Analysing the Schmid Operator for $\mathrm{SL}(2, \mathbb{R})$ and $\mathrm{SU}(2, 1)$

Master's thesis in Physics

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Abstract

The study of holomorphic discrete series on the special linear group $\mathrm{SL}(2,\mathbb{R})$ has been very fruitful in many areas in mathematics and physics, so there is a natural question to ask if they can be generalised to other Lie groups. There exists a differential operator called the Schmid operator which can be used to define discrete series representations of semisimple Lie groups. Understanding the Schmid operator is one of the main goals of this thesis. A second goal is to describe the Schmid operator in a tensorial formalism which is more closely related to differential geometry. We show that the Schmid operator before the projection is equivalent to a covariant derivative in the tensorial formalism. The Schmid operator is given explicitly for the Lie groups $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2,1)$. In the case of $\mathrm{SL}(2,\mathbb{R})$ the conditions for holomorphic and anti-holomorphic discrete series are retrieved. In the case of $\mathrm{SU}(2,1)$ the conditions for holomorphic, anti-holomorphic, and quaternionic discrete series are retrieved.

Keywords: Schmid operator, discrete series representation, holomorphic discrete series, quaternionic discrete series, symmetric space.

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Contents

Chapter 1

Introduction

The topic of this thesis is representation theory of Lie groups. More specifically the thesis is about a certain differential operator, called the *Schmid operator*, that is used to define *discrete series representations.* Before we are ready to formulate this idea more precisely, we need some background. In the background, we will first discuss what discrete series representations are, and then explain why they are important from the perspective of automorphic forms. This will then motivate the need to introduce the Schmid operator. At the end of the introduction we will discuss the aim and outline of this report.

1.1 Background

First let us introduce discrete series representations on a Lie group G. Consider a space X with a right group action $X \times G \to X$ defined as $x \mapsto xg$. Consider a space of functions $\mathcal{F} = \{f : X \to \mathbb{C}\}$ which transforms with the right regular group action:

$$f(x) \longmapsto f(xg). \tag{1.1}$$

A physicist would call f a scalar field, since it only transform in the argument. The space of functions $\mathcal{F}$ is a representation if it is closed under the group action. Since G is a continuous space, we would in general expect $\mathcal{F}$ to be a continuous space as well. However in certain instances, one can find spaces $\mathcal{F}$ which are discrete, then $\mathcal{F}$ is a discrete series representation.

Let us now discuss discrete series representations from the perspective of automorphic forms. Automorphic forms are certain types of functions with a discrete symmetry. Before defining automorphic forms, it is instructive to look at a very simple example of a discrete symmetry. Consider complex periodic functions f on the real number line. The discrete symmetry is:

$$f(x + 2\pi) = f(x), \quad \forall x \in \mathbb{R}. \tag{1.2}$$

All reasonable functions f can be expressed as a Fourier series:

$$f(x) = \sum_{n \in \mathbb{Z}} c_n \mathrm{e}^{inx}. \tag{1.3}$$

Where the basis functions e^{inx} are eigenfunctions to the Laplacian:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} \mathrm{e}^{inx} = -n^2 \mathrm{e}^{inx}. \tag{1.4}$$

The fact that the basis are eigenfunctions is very general and follows from Schur's lemma, which is discussed in chapter 2. Generalising the basis functions e^{inx} to the case of a general discrete symmetry Γ, will lead us to the notion of automorphic forms. The domain of automorphic forms is naturally a Lie group G which contains the discrete symmetry transformations Γ. Next we need to generalise the Laplacian. The Laplacian of a Lie group is a quadratic differential operator commuting with the group action. In general, there can also be higher order differential operators which also commute with the group action. These differential operators, including the Laplacian, are called Casimir operators. It is also useful to add a growth condition. The growth condition can depend on the application but the general idea is to limit the growth at infinity so one can do proper calculus. Let us summarise the three conditions defining automorphic forms φ.

1. Invariant under the action of a discrete subgroup $\Gamma \subset G$

$$\varphi(\gamma \cdot g) = \varphi(g), \quad \forall \gamma \in \Gamma. \tag{1.5}$$

2. An eigenfunction to all Casimir operators.

3. Have well-behaved growth conditions.

Automorphic forms appear in many branches of mathematics and physics such as string theory and number theory. One example from string theory is the predicted quantum corrections for a black hole [1].

There is a cousin to automorphic forms called modular forms. Modular forms are very similar to automorphic forms but have a weaker invariance property. A modular form is allowed to transform with a factor depending on the discrete transformation. We formulate modular forms in the special case of the Lie group $\mathrm{SL}(2,\mathbb{R})$, which consists of 2×2 real matrices of determinant 1. The Lie group $\mathrm{SL}(2,\mathbb{R})$ is closely related to the complex upper half plane:

$$\mathbb{H} = \{\tau \in \mathbb{C} : \operatorname{Im} \tau > 0\}. \tag{1.6}$$

This allows us to define functions on $\mathbb{H}$ rather than on $\mathrm{SL}(2,\mathbb{R})$. More about this in chapter 3. Let Γ be a discrete subgroup of $\mathrm{SL}(2,\mathbb{Z})$. A holomorphic modular form of weight $w \geq 0$ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ which satisfies:

$$f\Big(\frac{az+b}{cz+d}\Big) = (cz+d)^w f(z), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma. \tag{1.7}$$

And f is bounded by some polynomial as $\mathrm{Im}(z) \to \infty$.

One could similarly define a broader class of modular forms on SL(2,$\mathbb{R}$) without the holomorphic condition. However there is a reason to pay special attention to the holomorphic functions. Often the most interesting thing about modular forms are their Fourier coefficients. And the holomorphic functions have particularly nice Fourier coefficients. To see this, consider a modular form $f(z)$, with translation symmetry $f(z+1) = f(z)$. Let $z = x+iy$ with $y > 0$. The translation symmetry implies that f has a Fourier expansion in x:

$$f(z) = \sum_{n\in\mathbb{Z}} c_n(y) \mathrm{e}^{2\pi i n x}. \tag{1.8}$$

The holomorphic condition implies that the coefficients are $c_n(y) = b_n \mathrm{e}^{-2\pi n y}$. With this condition, f is a discrete series. The coefficients b_n can often contain valuable information. Let us consider a very instructive example of this from number theory, described in more detail in [2]. Let n be a fixed integer. Find the number of integer solutions $r(n)$ to the equation:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = n. \tag{1.9}$$

To solve this, we define a generating function:

$$\theta(\tau) = \sum_{n=0}^{\infty} r(n) q^n, \quad q = \mathrm{e}^{2\pi i \tau}. \tag{1.10}$$

This series converges for $\tau \in \mathbb{H}$. The generating function θ has two types of discrete symmetries. One obvious $\theta(\tau+1) = \theta(\tau)$. To find the other discrete symmetry we write θ as a product:

$$\theta(\tau) = \Big(\sum_{m\in\mathbb{Z}} q^{m^2}\Big)^4. \tag{1.11}$$

Using Poisson summation from Fourier analysis, one can show that θ has the following discrete symmetry:

$$\theta\big(\frac{\tau}{4\tau+1}\big) = (4\tau+1)^2 \theta(\tau). \tag{1.12}$$

We recognise θ as a holomorphic modular form of weight 2, where the discrete subgroup Γ is generated by the two group elements:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}). \tag{1.13}$$

It would be helpful if we could find all holomorphic modular forms of weight 2 with these symmetries. And then find their q expansions. Finding the space of functions satisfying the discrete symmetry is where the study of modular forms comes into play. It turns out there are only two linearly independent such functions, call them ϕ_1 and ϕ_2. We know $\theta(\tau)$ is some linear combination of ϕ_1 and ϕ_2, which can be determined by comparing manually the first two Fourier coefficents corresponding to q^0, q^1.

