma}^{\nu\dot{\alpha}\beta} - \sigma^\nu_{\alpha\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\beta}\right) \quad \bar{\sigma}^{\mu\nu\dot{\alpha}}{}_{\dot{\beta}} = \frac{1}{4}\left(\bar{\sigma}^{\mu\dot{\alpha}\alpha}\sigma^\nu_{\alpha\dot{\beta}} - \bar{\sigma}^{\nu\dot{\alpha}\alpha}\sigma^\mu_{\alpha\dot{\beta}}\right)$$

Introduce the two-component spinor ψ_α (a left-handed Weyl spinor), transforming in the $(\frac{1}{2}, 0)$ representation of the Lorentz group. Its conjugate is written $\bar{\psi}_{\dot{\alpha}} = \psi^\dagger_{\dot{\alpha}} = (\psi_\alpha)^\dagger$ and transforms as $(0, \frac{1}{2})$ (a right-handed Weyl spinor). The indices on two component spinors must be contracted according to

$$\xi\zeta = \xi^\alpha \zeta_\alpha = \xi^\alpha \zeta^\beta \epsilon_{\alpha\beta}, \quad \overline{\lambda\rho} = \bar{\lambda}_{\dot{\alpha}} \bar{\rho}^{\dot{\alpha}} = \bar{\lambda}_{\dot{\alpha}} \bar{\rho}_{\dot{\beta}} \epsilon^{\dot{\alpha}\dot{\beta}}, \quad \bar{\lambda}^{\dot{\alpha}} = (\lambda^\alpha)^\dagger$$

$$\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \epsilon^{\dot{\alpha}\dot{\beta}} = (\epsilon^{\alpha\beta})^*$$

Suppressed indices can always be reconstructed with these conventions, for example: $\psi\sigma^\mu\bar{\chi} = \psi^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}$. For a wealth of identities for two component spinors see [28].

A chiral representation is used for the 4-component Dirac spinors, where the gamma-matrices are

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{A.1}$$

Note that with this representation we can write a Dirac spinor as

$$\Psi_D = \begin{pmatrix} \psi_\alpha \\ \bar{\xi}^{\dot{\alpha}} \end{pmatrix} \tag{A.2}$$

Important integrals

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta)^n} = \frac{(-1)^n}{(4\pi)^{d/2}} \frac{\Gamma(n - d/2)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-d/2} \tag{A.3}$$

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^2}{(\ell^2 - \Delta)^n} = \frac{(-1)^{n-1}}{(4\pi)^{d/2}} \frac{id}{2} \frac{\Gamma(n - d/2 - 1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-d/2-1} \tag{A.4}$$

B

Ghosts

In section 2.2 it was briefly mentioned that the Yang-Mills Lagrangian in equation (2.7) is not quite the full story . To quantize it we will use the method of path integrals. Ignore the fermions and consider only the gluon field in the generating functional

$$Z = \int \mathcal{D}A_\mu \, e^{-S_{YM}[A_\mu]} \tag{B.1}$$

In the action a Wick rotation has been made ($x^0 \to ix^0$) to get an Euclidean metric and the minus sign. Along some directions of A_μ the action S_{YM} will be constant, because it is invariant under gauge transformation. Therefore when we try to compute the vacuum energy $\langle 0|0\rangle$ and integrate over all paths, we will get an infinite result along those directions. Such a divergence must be treated somehow, but it is not of the same type as those that appear in renormalization and cannot be removed in the same way. The way to handle it was proposed by Faddeev and Popov (see [33]) and the resulting Lagrangian is called the Faddeev-Popov Lagrangian.

To start out we make a naive analogy: the situation is similar to a normal double integral with the integrand only depending on one of the directions

$$Z = \int dx \, dy \, e^{-S[x]} \to \infty \tag{B.2}$$

Such a problem could be avoided in a number of ways. The most obvious is to skip the integration in y, but that would be very difficult to do for a path integral. Another possibility is to insert a delta function $\delta(y)$. This would certainly work for a functional measure and to make it more general we insert $\delta(f(x, y))$, such that $f(x, y) = 0$ for some path $y(x)$. The function $f(x, y)$ defines a gauge fixing along all possible gauge orbits, and the change of variables in the delta function brings about the Jacobian $(\partial f/\partial y|_x)$. The modified integral is

$$Z = \int dx\, dy\, \left.\frac{\partial f}{\partial y}\right|_x \delta(f(x,y)) e^{-S[x]} \tag{B.3}$$

Consider now the multidimensional generalization

$$Z = \int d^n X\, d^m Y\, \det\left(\frac{\partial f^a}{\partial Y^b}\right) \prod_{c=1}^{m} \delta(f^c(X,Y)) e^{-S_{YM}[X]} \tag{B.4}$$

The $f^c(X,Y)$ fixes the gauge in m directions. The determinant can be interpreted as a functional determinant, coming from a fermionic field

$$\det\left(\frac{\partial f^a}{\partial Y^b}\right) = \int d^m \eta\, d^m \bar{\eta}\, \exp[-\bar{\eta}_a \frac{\partial f^a}{\partial Y^b} \eta_b] \tag{B.5}$$

The η is called a BRST-ghost. Likewise, the delta function can be rewritten as an integral

$$\prod \delta(f^c(X,Y)) = \int d^m B\, e^{-iB^c f_c(X,Y)} \tag{B.6}$$

With these new fields the full path integral is

$$Z = \int d^n X\, d^m Y\, d^m \eta\, d^m \bar{\eta}\, d^m B\, \exp\left(-S_{YM}[X] - \bar{\eta}\frac{\partial f}{\partial Y}\eta - iB^T f(X,Y)\right) \tag{B.7}$$

The linear term is unphysical (it is a tadpole), but we can modify the integral by including a quadratic term $\frac{\xi}{2}B^2$ and complete the square, then integrate the B-fields away. The modification only changes the constant in front of the integral and that cancels when computing correlation functions. The generating functional can be written as

$$Z = \int d^n X\, d^m Y\, d^m \eta\, d^m \bar{\eta}\, \exp\left(-S_{YM}[X] - \bar{\eta}\frac{\partial f}{\partial Y}\eta + \frac{1}{2\xi} f^\dagger f\right) \tag{B.8}$$

For the Yang-Mills Lagrangian (equation (2.5)), S_{YM} is invariant under $A^a_\mu \to A^a_\mu + \partial_\mu \Lambda^a - g f^{abc} A^b_\mu \Lambda^c$. Some function must be chosen to fix the gauge and, theoretically, anything would do as long as it is not gauge invariant. In practice we still require something that is both Lorentz invariant and linear, and the canonical choice is

$$f^a = \partial_\mu A^{\mu a} = 0 \tag{B.9}$$

Note that under a gauge transformation this condition transforms as $\partial^\mu A^a_\mu \to \partial^\mu A^a_\mu + \partial^\mu D^{ab}_\mu \Lambda^b$ (with the covariant derivative defined as $D^{ab}_\mu = \delta^{ab}\partial_\mu - g f^{abc} A^c_\mu$), in which case

$$\frac{\partial f^a}{\partial Y^b} = \frac{\delta}{\delta \Lambda^b(y)} \partial^\mu A^a_\mu(x) = \partial^\mu D^{ab}_\mu \delta(x-y) \tag{B.10}$$

The η-integral can be written properly with the above information and the new field c

$$\det\left(\frac{\partial f^a}{\partial Y^b}\right) = \int \mathcal{D}c\,\mathcal{D}\bar{c}\,\exp\left(\int d^4x\,\bar{c}^a\partial^\mu D^{ab}_\mu c^b\right) \tag{B.11}$$

The final form of the gauge sector in the Yang-Mills action is

$$S_{YM} = \int d^4x\left[-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \bar{c}^a(\partial^\mu D^{ab}_\mu)c^b - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2\right] \tag{B.12}$$

The end result is that a new field c^a has appeared. This is not a physical field, it is a theoretical tool to help kill of the extra degrees of freedom coming from gauge invariance. However it does provide a new interaction with the gluon field and this must be taken into account in the same way as any other vertex.

The second term, $\frac{1}{2\xi}(\partial^\mu A^a_\mu)^2$, is a correction to the propagator and we have to take it into account when computing the full expression for the gluon two-point correlation function

$$\langle\Omega|\,TA^a_\mu A^b_\nu|\,\Omega\rangle = \frac{-i\delta^{ab}}{k^2+i\epsilon}\left(\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right) \tag{B.13}$$

.

In this thesis $\xi = 1$, called the Feynman-'t Hooft gauge. The physics must be independent of ξ but depending on what is described other choices may be preferable.