The study of holomorphic modular forms f on $\mathrm{SL}(2, \mathbb{R})$ has been very fruitful in many areas in mathematics and physics, so there is a natural question to ask if they can be generalised to other Lie groups. The holomorphic property can be written explicitly with Cauchy Riemann equations:

$$\frac{\partial}{\partial \bar{z}} f(z) = 0. \tag{1.14}$$

Some Lie groups have other structures than complex. Perhaps then the differential operator $\partial_{\bar{z}}$ can be replaced with something more general. In 1970, Wilfried Schmid gave a possible answer to this question, by introducing the Schmid operator [3]. The Schmid operator $\mathbf{D}$ is a differential operator acting on for instance automorphic forms φ. And requiring that the automorphic forms are annihilated by the Schmid operator $\mathbf{D}$, restricts the automorphic forms to a discrete series representation, that can be of greater interest.

1.2 Aim

Formulating and understanding the Schmid operator is one of the main goals of this thesis. An interesting special case is when the Lie group has a so called *quaternionic structure*. Then the requirement that the Schmid operator annihilates the automorphic forms can be used to define a quaternionic discrete series representation. These quaternionic discrete series representations have for instance been studied by Aaron Pollack [4]. One example that will be of interest in this thesis is the Lie group $\mathrm{SU}(2, 1)$. It is a group with only rank 2 and it has both a quaternionic and complex structure. The automorphic forms on the group $\mathrm{SU}(2, 1)$ also have interesting applications in string theory [5].

An advantage of the Schmid operator is that it is formulated in a very general manner. But this also makes it abstract. It would therefore be enlightening to formulate it in another setting. There is an alternative method of creating differential operators on automorphic forms using a tensorial formalism, that for instance is used in the article [6]. The tensorial formalism is based on coordinates on the group and is more closely related to differential geometry. The main idea is to consider a covariant derivative $\mathcal{D}$, from [6]. A second goal of the thesis is to relate the Schmid operator to the tensorial formalism.

1.3 Outline of the thesis

In chapter 2, some theory of Lie groups and their representation theory is presented. In chapter 3 and 4 we take a closer look at the Lie groups $\mathrm{SL}(2,\mathbb{R})$ and SU(2,1) respectively. These two groups will serve as examples when studying the Schmid operator in chapter 5. At the end of chapter 5, we make the connection between the Schmid operator and the tensorial formalism. Finally in chapter 6, we discuss the possible continuations of this project.

Chapter 2

Theory

In this chapter, some elements of Lie theory and representation theory are presented that will be useful in the later chapters. Most of the material can for instance be found in [7]. The experienced reader can skip this chapter or perhaps use it as a dictionary when needed.

2.1 Lie theory

The natural place to start is to introduce groups.

Definition 1 (Group)**.** A group $(G, \cdot)$ is a set G together with a binary operation $(\cdot)$ which satisfies four properties:

(i) Closed: $a, b \in G \Rightarrow ab \in G$.

(ii) Neutral element exists: $\exists e \in G : ea = ae = a, \forall a \in G$.

(iii) Inverse exists: $\forall a \in G, \exists a^{-1} \in G : a^{-1}a = aa^{-1} = e$.

(iv) Associative: $(ab)c = a(bc), \quad \forall a, b, c \in G$.

A subgroup H is a subset of G which is also a group. One can form a left coset of H by gH, where $g \in G$. The space of cosets is called the quotient space G/H. The quotient space is a set of equivalence classes of G with the equivalence relation $g \sim gh$, where $h \in H$ and $g \in G$. Note that the quotient space is in general not a group. Groups are interesting objects to study by themselves, but it is often interesting to see how a group acts as transformations.

Definition 2 (Group action)**.** Let G be a group and X be a set. A group action of G on X is an operation $* : G \times X \to X$ such that:

1. The neutral element $e \in G$ leaves any $x \in X$ invariant:

$$e * x = x. \tag{2.1}$$

2. The group action is compatible with group multiplication of any two elements $g', g \in G$:
$$(g'g) * x = g' * (g * x). \tag{2.2}$$

The next step will be to introduce Lie groups. Recall that a real manifold is a space that is locally Euclidean. And a smooth manifold is a space that locally resembles a vector space. From a practical point of view Lie groups can be seen as groups that can be parameterised by some set of continuous coordinates, and there exists partial derivatives with respect to these coordinates.

Definition 3 (Real Lie group)**.** A real Lie group is a group which is also a real smooth manifold. And the group multiplication and inversion are smooth maps. I.e. the map $\phi : G \times G \to G$ defined as

$$\phi(x, y) = x^{-1}y, \tag{2.3}$$

is smooth.

Similarly one can define complex Lie groups. In this thesis the groups of interests are matrix groups. For matrix groups the topological notations of neighbourhoods and compactness have very concrete meaning. A matrix group of $n \times n$ real matrices can be seen as a space embedded in $\mathbb{R}^{n\times n}$ Euclidean space. A compact subgroup is then a subspace of $\mathbb{R}^{n\times n}$ which is closed and bounded. The concept of a maximal compact subgroup will be of great interest in this thesis.

Definition 4 (Maximal compact subgroup)**.** Let G be a Lie group. A maximal compact subgroup K, is a compact subgroup which is not a proper subgroup of any other compact subgroup.

Remark. The maximal compact subgroup is unique up to conjugation, by the Cartan-Iwasawa-Malcev theorem.

Lie Algebra

A Lie group is a smooth manifold. Meaning that locally it resembles a vector space. The vector space at the identity is the Lie algebra.

Definition 5 (Lie Algebra)**.** A Lie Algebra $\mathfrak{g}$ is a vector space together with a product called Lie bracket. The Lie bracket is anti-symmetric, bilinear and obeying the Jacobi identity.

It is useful to categorise different types of Lie algebras depending on how the Lie bracket behave. Consider a Lie algebra $\mathfrak{g}$. A subalgebra $\mathfrak{h}$ is a subspace closed under the bracket $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$. An ideal I is a subalgebra which is stable under the adjoint action: $[\mathfrak{g}, I] \subseteq I$. For any Lie algebra $\mathfrak{g}$, one can construct a series of ideals I_n called the derived series:

$$I_{n+1} = [I_n, I_n], \quad I_1 = [\mathfrak{g}, \mathfrak{g}]. \tag{2.4}$$

A Lie algebra $\mathfrak{g}$ is solvable if $I_n = 0$ for large enough n. For matrix algebras, solvable means upper triangular matrices in some basis. The largest solvable ideal in $\mathfrak{g}$ is called the radical and is denoted $\mathrm{rad}(\mathfrak{g})$. A Lie algebra $\mathfrak{g}$ is called semisimple if it contains no nonzero solvable ideals. The following theorem explains why the semisimple Lie algebras are important.