C

QCD Counterterms

C.1 The Gluon Counterterm

The gluon propagator diagrams to one loop are given in figure C.1

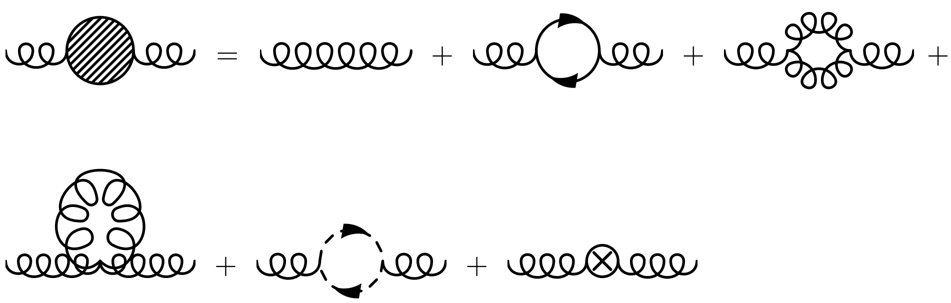
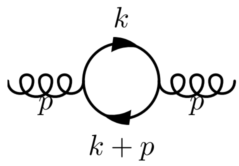


Figure C.1: All gluon propagator diagrams to one loop.

The only way to specify the counterterm is to compute all of them. With the definitions in figure 2.1 we may write

$$i\Pi^{ab\mu\nu} = \mathcal{M}_{g1}^{ab\mu\nu} + \mathcal{M}_{g2}^{ab\mu\nu} + \mathcal{M}_{g3}^{ab\mu\nu} + \mathcal{M}_{g4}^{ab\mu\nu} - i\delta_A(p^2\eta^{\mu\nu} - p^\mu p^\nu)\delta^{ab} \quad \text{(C.1)}$$

With a, b the color indices and $\mathcal{M}_g$ corresponds to the loops in the order specified by the figure. The color dependence on the corrections should be Kronecker deltas, otherwise the gluon field would not be properly diagonalized. Compute the diagrams in the order specified by figure C.1. The first one is the fermion loop



Application of the Feynman rules gives the expression

$$\mathcal{M}^{ab\mu\nu}_{g1} = -\mathrm{tr}[T^a T^b] \int \frac{d^d k}{(2\pi)^d} \,\mathrm{tr}\left[(ig\mu^\varepsilon \gamma^\mu) i \frac{\not{k} + \not{p} + m}{(k+p)^2 - m^2 + i\epsilon} (ig\mu^\varepsilon \gamma^\nu) \times i \frac{\not{k} + m}{k^2 - m^2 + i\epsilon} \right] \tag{C.2}$$

The first minus sign is needed because it is a fermion circling the loop. The Lie algebra factor is $\mathrm{tr}\, T^a T^b$ and can be chosen proportional to the identity matrix times some constant $C(r)\delta^{ab}$, different for each representation (see [11]). Remove the Kronecker delta to make the derivation cleaner. Using the Feynman parameters and shifting the momentum to $\ell^\mu = k^\mu + xp^\mu$ brings the integral to the usual form, and all odd powers of ℓ can be dropped in the numerator. The resulting expression is

$$\mathcal{M}^{\mu\nu}_{g1} = -g^2 \mu^{2\varepsilon} C(r) \int_0^1 dx \int \frac{d^d \ell}{(2\pi)^d} \frac{N^{\mu\nu}}{(\ell^2 - \Delta)^2} \tag{C.3}$$
$$N^{\mu\nu} = dm^2 \eta^{\mu\nu} + d\left[2\ell^\mu \ell^\nu - 2x(1-x)p^\mu p^\nu - (\ell^2 - x(1-x)p^2)\eta^{\mu\nu} \right]$$
$$\Delta = x(x-1)p^2 + m^2 - i\epsilon$$

Inside the integral Lorentz invariance demands that $\ell^\mu \ell^\nu = A\eta^{\mu\nu}\ell^2$ and contracting the indices on both sides shows $A = 1/d$, after which it can be computed as before. Tables for the integrals can be found in the appendix of [3] and the result is

$$\mathcal{M}^{\mu\nu}_{g1} = -i \frac{g^2 \mu^{2\varepsilon} C(r)}{(4\pi)^{d/2}} \int_0^1 dx\, d \left[\left(m^2 \eta^{\mu\nu} - x(1-x)(2p^\mu p^\nu - p^2 \eta^{\mu\nu}) \right) \times \frac{\Gamma(2 - d/2)}{\Delta^{2-d/2}} - \eta^{\mu\nu} (\frac{d}{2} - 1) \frac{\Gamma(1 - d/2)}{\Delta^{1-d/2}} \right] \tag{C.4}$$

The last term is the apparent quadratic divergence[1] but if the Gamma function is expanded we get

[1]It is called a quadratic divergence because it is divergent for $d = 2$.

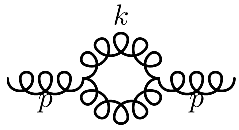
$$(\frac{d}{2}-1)\Gamma(1-d/2) = -(1-d/2)\frac{\Gamma(2-d/2)}{1-d/2} \tag{C.5}$$

The factor in front cancels the quadratic divergence. Combining the two contributions (by multiplying the latter by Δ/Δ) yields

$$\mathcal{M}^{\mu\nu}_{g1} = i\frac{g^2\mu^{2\varepsilon}C(r)}{(4\pi)^{d/2}}\left(p^\mu p^\nu - p^2\eta^{\mu\nu}\right)\int_0^1 dx\, 2d\frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}\left[x(1-x)\right] \tag{C.6}$$

Note that the tensor structure $p^\mu p^\nu - p^2\eta^{\mu\nu}$ is transverse to p_μ. This is not a coincidence but actually a requirement forced onto the propagator by the Ward identity. The tensor part of the gluon propagator MUST be of this form, otherwise it violates gauge invariance. Each flavor of quark contributes to this diagram so we must multiply the above result by n_f.

The second diagram is



Application of the Feynman rules give

$$\begin{aligned}\mathcal{M}^{ab\mu\nu}_{g2} &= -\frac{g^2\mu^{2\varepsilon}}{2}\int\frac{d^dk}{(2\pi)^d}f^{cad}\left[\eta^{\rho\mu}(k-p)^\sigma + \eta^{\mu\sigma}(2p+k)^\rho - \eta^{\sigma\rho}(2k+p)^\mu\right]\\ &\times\frac{\eta_{\sigma\kappa}\delta^{dd'}}{(k+p)^2+i\epsilon}f^{d'bc'}\left[\eta^{\kappa\nu}(2p+k)^\lambda + \eta^{\nu\lambda}(k-p)^\kappa - \eta^{\lambda\kappa}(2k+p)^\nu\right]\frac{\eta_{\lambda\rho}\delta^{cc'}}{k^2+i\epsilon}\end{aligned} \tag{C.7}$$

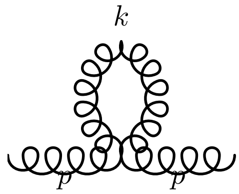
The factor of 1/2 is a symmetry factor and needed because the gluon propagators inside the loop are equivalent. The Lie algebra factor is the Casimir operator but for the adjoint representation, denoted by $r = G$, so $f^{acd}f^{bcd} = C_2(G)\delta^{ab}$. Shift momentum to $\ell^\mu = k^\mu + xp^\nu$ and drop all odd powers of ℓ

$$\begin{aligned}\mathcal{M}^{\mu\nu}_{g2} &= \frac{g^2\mu^{2\varepsilon}C_2(G)}{2}\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\frac{N^{\mu\nu}}{(\ell^2-\Delta)^2} \qquad \text{(C.8)}\\ N^{\mu\nu} &= 6(\ell^2\eta^{\mu\nu} - \ell^\mu\ell^\nu) + ((1+x)^2 + (2-x)^2)\eta^{\mu\nu}p^2\\ &+ p^\mu p^\nu\left((d-2)(1-2x)^2 - 2(1+x)(2-x)\right)\\ \Delta &= x(x-1)p^2 - i\epsilon\end{aligned}$$

The integration can be performed in the familiar way and the result is

$$\mathcal{M}_{g2}^{\mu\nu} = \frac{ig^2\mu^{2\varepsilon}C_2(G)}{2(4\pi)^{d/2}}\int_0^1 \frac{dx}{\Delta^{2-d/2}}\left[3(1-d)\Delta\eta^{\mu\nu}\Gamma(1-d/2) + \Xi^{\mu\nu}\Gamma(2-d/2)\right] \tag{C.9}$$

The third diagram is



The Feynman rules give the amplitude

$$\begin{aligned}\mathcal{M}_{g3}^{ab\mu\nu} = -i\frac{g^2\mu^\varepsilon}{2}&\Big[f^{cde}f^{abe}\left(\eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\rho\nu}\eta^{\sigma\mu}\right) + f^{cae}f^{dbe}\left(\eta^{\rho\sigma}\eta^{\mu\nu} - \eta^{\rho\nu}\eta^{\sigma\mu}\right)\\ &+f^{cbe}f^{dae}\left(\eta^{\rho\sigma}\eta^{\mu\nu} - \eta^{\rho\mu}\eta^{\sigma\nu}\right)\Big]\int\frac{d^dk}{(2\pi)^d}\frac{-i\eta_{\rho\sigma}\delta^{cd}}{k^2+i\epsilon}\end{aligned} \tag{C.10}$$