Theorem 1 (Levi)**.** *Any Lie algebra $\mathfrak{g}$ can be written as a direct sum (as vector spaces) of the radical and a semisimple subalgebra $\mathfrak{g}_{\mathrm{ss}}$.*

$$\mathfrak{g} = \mathrm{rad}(\mathfrak{g}) \oplus \mathfrak{g}_{\mathrm{ss}}. \tag{2.5}$$

For instance the Lie algebra $\mathfrak{u}(n) = \mathfrak{u}(1) \oplus \mathfrak{su}(n)$ is the sum of the radical $\mathfrak{u}(1)$ and the semisimple subalgebra $\mathfrak{su}(n)$. Most Lie algebras considered in this thesis will be semisimple. Semisimple Lie algebras can be decomposed further into a sum of simple Lie algebras. A simple Lie algebra is non-abelian and contains no proper non-zero ideals. One can add additional structure to a Lie algebra by introducing a metric.

Definition 6 (Invariant bilinear form)**.** A bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ on a Lie algebra $\mathfrak{g}$ is called invariant if:

$$B([Z, X], Y) + B(X, [Z, Y]) = 0, \quad \forall X, Y, Z \in \mathfrak{g}. \tag{2.6}$$

Definition 7 (Killing form)**.** Let $\mathfrak{g}$ be a Lie algebra. Define the *Killing form* $K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ as:

$$K(X, Y) := \mathrm{tr}(\mathrm{ad}\, X \circ \mathrm{ad}\, Y). \tag{2.7}$$

It is easy to see that the Killing form is invariant. The following theorem says that the Killing form is suitable as metric for semisimple Lie algebras.

Theorem 2 (Cartan criterion of semisimplicity)**.** *A Lie algebra is semisimple if and only of the Killing form is non-degenerate.*

For real Lie algebras, the sign of the Killing form contain valuable information about the compactness of the Lie group. For instance consider the Lie algebra $\mathfrak{so}(n, \mathbb{R})$. The antisymmetry implies that the Killing form is negative definite and exponentiation of an anti-symmetric matrix yields a bounded result. That example is a special case of the following theorem.

Theorem 3 (Compactness)**.** *Let G be a connected real Lie group, with a semisimple Lie algebra $\mathfrak{g}$. Then G is compact if and only if the Killing form on $\mathfrak{g}$ is negative definite.*

Theorem 3 is a useful tool to test compactness. Say that we are working with some Lie group G and want to find a maximal compact subgroup K. By theorem 3 it is sufficient to find a maximal subalgebra $\mathfrak{k}$ with negative definite Killing form.

Cartan involution

Finding a maximal subalgebra $\mathfrak{k}$ can be done in a systematic way, by introducing the concept of Cartan involution.

Definition 8 (Cartan involution)**.** Let $\mathfrak{g}$ be a real semisimple Lie algebra and let K be the Killing form. A Cartan involution θ is an involution and automorphism satisfying:

$$K(X, \theta(X)) < 0, \quad \forall X \in \mathfrak{g}. \tag{2.8}$$

The Cartan involution can be seen as a generalisation of minus hermitian conjugate. One can prove that a Cartan involution always exists and is unique up to an inner automorphism. The Cartan involution has two possible eigenvalues ± 1, because $\theta^2 = 1$. Since θ is symmetric with respect to an inner product: $-K(X, \theta(Y))$, we can use the spectral theorem to decompose the vector space $\mathfrak{g}$ accordingly into $\mathfrak{k}$ $(+1)$ and $\mathfrak{p}$ (-1). The vector space decomposition is:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}. \tag{2.9}$$

where the two eigenspaces are orthogonal

$$K(\mathfrak{k}, \mathfrak{p}) = 0. \tag{2.10}$$

Notice that the Killing form is non-degenerate on the subspaces $\mathfrak{k}$ and $\mathfrak{p}$. The following relations:

$$1 \cdot 1 = 1, \quad (-1) \cdot (-1) = 1, \quad (-1) \cdot 1 = -1,$$

imply that:

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}, \quad [\mathfrak{p}, \mathfrak{k}] \subseteq \mathfrak{p}. \tag{2.11}$$

The last relation $[\mathfrak{p}, \mathfrak{k}] \subseteq \mathfrak{p}$ shows that $\mathfrak{p}$ is a representation of $\mathfrak{k}$. In the fundamental matrix representation, this action also exists at the group level.

Proposition 1. $\mathfrak{p}$ *is a representation of the compact subgroup K, via the action:*

$$k^{-1} \mathfrak{p} k \subseteq \mathfrak{p}. \tag{2.12}$$

Proof. This can be seen by writing k as:

$$k = \mathrm{e}^X \approx \big(1 + \frac{X}{n}\big)^n, \tag{2.13}$$

for some large integer n and $X \in \mathfrak{k}$. It is easy to see:

$$\big(1 - \frac{X}{n}\big)\mathfrak{p}\big(1 + \frac{X}{n}\big) + \mathcal{O}\big(\frac{1}{n^2}\big) \subseteq \mathfrak{p}, \tag{2.14}$$

where $\mathcal{O}$ is the big ordo notation. Equation 2.12 then follows by induction. □

An important consequence of the Cartan involution is that $\mathfrak{p}$ is equipped with a positive definite inner product. This implies that the quotient space G/K is a Riemannian manifold. Moreover the Cartan involution induces an inversion symmetry which implies that G/K is in fact a symmetric space.

Root system

A Lie algebra is completely defined by a basis T_i and the structure constants f_{ij}^k:

$$[T_i, T_j] = f_{ij}^k T_k. \tag{2.15}$$

However when classifying a Lie algebra, it is useful to introduce a language which is more basis independent. This will lead us to the notion of a root system. Consider a complex semisimple Lie Algebra $\mathfrak{g}$. Take some element $h_1 \in \mathfrak{g}$ with $\operatorname{ad} h_1$ diagonalisable. We would want to classify elements $X \in \mathfrak{g}$ by their eigenvalue with $\operatorname{ad} h_1$:

$$\operatorname{ad} h_1(X) = [h_1, X] = \lambda_1 X. \tag{2.16}$$

In general the eigenvalues will be degenerate. So we need more operators to classify the elements. Introduce a Cartan subalgebra.

Definition 9. Let $\mathfrak{g}$ be a semisimple Lie algebra. A Cartan subalgebra $\mathfrak{h}$ is a maximal abelian subalgebra such that $\operatorname{ad} h, \forall h \in \mathfrak{h}$, is simultaneously diagonalisable.

The elements in the Cartan subalgebra $\mathfrak{h}$ share a common eigenbasis $\{v\}$. Take some eigenvector v and $h \in \mathfrak{h}$, the corresponding array of eigenvalues $\alpha(h)$ is called a root.

$$[h, v] = \alpha(h)v. \tag{2.17}$$

Let us decompose the vector space $\mathfrak{g}$ into eigenspaces:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha, \tag{2.18}$$

where

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} : [h, X] = \alpha(h)X, \ \forall h \in \mathfrak{h}\}. \tag{2.19}$$

The set of roots $\{\alpha\}$ is called the root system. The roots are vectors in $\mathfrak{h}^*$, i.e. the dual space of $\mathfrak{h}$. The metric on $\mathfrak{h}$ is the Killing form of $\mathfrak{g}$ and is by construction non-degenerate. One can show that the Killing form, restricted to the real vector space generated by the roots, is real and positive definite. Denote the induced inner product by $\langle , \rangle$. This allows us to define lengths and relative angles between roots.