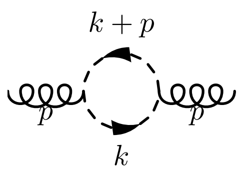
The factor of 1/2 is again due to symmetry. To make the amplitude look like the ones already computed we multiply by $(k+p)^2/(k+p)^2$ and shift momentum. The expression simplifies to

$$\mathcal{M}_{g3}^{\mu\nu} = (1-d)g^2\mu^\varepsilon C_2(G)\eta^{\mu\nu}\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\frac{\ell^2+(1-x)^2p^2}{(\ell^2-\Delta)^2} \tag{C.11}$$

Again the color delta has been taken away. The integral solves to

$$\begin{aligned}\mathcal{M}_{g3}^{\mu\nu} = i\frac{g^2C_2(G)}{(4\pi)^2}(1-d)\eta^{\mu\nu}\int_0^1\frac{dx}{\Delta^{2-d/2}}&\Big[(1-x)^2p^2\Gamma(2-d/2)\\ &-\frac{d}{2}\Delta\Gamma(1-d/2)\Big]\end{aligned} \tag{C.12}$$

This does not fix the any of the problems that came up. The only hope is the final diagram, the ghost contribution



The matrix element is

$$\mathcal{M}_{g4}^{ab\mu\nu} = -\int \frac{d^dk}{(2\pi)^d}(-g\mu^\varepsilon f^{dac})\frac{i(k+p)^\mu}{(k+p)^2+i\epsilon}(-g\mu^\varepsilon f^{cbd})\frac{ik^\nu}{k^2+i\epsilon} \tag{C.13}$$

The ghost are fermions which explains the extra minus sign. Shift momentum and simplify as before to get the expression

$$\mathcal{M}_{g4}^{\mu\nu} = -g^2\mu^{2\varepsilon}C_2(G)\int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d}\frac{\ell^\mu\ell^\nu - x(1-x)p^\mu p^\nu}{(\ell^2-\Delta)^2} \tag{C.14}$$

The final form of the ghost contribution is

$$\begin{aligned}\mathcal{M}_{g4}^{\mu\nu} = i\frac{g^2\mu^{2\varepsilon}C_2(G)}{(4\pi)^2}\int_0^1 \frac{dx}{\Delta^{2-d/2}}\Big[&x(1-x)p^\mu p^\nu\Gamma(2-d/2) \\ &+\eta^{\mu\nu}\frac{\Delta}{2}\Gamma(1-d/2)\Big]\end{aligned} \tag{C.15}$$

Now we sum up the four contributions and take a closer look at the coefficient in front of the seemingly quadratic divergence

$$\begin{aligned}\frac{1}{2}\left(3(1-d)-d(1-d)+1\right)\Gamma(1-d/2) &= 2\left(1-\frac{d}{2}\right)^2\Gamma(1-d/2) \\ &= 2\left(1-\frac{d}{2}\right)\Gamma(2-d/2)\end{aligned} \tag{C.16}$$

It cancels! This would not have happened without the ghost diagram. The second thing that needs checking is that the tensor structure has the proper form. The coefficients in front of the different parts are

$$p^2\eta^{\mu\nu} : 2x(x-1)(1-d/2)+(1-d)(1-x)^2+\frac{1}{2}\left((1+x)^2+(2-x)^2\right)$$
$$p^\mu p^\nu : (\frac{d}{2}-1)(1-2x)^2-2$$

At first glance they do not look the same. The brute force way of showing it is to perform the integrals in x and both turn out to be $\pm\frac{7}{3}\mp\frac{d}{6}$. A more elegant way to do it is by realizing that the expression is symmetric under $x\to 1-x$, a consequence of the Feynman parameters, in which case we may simplify by letting $x \to \frac{1}{2}x+\frac{1}{2}(1-x)=\frac{1}{2}$. For the coefficient in front of $p^2\eta_{\mu\nu}$, it yields

$$\begin{aligned}p^2\eta^{\mu\nu} : -\frac{d}{2}\Big[(1-2x)^2+1-2x\Big]+(1-2x)^2+2+\frac{1}{2}-x \\ \to -(\frac{d}{2}-1)(1-2x)^2+2\end{aligned} \tag{C.17}$$

Note that dimensional regularization is a requirement for this to work. Unlike the quark counterterm, here it would not be possible to both satisfy the Ward identities and get the correct tensor structure using a Pauli-Villars prescription. The counterterm can now be specified by summing up the four diagrams and using equation (C.1) at $p^2 = \mu^2$

$$\delta_A = \frac{g^2}{16\pi^2}\int_0^1 dx \left[\frac{1}{\varepsilon}\left(1 + \varepsilon\log\left(\frac{\mu^2}{\Delta}\bigg|_{p^2=\mu^2}\right) + \mathcal{O}(\varepsilon^2)\right) \times \Big((2-(1-2x)^2)C_2(G) - 8n_f x(1-x)C(r)\Big)\right] \tag{C.18}$$

The divergent part is

$$\lim_{\varepsilon\to 0}\varepsilon\delta_A = \frac{g^2}{16\pi^2}\frac{5C_2(G) - 4n_f C(r)}{3} \tag{C.19}$$

C.2 The Vertex Counterterm

The diagrams are given by figure C.2. Write the propagator as

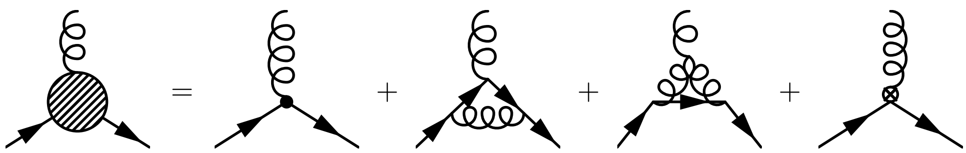
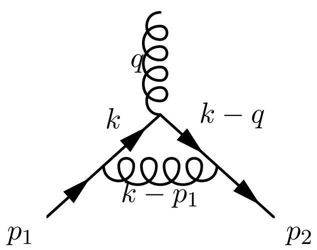


Figure C.2: The vertex correction diagrams to one loop.

$$ig\mu^\varepsilon\Gamma^\mu D^a = ig\gamma^\mu T^a + \mathcal{M}_{v1}^{a\mu} + \mathcal{M}_{v2}^{a\mu} + ig\mu^\varepsilon\delta_g\gamma^\mu T^a \tag{C.20}$$

The first diagram is



$$\mathcal{M}_{v1}^{a\mu} = g^3\mu^{3\varepsilon}T^bT^aT^b\int\frac{d^dk}{(2\pi)^d}\gamma^\rho\frac{\not{k}+m}{k^2-m^2+i\epsilon}\gamma^\mu\frac{\not{k}-\not{q}+m}{(k-q)^2-m^2+i\epsilon}\gamma^\sigma \times\frac{\eta_{\sigma\rho}}{(k-p_1)^2+i\epsilon} \tag{C.21}$$

The Lie algebra factor can be simplified by commuting T^b through

$$T^bT^aT^b = C_2(r)T^a + if^{bac}T^cT^b = \Big(C_2(r) - \frac{1}{2}C_2(G)\Big)T^a \tag{C.22}$$

In the last step the anti-symmetry of f^{abc} was used to write $f^{bac}T^cT^b = \frac{1}{2}f^{bac}(T^cT^b - T^bT^c)$. Shift the momentum to $k = \ell - yq - zp_1$ to get

$$\begin{aligned}
\mathcal{M}^\mu_{v1} &= g^3\mu^{3\varepsilon}\Big(C_2(r) - \frac{1}{2}C_2(G)\Big)\int dx\,dy\,dz\,\delta(x+y+z-1)\\
&\quad\times\int\frac{d^d\ell}{(2\pi)^d}\frac{2N^\mu}{(\ell^2-\Delta)^3}\\
\Delta &= (x+y)m^2 + y(y-1)q^2 + z(z-1)p_1^2 + 2yzqp_1 - i\epsilon\\
N^\mu &= \gamma^\rho(\not{\ell} + y\not{q} + z\not{p}_1 + m)\gamma^\mu(\not{\ell} + (y-1)\not{q} + z\not{p}_1)\gamma_\rho
\end{aligned}$$