Theorem 4 (Root system). *Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and root decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha$ then:*

1. *The subspaces $\mathfrak{g}_\alpha$ are 1-dimensional.*

2. *If α is a root, then so is $-\alpha$.*

3. If α, β and $\alpha+\beta$ are roots, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

The first part implies that there is a 1-1 correspondance between Lie algebra elements $\mathfrak{g} \ominus \mathfrak{h}$ and roots. The second part allows us to divide the roots into two categories.

Definition 10 (Positive roots)**.** Let Δ be the set of non-zero roots. A set of positive roots $\Delta^+ \subset \Delta$ is such that:

1. For each $\alpha \in \Delta$, exactly one of α and $-\alpha$ is in Δ^+.
2. For any two distinct $\alpha, \beta \in \Delta^+$ and $\alpha+\beta \in \Delta$, then $\alpha+\beta \in \Delta^+$.

As an example it is illustrative to consider the fundamental matrix representation. If the Cartan subalgebra is diagonal, the positive roots can be associated to the strictly upper triangular elements, and the negative roots to the strictly lower triangular elements.

2.2 Group action on functions

Consider functions f defined on some Lie group G. The functions f can be viewed as elements in a representation module, by introducing a group action on f called the right-regular action.

Definition 11 (Right-regular action)**.** Let G be a Lie group and f a function on G. The right-regular action $*$ is defined as:

$$(h * f)(g) = f(gh), \quad g, h \in G. \tag{2.20}$$

This is a group action since $\forall h, h', g \in G$:

$$h * (h' * f)(g) = h' * f(gh) = f(ghh') = (hh' * f)(g). \tag{2.21}$$

Assuming G is some matrix group then the corresponding Lie algebra action is given by:

$$(X * f)(g) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(g\,\mathrm{e}^{tX}), \quad X \in \mathfrak{g}, g \in G. \tag{2.22}$$

Group actions can by combined with Lie algebra actions. For instance $h^{-1}Xh$ where $X \in \mathfrak{g}$ and $h \in G$:

$$([h^{-1}Xh] * f)(g) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(g\mathrm{e}^{th^{-1}Xh}) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(gh^{-1}\mathrm{e}^{tX}h). \tag{2.23}$$

Maurer Cartan form

It is useful to relate the Lie algebra action in equation (2.22), to a specific parametrisation of a matrix Lie group G. Say a generic element $g = g(t)$ depends on a coordinate t. We would like to know what Lie algebra element T corresponds to the partial derivative ∂_t.

$$
\begin{aligned}
(T * f)(g) := (\partial_t f)\big(g(t)\big) &:= \lim_{\delta t \to 0} \frac{f\big(g(t+\delta t)\big) - f\big(g(t)\big)}{\delta t} \\
&= \lim_{\delta t \to 0} \frac{g^{-1}(t) g(t+\delta t) - 1}{\delta t} * f\big(g(t)\big) = g^{-1}(t)\partial_t g(t) * f\big(g(t)\big). \quad (2.24)
\end{aligned}
$$

Hence

$$T = g^{-1}\partial_t g. \quad (2.25)$$

Here we used that g can be expanded as a power series in δt. This calculation motivates the following definition.

Definition 12 (Maurer Cartan form)**.** Let G be a matrix Lie group, with generic element g. Let x^μ be a parametrisation of G. The Maurer Cartan form is the one-form:

$$g^{-1}\mathrm{d}g = g^{-1}(x)\partial_\mu g(x)\mathrm{d}x^\mu. \quad (2.26)$$

The Maurer Cartan form is for instance useful when writing Casimir elements as differential operators.

2.3 Representation theory

In this section, some useful elements of representation theory for Lie algebras and Lie groups, are discussed.

Definition 13 (Group representation)**.** A finite-dimensional representation (ρ, V) of a Lie group G is a vector space V together with a group homomorphism $\rho : G \to \mathrm{GL}(V)$. I.e. a map compatible with group multiplication:

$$\rho(gh) = \rho(g)\rho(h), \quad \forall g, h \in G. \quad (2.27)$$

Definition 14 (Lie algebra representation)**.** A finite-dimensional representation (σ, V) of a Lie algebra $\mathfrak{g}$ is a vector space V together with a Lie algebra homomorphism $\sigma : \mathfrak{g} \to \mathrm{GL}(V)$. I.e. a linear map compatible with the Lie bracket:

$$\sigma([X, Y]) = [\sigma(X), \sigma(Y)], \quad \forall X, Y \in \mathfrak{g}. \quad (2.28)$$

It is often sufficient to consider Lie algebra representations to categorise Lie group representations. More precisely, if G is a simply connected Lie group with Lie algebra $\mathfrak{g}$, then there is a 1-1 correspondence between representations of G and of $\mathfrak{g}$. For matrix groups, this correspondence is very explicit. If we have a representation σ of $\mathfrak{g}$ then

$$\rho(\mathrm{e}^{X}) = \mathrm{e}^{\sigma(X)}, \tag{2.29}$$

is a representation of G. Conversely if ρ is a representation of G then

$$\sigma(X) = \frac{\mathrm{d}}{\mathrm{d}t}\rho(\mathrm{e}^{tX})\big|_{t=0}, \tag{2.30}$$

is a representation of $\mathfrak{g}$. Another important correspondence is for complex and real Lie algebras. If σ is a representation of $\mathfrak{g}$ then there is an induced representation $\sigma_{\mathbb{C}}$ of the complexification $\mathfrak{g}_{\mathbb{C}}$ given by:

$$\sigma_{\mathbb{C}}(X + iY) = \sigma(X) + i\sigma(Y), \quad \forall X, Y \in \mathfrak{g}. \tag{2.31}$$

This is useful since representation theory of complex Lie algebras are in many ways easier than that of real Lie algebras.

Casimir operators

All representations of a Lie algebra share a common algebraic structure called the *Universal enveloping algebra.*

Definition 15 (Universal enveloping algebra)**.** Let $\mathfrak{g}$ be a Lie algebra. The universal enveloping algebra $\mathcal{U}_{\mathfrak{g}}$ is the tensor algebra:

$$\mathcal{T}_{\mathfrak{g}} = \bigoplus_{n\geq 0} \mathfrak{g}^{\otimes n}, \tag{2.32}$$

with the extra condition:

$$xy - yx = [x, y], \quad \forall x, y \in \mathfrak{g}. \tag{2.33}$$

The set of elements in $\mathcal{U}_{\mathfrak{g}}$ which commute with $\mathfrak{g}$ is called the center and is denoted $\mathcal{Z}_{\mathfrak{g}}$. The defining property is $[\mathcal{Z}_{\mathfrak{g}}, \mathfrak{g}] = 0$. The dimension of $\mathcal{Z}_{\mathfrak{g}}$ is equal to the rank of $\mathfrak{g}$, by Racah's theorem. For a finite semisimple Lie algebra the center is spanned by homogeneous polynomials, called *Casimir elements.*

Theorem 5 (Quadratic Casimir element)**.** *Let $\mathfrak{g}$ be a simple Lie algebra, with basis x_i and dual basis x^i w.r.t. the Killing form. There is a unique (up to a constant) quadratic Casimir element which is given by:*

$$C_2 = x_i x^i. \tag{2.34}$$

Similarly one can construct higher order Casimir elements. C_m is a Casimir element of order m:

$$C_m = \mathrm{str}(\mathrm{ad}\, x_{i_1}\, \mathrm{ad}\, x_{i_2} \ldots \mathrm{ad}\, x_{i_m}) x^{i_1} x^{i_2} \ldots x^{i_m}, \tag{2.35}$$

here str is the symmetrised trace. For semisimple Lie algebras, the Casimir elements can be found by using the decomposition into simple Lie algebras. The reason why the center $\mathcal{Z}$ is so important is explained by Schur's lemma.

Theorem 6 (Schur's lemma)**.** *Let V, W be irreducible complex representations of a Lie group G. Where W is not isomorphic to V. Then:*

1. *any linear map $\phi : V \to V$ commuting with the group action, is a scalar in $\mathbb{C}$.*

2. *and, any linear map $\phi : V \to W$ commuting with the group action, is zero: $\phi = 0$.*

Proof can for instance be found here [7]. The first part of Schur's lemma implies that for any irreducible representation the Casimir elements will act as scalars. In other words, all states in a given representation satisfy the same eigenvalue equation. In general, these eigenvalues depend on the representation, so they can be used to label different irreducible representations.

Weight space

Just like a root system is useful to define a Lie algebra, there is a weight space that defines the finite representations. We focus on complex semisimple Lie algebras.

Definition 16 (Weight)**.** Let (ρ, V) be a representation of a Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$. An element $w \in \mathfrak{h}^*$ satisfying for some $\boldsymbol{v} \in V$:

$$\rho(h)\boldsymbol{v} = w(h)\boldsymbol{v}, \quad \forall h \in \mathfrak{h}, \tag{2.36}$$

is called a weight. The set of weights is called the weight space.

The weight space is closely related to the root system. In fact the weight space is a subset of the root lattice. More precisely, define coroots $\{\alpha^\vee\}$ by renormalising the roots $\{\alpha\}$:

$$\alpha^\vee := \frac{2}{\langle \alpha, \alpha \rangle} \alpha, \tag{2.37}$$

where $\langle , \rangle$ is the induced inner product on $\mathfrak{h}^*$ from the Killing form. It turns out that the weight space is the dual space to the coroot lattice. The weight space naturally classifies finite representations. Let us make this more precise. Let Δ^+

be the positive roots. For a finite representation (ρ, V) there must be a highest weight λ, i.e. such that no one of:

$$\lambda + \alpha, \quad \alpha \in \Delta^+, \tag{2.38}$$

are weights. The corresponding vector $\boldsymbol{v} \in V$ such that $\rho(h)\boldsymbol{v} = \lambda(h)\boldsymbol{v}$, is called a highest weight vector. A representation, generated by a highest weight vector, is called a highest weight representation.

Theorem 7 (Highest weight)**.** *Every irreducible finite-dimensional representation of a complex semisimple Lie algebra is a highest weight representation.*

See proof for instance in here [7].

Representation theory of SU(2) and U(1)

The representation theory of SU(2) and U(1) will be relevant later. First consider irreducible representations of U(1). Since the group is abelian all complex irreducible representations are 1 dimensional by Schur's lemma. They are all on the form:

$$\mathrm{e}^{i\theta} \longmapsto \mathrm{e}^{iM\theta}, \quad M \in \mathbb{Z}. \tag{2.39}$$

The tensor product between two such representations is:

$$\mathrm{e}^{iM\theta} \otimes \mathrm{e}^{iM'\theta} \cong \mathrm{e}^{i(M+M')\theta}. \tag{2.40}$$

Now consider the finite irreducible representations of SU(2). The root lattice for SU(2) is isomorphic to $2\mathbb{Z}$ and the weight lattice is then isomorphic to $\mathbb{Z}$. Hence each irreducible representation can be labelled by a highest weight $N \in \mathbb{Z}_{\geq 0}$. Denote the corresponding module V_N. One can show that V_N is spanned by $N+1$ weight vectors with weights:

$$-N, 2-N, \ldots, 2n-N, \ldots, N, \tag{2.41}$$

respectively. Consider two irreducible representations of SU(2) labelled by highest weights N and 1. Denote the modules V_N and V_1 respectively. The tensor product between them decomposes as:

$$V_N \otimes V_1 = V_{N-1} \oplus V_{N+1}. \tag{2.42}$$

This can easily be seen by counting the dimensions of the weight spaces and then utilising theorem 7.

2.4 Discrete series representation

Let G be a Lie group. Consider complex functions defined on the group $f : G \to \mathbb{C}$. As was mentioned in section 2.2, the space of functions

$$\mathcal{F} = \{f : G \to \mathbb{C}\}, \tag{2.43}$$

is a representation module of G, transforming by the right-regular action $*$. It turns out this representation can be decomposed into subrepresentations, which is described by Plancherel's theorem. The theory behind Plancherel's theorem is quite heavy and beyond the scope of this thesis, see for instance [8] for more details. Plancherel's theorem states that the space of square integrable functions can be decomposed into a direct integral with respect to a measure μ called Plancherel measure.

Definition 17 (Discrete series)**.** A discrete series representation is a discrete set of atoms of the Plancherel measure μ, which is also a representation.

Let us look at some simple examples. For instance on the unit circle $\mathrm{U}(1)$ the Fourier series:

$$f(x) = \sum_{n\in\mathbb{Z}} c_n \mathrm{e}^{inx}, \tag{2.44}$$

is a discrete series representation. Here all square integrable functions f can be written as a linear combination of unitary irreducible functions e^{inx}. In contrast, in the case of $f : \mathbb{R} \to \mathbb{C}$ the representation as a Fourier transform is not a discrete series:

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(k) \mathrm{e}^{ikx} \mathrm{d}k. \tag{2.45}$$

The continuous spectrum is due to the fact that $\mathbb{R}$ is non–compact. Recall that for compact groups the spectrum is discrete by the Peter–Weyl theorem [9]. For semisimple Lie groups there is an interesting theorem by Harish-Chandra [10].

Theorem 8 (Harish-Chandra)**.** *Let G be a connected semisimple Lie group, with a maximal compact subgroup K. G has a discrete series if and only if rank G = rank K.*

From this theorem, it follows for instance that $\mathrm{SL}(n, \mathbb{R})$ has a discrete series representation if and only if $n = 2$.

Chapter 3

The special linear group $\mathrm{SL}(2,\mathbb{R})$

In this chapter we consider the special linear group $\mathrm{SL}(2,\mathbb{R})$. We will derive the differential operators on the group and explain the connection to the upper half plane. We also consider automorphic forms, and in particular the condition for holomorphic discrete series. The group $\mathrm{SL}(2,\mathbb{R})$ will be a used as a concrete example later on, and will help us build understanding of the Schmid operator and the tensorial formalism.

The group structure

The special linear group $G = \mathrm{SL}(2,\mathbb{R})$ consists of 2×2 real matrices of determinant 1.

$$\mathrm{SL}(2,\mathbb{R}) = \Big\{ g \in \mathbb{R}^{2\times 2} \;:\; \det g = 1 \Big\}.$$

And the fundamental representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ consists of 2×2 real traceless matrices.

$$\mathfrak{sl}(2,\mathbb{R}) = \big\{ X \in \mathbb{R}^{2\times 2} \;:\; \mathrm{tr}\, X = 0 \big\}.$$

In the Chevalley-Serre basis $\mathfrak{sl}(2,\mathbb{R})$ takes the form:

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{3.1}$$

A subgroup of $\mathrm{SL}(2,\mathbb{R})$ is the group of 2D rotations $\mathrm{SO}(2)$. In fact we have the following proposition.