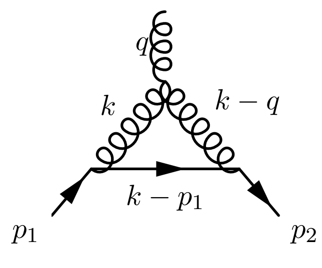
The numerator can be simplified but it is more complicated here than it was for the propagators. This is because there will be a wealth of finite terms and they must be simplified to a form where the renormalization condition in (2.14) can be used. We can argue for both Lorentz invariance and parity, in which case the only possible terms are

$$\mathcal{M}^\mu = A\gamma^\mu + B(p_1+p_2)^\mu + Cq^\mu \tag{C.23}$$

But it doesn't help off-shell because the coefficients will depend not only on m^2, p_1^2 and p_2^2, but also on $\not{p}_1$ and $\not{q}$. Therefore it is far from obvious how the renormalization condition should be used. As only the divergent part is important for the β-function we can isolate it and go on without specifying the finite contribution further, meaning that we only have to compute $A\gamma^\mu$. The resulting expression is

$$\begin{aligned}
\mathcal{M}^\mu_{v1} = i\frac{g^3\mu^{3\varepsilon}}{16\pi^2}\Big(C_2(r) - \frac{1}{2}C_2(G)\Big)\gamma^\mu\int dx\,dy\,dz\,\delta(x+y+z-1)\\
\times\frac{(2-d)^2}{2}\frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}
\end{aligned} \tag{C.24}$$

The second diagram is



$$\mathcal{M}^{a\mu}_{v2} = ig^3\mu^{3\varepsilon}\int \frac{d^dk}{(2\pi)^d}\left[-\eta^{\mu\nu}(q+k)^\mu + \eta^{\nu\rho}(2k-q)^\mu + \eta^{\rho\mu}(2q-k)^\nu\right]$$
$$\times f^{abc}T^bT^c\gamma^\lambda \frac{\not{k}-\not{p}_1+m}{(k-p_1)^2-m^2+i\epsilon}\gamma^\sigma \frac{\eta_{\nu\lambda}}{k^2+i\epsilon}\frac{\eta_{\rho\sigma}}{(k-q)^2+i\epsilon} \quad \text{(C.25)}$$

The Lie algebra factor simplifies to $\frac{i}{2}C_2(G)T^a$. Shift momentum to $\ell = k - yq - zp_1$ and combine the denominator

$$\mathcal{M}^{\mu}_{v2} = -g^3\mu^{3\varepsilon}\frac{C_2(G)}{2}\int dx\,dy\,dz\,\delta(x+y+z-1)\int \frac{d^d\ell}{(2\pi)^2}\frac{2N^\nu}{(\ell^2-\Delta)^3}$$
$$\Delta = xm^2 + y(y-1)q^2 + z(z-1)p_1^2 + 2yzqp_1 - i\epsilon$$
$$N^\mu = -\gamma^\mu(\not{\ell} + y\not{q} + (z-1)\not{p}_1 + m)(\not{\ell} + (1+y)\not{q} + z\not{p}_1)$$
$$-(\not{\ell} + (y-2)\not{q} + z\not{p}_1)(\not{\ell} + y\not{q} + (z-1)\not{p}_1 + m)\gamma^\mu$$
$$+\gamma^\rho(\not{\ell} + y\not{q} + (z-1)\not{p}_1 + m)\gamma_\rho(2\ell + (2y-1)q + 2zp_1)^\mu$$

As before we are only interested in the divergent part and it integrates to

$$\mathcal{M}^{\mu}_{v2} = -i\frac{g^3\mu^{3\varepsilon}C_2(G)}{16\pi^2}\gamma^\mu(1-d)\int dx\,dy\,dz\,\delta(x+y+z-1)\frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} \quad \text{(C.26)}$$

With the counterterm specified as in equation (C.20), with $x = 1 - z - y$

$$\delta_g = -\frac{g^2\mu^{2\varepsilon}}{16\pi^2}\int dy\,dz\,\frac{1}{\varepsilon}\left[1 + \varepsilon\log\left(\left.\frac{\mu^2}{\Delta}\right|_{p^2=\mu^2}\right) + \mathcal{O}(\varepsilon^2)\right]$$
$$\times 2\left(C_2(r) + C_2(G)\right) \quad \text{(C.27)}$$

The divergent part is

$$\lim_{\varepsilon\to 0}\varepsilon\delta_g = -\frac{g^2}{16\pi^2}\left(C_2(r) + C_2(G)\right) \quad \text{(C.28)}$$