Proposition 2. $\mathrm{SO}(2)$ *is a maximal compact subgroup of* $\mathrm{SL}(2,\mathbb{R})$.

Proof. Use the Cartan involution $\Theta(X) = -X^{\mathrm{T}}$. Notice that:

$$\Theta(e-f) = e-f, \quad \Theta(h) = -h, \quad \Theta(e+f) = -(e+f). \tag{3.2}$$

Hence we get the Cartan decomposition:

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}, \tag{3.3}$$

where $\mathfrak{k} = \mathbb{R}(e - f)$. Exponentiation of $\mathfrak{k}$ gives:

$$\mathrm{e}^{\theta(e-f)} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \sin\theta \end{pmatrix}, \tag{3.4}$$

showing that SO(2) is a maximal compact subgroup. □

There is a common way to parameterise $\mathrm{SL}(2, \mathbb{R})$ via the Iwasawa decomposition:

$$g = n(x)a(y)k(\theta) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{3.5}$$

where $y > 0$ and $x, \theta \in \mathbb{R}$. The intuition for this decomposition is that the SO(2) matrix can rotate away the lower left component.

3.1 Differential operators

We can express the differential operators $\partial_x, \partial_y, \partial_\theta$ in terms of the Lie algebra using the Maurer Cartan form. Notice that these differential operators belong to the right regular representation of the Lie algebra. Start with ∂_x:

$$\begin{aligned} g^{-1}\partial_x g = g^{-1}\partial_x \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} ak = g^{-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} ak \\ = \frac{1}{2y}\Big(-\sin(2\theta)h + \cos(2\theta)(e + f) + (e - f)\Big). \end{aligned} \tag{3.6}$$

Next calculate ∂_y:

$$g^{-1}\partial_y g = k^{-1}a^{-1} \begin{pmatrix} \frac{1}{2\sqrt{y}} & 0 \\ 0 & -\frac{1}{2y\sqrt{y}} \end{pmatrix} k = \frac{1}{2y}\Big(\cos 2\theta\, h + \sin 2\theta(e + f)\Big). \tag{3.7}$$

Lastly calculate ∂_θ:

$$g^{-1}\partial_\theta g = k^{-1}\partial_\theta k = k^{-1} \begin{pmatrix} -\sin\theta & \cos\theta \\ -\cos\theta & -\sin\theta \end{pmatrix} = e - f. \tag{3.8}$$

Let us summarise these three equations:

$$\begin{pmatrix} 2y\partial_x \\ 2y\partial_y \\ \partial_\theta \end{pmatrix} = \begin{pmatrix} -\sin 2\theta & \cos 2\theta & 1 \\ \cos 2\theta & \sin 2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} h \\ e + f \\ e - f \end{pmatrix}, \tag{3.9}$$

where σ denotes the right regular representation. Solve for $h, e+f, e-f$:

$$\sigma \begin{pmatrix} h \\ e+f \\ e-f \end{pmatrix} = \begin{pmatrix} -\sin 2\theta & \cos 2\theta & \sin 2\theta \\ \cos 2\theta & \sin 2\theta & -\cos 2\theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2y\partial_x \\ 2y\partial_y \\ \partial_\theta \end{pmatrix} \tag{3.10}$$

Next let us calculate the Casimir operator. Denote the Lie algebra $X_i = (h, e, f)_i$ for $i = 1, 2, 3$. Choose the normalisation of the metric as:

$$\gamma_{ij} = \frac{1}{2} K(X_i, X_j) = \frac{1}{2} \operatorname{tr}(\operatorname{ad} X_i \circ \operatorname{ad} X_j) = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}_{ij}. \tag{3.11}$$

Where the inverse metric is:

$$\gamma^{ij} = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}^{ij}. \tag{3.12}$$

And the quadratic Casimir operator is:

$$C_2 = X_i X^i = \gamma^{ij} X_i X_j = \frac{1}{4} h^2 + \frac{1}{2}(ef + fe). \tag{3.13}$$

In the right regular representation we denote the Casimir operator: $\sigma(C_2) = \Delta_{\mathrm{SL}(2,\mathbb{R})}$.

Proposition 3. *The Casimir operator takes the following form in the right regular representation:*

$$\Delta_{\mathrm{SL}(2,\mathbb{R})} = y^2(\partial_x^2 + \partial_y^2) - y\partial_x \partial_\theta. \tag{3.14}$$

Proof. When calculating the $\Delta_{\mathrm{SL}(2,\mathbb{R})}$ it is useful to use the following fact:

$$k^{-1}(e-f)k = e - f. \tag{3.15}$$

Rewrite C_2:

$$C_2 = \frac{1}{4} h^2 + \frac{1}{2}(ef + fe) = \frac{1}{4} h^2 - \frac{1}{2} h + e^2 - e(e-f). \tag{3.16}$$

Utilise the fact that the Casimir operator commutes with the group action k.

$$C_2 = \frac{1}{4}(k^{-1}hk)^2 - \frac{1}{2} k^{-1}hk + (k^{-1}ek)^2 - k^{-1}ek(e-f). \tag{3.17}$$

From the calculations above we see that:

$$\partial_x = \sigma\big(\frac{1}{y} k^{-1} ek\big), \quad \partial_y = \sigma\big(k^{-1} \frac{h}{2y} k\big), \quad \partial_\theta = \sigma(e-f). \tag{3.18}$$

Hence:

$$\Delta_{\mathrm{SL}(2,\mathbb{R})} = (y\partial_y)^2 - y\partial_y + (y\partial_x)^2 - y\partial_x\partial_\theta = y^2(\partial_x^2 + \partial_y^2) - y\partial_x\partial_\theta. \tag{3.19}$$

□

3.2 The quotient space $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$

The Lie group $G = \mathrm{SL}(2,\mathbb{R})$ is closely related to the complex upper half plane:

$$\mathbb{H} = \{\tau \in \mathbb{C} : \operatorname{Im}\tau > 0\}. \tag{3.20}$$

In particular we can define a group action on $\mathbb{H}$.

Proposition 4. *The operation $G \times \mathbb{H} \to \mathbb{H}$ defined by:*

$$g * z := \frac{az+b}{cz+d}, \quad \textit{where } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, \tag{3.21}$$

is a group action.

Proof. First check that $G * \mathbb{H} \subseteq \mathbb{H}$.

$$\operatorname{Im}(g*z) = \operatorname{Im}\frac{az+b}{cz+d} = \operatorname{Im}\frac{(az+b)\overline{(cz+d)}}{|cz+d|^2} = \frac{(ad-bc)\operatorname{Im} z}{|cz+d|^2} = \frac{\operatorname{Im} z}{|cz+d|^2} > 0.$$

Next check that the neutral element leaves z invariant.

$$1 * z = \frac{1\cdot z + 0}{0 \cdot z + 1} = z. \tag{3.22}$$

Finally check the compatibility with group multiplication:

$$g' * (g * z) = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} * \frac{az+b}{cz+d} = \frac{a'\frac{az+b}{cz+d} + b'}{c'\frac{az+b}{cz+d} + d'} = \frac{(a'a+b'c)z + a'b + b'd}{(c'a+d'c)z + c'b + d'd} = g'g * z.$$

Thus $*$ is a group action. □

Consider the quotient space $G/K = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. It is not a group since $\mathrm{SO}(2)$ is not normal. However G/K is still a space with a topology. The next proposition shows that the quotient space G/K is topologically equivalent to the complex upper half plane $\mathbb{H}$.

Proposition 5. *The quotient space $G/K = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ is homeomorphic to $\mathbb{H}$.*

Proof. We need to find a bijection $\phi : G/K \to \mathbb{H}$. Where both ϕ and ϕ^{-1} are continuous. Use the group action $*$ defined in proposition 4. First notice that $G * i = \mathbb{H}$, since

$$g * i = \begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} * i = x + iy, \tag{3.23}$$

parameterise $\mathbb{H}$. Next show that: $\text{Stab}\, i = K$.

$$g \in \text{Stab}\, i \Leftrightarrow \quad g * i = \begin{pmatrix} a & b \\ c & d \end{pmatrix} * i = i \quad \Leftrightarrow \quad \begin{cases} a = d \\ b = -c \end{cases} \quad \Leftrightarrow \quad g \in K. \tag{3.24}$$

It is now sufficient to show the homeomorphism $G/\text{Stab}\, i \cong G * i$. Consider the map:

$$\phi : G/\text{Stab}\, i \to G * i, \tag{3.25}$$

defined by:

$$[g] \mapsto g * i. \tag{3.26}$$

The map is surjective by definition of $G*i$. And ϕ is well-defined ($\Rightarrow$) and injective ($\Leftarrow$) by the equivalence:

$$[g'] = [g] \quad \Leftrightarrow \quad g^{-1}g' \in \text{Stab}\, i \quad \Leftrightarrow \quad g' \cdot i = g \cdot i. \tag{3.27}$$

Hence ϕ is a well-defined bijective map. Finally check the continuity. Notice that G/K is equipped with the quotient topology defined by:

$$\{[g]\} \text{ open } \Leftrightarrow \{g\} \text{ open}. \tag{3.28}$$

The transformation:

$$g \mapsto g * i = \frac{ai + b}{ci + d}, \tag{3.29}$$

is continuous with respect to the parameters a, b, c, d. Hence:

$$\{g * i\} \text{ open } \Rightarrow \{g\} \text{ open } \Rightarrow \{[g]\} \text{ open}. \tag{3.30}$$

Thus ϕ is continuous. Proving that ϕ^{-1} is continuous, takes a bit more work and is therefore left out. See proof for instance in [11]. □

3.3 Constructing automorphic forms from modular forms

Next we would like to construct an automorphic form from a holomorphic modular form. In particular, we want to find the differential equation corresponding to Cauchy Riemann equations. A holomorpic modular form of weight $w \geq 0$ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ which transforms according to:

$$f(\gamma * z) = f\Big(\frac{az + b}{cz + d}\Big) = (cz + d)^w f(z), \tag{3.31}$$

under the discrete action $\gamma \in \Gamma \subseteq \text{SL}(2, \mathbb{Z})$.

Definition 18 (Lift)**.** Given a modular form $f : \mathbb{H} \to \mathbb{C}$ with weight w, define the lift function $\varphi : \mathrm{SL}(2,\mathbb{R}) \to \mathbb{C}$ by:

$$\varphi(g) = (ci+d)^{-w} f(g * i), \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}). \tag{3.32}$$

We can find an explicit expression for φ from the Iwasawa decomposition:

$$ci + d = -\frac{i}{\sqrt{y}} \sin\theta + \frac{1}{\sqrt{y}} \cos\theta = \frac{1}{\sqrt{y}} \mathrm{e}^{-i\theta}. \tag{3.33}$$

Then φ takes the form:

$$\varphi(g) = \mathrm{e}^{iw\theta} y^{\frac{w}{2}} f(x+iy). \tag{3.34}$$

Proposition 6. *The lift function* φ *is an automorphic form on* $\mathrm{SL}(2,\mathbb{R})$*.*

Proof. We need to check four conditions:

1. Discrete symmetry condition.
2. φ is a finite dimensional representation of $K = \mathrm{SO}(2)$.
3. φ is an eigenfunction to the Laplacian.
4. Moderate growth condition.

The moderate growth condition is inherited from the modular form f. We see directly that the K-representation is one-dimensional from how φ depend on θ in equation 3.34. Next check the invariance under the discrete action $\gamma \in \Gamma$. Consider three elements:

$$\gamma g = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}, \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.35}$$

Use the transformation law of the modular form f:

$$\begin{aligned} \varphi(\gamma g) &= (c'i + d')^{-w} f(\gamma g * i) \\ &= (c'i + d')^{-w} (Cg * i + D)^{w} f(g * i) \\ &= (ci + d)^{-w} f(g * i) = \varphi(g). \end{aligned} \tag{3.36}$$

In the second to last step we used that:

$$(ci+d)(Cg * i + D) = C(ai+b) + D(ci+d) = (Ca + Dc)i + (Cb + Dd) = c'i + d'.$$

Finally check that φ is an eigenfunction to the Casimir operator in equation 3.14:

$$\Delta_{\mathrm{SL}(2,\mathbb{R})} = y^2(\partial_x^2 + \partial_y^2) - y\partial_x\partial_\theta = 4y^2\partial_z\partial_{\bar{z}} - y\partial_x\partial_\theta. \tag{3.37}$$

Move the factor $\mathrm{e}^{iw\theta}$ to the left:

$$\Delta_{\mathrm{SL}(2,\mathbb{R})}\varphi = \mathrm{e}^{iw\theta}\Big[4y^2\partial_z\partial_{\bar{z}} - iwy\partial_x\Big]y^{\frac{w}{2}}f(z). \tag{3.38}$$

Since f is holomorphic it will pass through the $\partial_{\bar{z}}$ derivative and we also utilise that $\partial_x = \partial_z + \partial_{\bar{z}}$.

$$\begin{aligned}\Delta_{\mathrm{SL}(2,\mathbb{R})}\varphi &= \mathrm{e}^{iw\theta}\Big[4y^2\partial_z\big(f(z)\partial_{\bar{z}}y^{\frac{w}{2}}\big) - iwy^{\frac{w}{2}+1}\partial_x f(z)\Big] \\ &= \mathrm{e}^{iw\theta}f(z)\Big[iwy^2\partial_z y^{\frac{w}{2}-1}\Big] \\ &= \frac{w}{2}(\frac{w}{2}-1)\varphi. \end{aligned} \tag{3.39}$$

Hence φ is an eigenfunction with eigenvalue $\frac{w}{2}(\frac{w}{2}-1)$. □

We have constructed an automorphic form φ from a holomorphic modular form f. Similarly one can consider an anti-holomorphic modular form $\tilde{f}(z) := f(\bar{z})$. And then construct an automorphic form by:

$$\tilde{\varphi} = \mathrm{e}^{-iw\theta}y^{\frac{w}{2}}f(\bar{z}). \tag{3.40}$$

The holomorphic function f satisfies Cauchy-Riemann equations:

$$\frac{\partial}{\partial\bar{z}}f = \frac{1}{2}(\partial_x + i\partial_y)f = 0. \tag{3.41}$$

There is a corresponding equation for the associated automorphic form φ, described by the following proposition.