D

Calculus in Superspace

This is a short list of the most common conventions for integration and differentiation with Grassman numbers. Actually, calculus is much simpler for fermionic numbers than for normal coordinates, because anticommutivity guarantees that the only functions that appear are constants and first order polynomials. The basic rules for differentiation are

$$
\begin{aligned}
&\partial_\alpha = \frac{\partial}{\partial\theta^\alpha}, \quad \partial^\alpha = \frac{\partial}{\partial\theta_\alpha} = -\epsilon^{\alpha\beta}\partial_\beta \\
&\bar{\partial}^{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}}, \quad \bar{\partial}_{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} = -\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\partial}^{\dot{\beta}} \\
&\partial_\alpha\theta^\beta = \delta^\beta_\alpha, \quad \bar{\partial}^{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} = \delta^{\dot{\alpha}}_{\dot{\beta}} \\
&\partial_\alpha(\theta\theta) = 2\theta_\alpha, \quad \bar{\partial}_{\dot{\alpha}}(\overline{\theta\theta}) = -2\bar{\theta}_{\dot{\alpha}} \\
&\partial^2(\theta\theta) = 4, \quad \bar{\partial}^2(\overline{\theta\theta}) = 4
\end{aligned}
$$

Integration is also very simple. It is called Berezin integration, due to its founder

$$
\begin{aligned}
&\int d\theta^1\,\theta^1 = \int d\theta^2\,\theta^2 = 1 \\
&\int d\theta^1\,\theta^2 = \int d\theta^2\,\theta^1 = \int d\theta^\alpha = 0
\end{aligned}
$$

Likewise for $\bar{\theta}$. Note that Berezin integration produces the same result as differentiation and therefore an integral can always be swapped for a derivative. The basic reason integration is defined in this way is to get translational invariance, in analogy to integration in normal space. It is easy to see that it is invariant: make a change of coordinates $\theta \to \theta + \zeta$, where ζ is a constant in superspace, and we have

$$\int d(\theta+\zeta)\, f(\theta+\zeta) = \int d\theta\, f(\theta) \tag{D.1}$$

Note that if the coordinates were assigned a unit, the differential $d\theta^\beta$ would acquire the inverse unit. Introduce the following conventions for the differential

$$\begin{aligned} d^2\theta &= -\frac{1}{4} d\theta^\alpha d\theta^\beta \epsilon_{\alpha\beta} \\ d^2\bar{\theta} &= -\frac{1}{4} d\bar{\theta}_{\dot{\alpha}} d\bar{\theta}_{\dot{\beta}} \epsilon^{\dot{\alpha}\dot{\beta}} \\ d^4\theta &= d^2\theta\, d^2\bar{\theta} \end{aligned} \tag{D.2}$$

The reason these conventions are preferable is the identity

$$\int d^2\theta\, \theta\theta = 1, \qquad \int d^2\bar{\theta}\, \overline{\theta\theta} = 1 \tag{D.3}$$

The expression for the SUSY generators and the covariant derivatives are

$$Q_\alpha = \partial_\alpha - i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu, \qquad \overline{Q}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} + i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \tag{D.4}$$

$$D_\alpha = \partial_\alpha + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu, \qquad \overline{D}\dot{\alpha} = -\bar{\partial}_{\dot{\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \tag{D.5}$$

For chiral superfields it is often convenient to use the bosonic coordinates y and $y^\dagger$. In terms of these the generators and covariant derivatives are

$$Q_\alpha(y) = \partial_\alpha, \qquad \overline{Q}_{\dot{\alpha}}(y) = -\bar{\partial}_{\dot{\alpha}} + 2i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\frac{\partial}{\partial y^\mu} \tag{D.6}$$

$$Q_\alpha(y^\dagger) = \partial_\alpha - 2i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial y^{\dagger\mu}}, \qquad \overline{Q}_{\dot{\alpha}}(y^\dagger) = -\partial_{\dot{\alpha}} \tag{D.7}$$

$$D_\alpha(y) = \partial_\alpha + 2i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial y^\mu}, \qquad \overline{D}_{\dot{\alpha}}(y) = -\bar{\partial}_{\dot{\alpha}} \tag{D.8}$$

$$D_\alpha(y^\dagger) = \partial_\alpha, \qquad \overline{D}_{\dot{\alpha}}(y^\dagger) = -\bar{\partial}_{\dot{\alpha}} - 2i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\frac{\partial}{\partial y^{\dagger\mu}} \tag{D.9}$$

The properties of the covariant derivatives are

$$\{D_\alpha, \overline{D}_{\dot{\alpha}}\} = -2i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \tag{D.10}$$

$$\{D_\alpha, D_\beta\} = \{\overline{D}_{\dot{\alpha}} = \overline{D}_{\dot{\beta}}\} = 0 \tag{D.11}$$

$$\{D_\alpha, Q_\beta\} = \{D_\alpha, \overline{Q}_{\dot{\beta}}\} = \{\overline{D}_{\dot{\alpha}}, Q_\beta\} = \{\overline{D}_{\dot{\alpha}}, \overline{Q}_{\dot{\beta}}\} = 0 \tag{D.12}$$

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