Proposition 7. *The holomorphic condition on f implies that the associated automorphic form φ satisfies:*

$$-2i\mathrm{e}^{-2i\theta}\big(y\partial_{\bar{z}} - \frac{1}{4}\partial_\theta\big)\varphi = 0. \tag{3.42}$$

Similarly the anti-holomorphic condition on $\tilde{f}$ implies that the associated automorphic $\tilde{\varphi}$ satisfies:

$$2i\mathrm{e}^{2i\theta}\big(y\partial_z - \frac{1}{4}\partial_\theta\big)\tilde{\varphi} = 0. \tag{3.43}$$

Proof. The two statements are equivalent so focus on first one. $\varphi = \mathrm{e}^{iw\theta}y^{\frac{w}{2}}f(z)$ satisfies the equation 3.42 since:

$$\big(y\partial_{\bar{z}} - \frac{1}{4}\partial_\theta\big)\mathrm{e}^{iw\theta}y^{\frac{w}{2}} = 0. \tag{3.44}$$

□

Equation 3.42 and 3.43 can also be expressed using Lie algebra elements using equation 3.10:

$$\frac{1}{2}(h - i(e + f))\varphi = 0, \tag{3.45}$$

and

$$\frac{1}{2}(h + i(e + f))\tilde{\varphi} = 0, \tag{3.46}$$

respectively. Here the Lie algebra acts in the right regular representation.

Chapter 4

The special unitary group $\mathrm{SU}(2,1)$

In this chapter, we consider the special unitary group $\mathrm{SU}(2,1)$. First we write down the Lie algebra and its complexification. Then we consider the root space and find a basis for representations of the maximal compact subalgebra. $\mathrm{SU}(2,1)$ has both a quaternionic structure and a complex structure and will therefore serve as an illustrative example when discussing the Schmid operator in chapter 5.

4.1 The group structure

The real Lie group $\mathrm{SU}(2,1)$ consists of 3×3 traceless complex matrices which preserve a hermitian form η of signature $(2,1)$. I.e. η is some hermitian 3×3 matrix with eigenvalues $1, 1, -1$, and the group is defined:

$$\mathrm{SU}(2,1) = \Big\{ g \in \mathbb{C}^{3\times 3} \,:\, \det g = 1,\ g^\dagger \eta g = \eta \Big\}. \tag{4.1}$$

The Lie algebra is:

$$\mathfrak{su}(2,1) = \Big\{ X \in \mathbb{C}^{3\times 3} \,:\, \operatorname{tr} X = 0,\ X^\dagger \eta + \eta X = 0 \Big\}. \tag{4.2}$$

Choose the Hermitian form η to be:

$$\eta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{4.3}$$

A generic Lie algebra element can be written as:

$$X = \begin{pmatrix} A & B \\ B^\dagger & -\operatorname{tr} A \end{pmatrix}, \tag{4.4}$$

where $A \in \mathbb{C}^{2\times 2}$ is an anti-hermitian matrix, and $B \in \mathbb{C}^{2\times 1}$. Notice that A and B both have 4 real degrees of freedom. In total there are 8 real degrees of freedom.

Proposition 8. U(2) *is a maximal compact subgroup of* SU(2, 1).

Proof. Use the Cartan involution $\theta(X) = -X^\dagger$. This is a valid Cartan involution with respect to the trace form $\operatorname{tr}(XY)$ since:

$$\operatorname{tr}(X\theta(X)) = -\operatorname{tr}(XX^\dagger) < 0. \tag{4.5}$$

Notice that:

$$\begin{pmatrix} A & 0 \\ 0 & -\operatorname{tr} A \end{pmatrix}^\dagger = -\begin{pmatrix} A & 0 \\ 0 & -\operatorname{tr} A \end{pmatrix}, \quad \begin{pmatrix} 0 & B \\ B^\dagger & \end{pmatrix}^\dagger = \begin{pmatrix} 0 & B \\ B^\dagger & \end{pmatrix}. \tag{4.6}$$

Hence we get the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$:

$$\mathfrak{k} = \begin{pmatrix} A & 0 \\ 0 & -\operatorname{tr} A \end{pmatrix}, \quad \mathfrak{p} = \begin{pmatrix} 0 & B \\ B^\dagger & 0 \end{pmatrix}. \tag{4.7}$$

Since A is anti-hermitian, $\mathfrak{k}$ is isomorphic to $\mathfrak{u}(2)$. Exponentiation of $\mathfrak{k}$ gives:

$$K \cong \mathrm{U}(2) \tag{4.8}$$

showing that U(2) is a maximal compact subgroup. □

Remark. Later it will be useful to decompose $\mathfrak{u}(2)$ into a traceless part and a trace:

$$\mathfrak{k} \cong \mathfrak{u}(2) \cong \mathfrak{su}(2) \oplus \mathfrak{u}(1).$$

The non-compact part $\mathfrak{p}$ transforms under $\mathfrak{k}$ in a canonical way as:

$$\mathfrak{p} \longmapsto [\mathfrak{k}, \mathfrak{p}]. \tag{4.9}$$

Calculate this explicitly:

$$\left[\begin{pmatrix} A & 0 \\ 0 & -\operatorname{tr}(A) \end{pmatrix}, \begin{pmatrix} 0 & B \\ B^\dagger & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & AB + B\operatorname{tr}(A) \\ -\operatorname{tr}(A)B^\dagger - B^\dagger A & 0 \end{pmatrix}. \tag{4.10}$$

We see that B transforms in the fundamental representation of $\mathfrak{su}(2)$.

$$B \overset{\mathfrak{su}(2)}{\longmapsto} \big(A - \frac{1}{2}\operatorname{tr} A\big)B. \tag{4.11}$$

Under $\mathfrak{u}(1)$, B transforms 3 times the fundamental representation.

$$B \overset{\mathfrak{u}(1)}{\longmapsto} \frac{3}{2}\operatorname{tr}(A)B. \tag{4.12}$$

Let us find a basis for $\mathfrak{p}$. Introduce a set of 3×3 matrices which are 1 at entry $\mu\nu$ and else 0, this can be written with the Kronecker delta symbol as:

$$(X_{\mu\nu})^m_n = \delta^m_\mu \delta_{\nu n}. \tag{4.13}$$

where $\mu, \nu, m, n = 1, 2, 3$. A basis $\{Y_i\}_{i=1}^4$ for $\mathfrak{p}$ is:

$$\{Y_i\}_{i=1}^4 = \big\{X_{13} + X_{31},\, i(X_{13} - X_{31}),\, X_{23} + X_{32},\, i(X_{23} - X_{32})\big\}. \tag{4.14}$$

The Killing form is proportional to:

$$\operatorname{tr}(Y_i Y_j) = 2\delta_{ij}. \tag{4.15}$$

4.2 Complexification

As was mentioned in section 2.3, there is a correspondence between the representation theory of real and complex Lie algebras. In fact, the Schmid operator is actually partly defined in terms of the complexified Lie algebra. Let us therefore consider the complexified Lie algebra of $\mathfrak{su}(2,1)$, which is $\mathfrak{sl}(3,\mathbb{C})$. The Cartan decomposition is complexified in the natural way:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \quad \longmapsto \quad \mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}} \,. \tag{4.16}$$

We will still call $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{p}_{\mathbb{C}}$ the compact and non-compact part respectively. Notice that $\mathfrak{p}_{\mathbb{C}}$ takes the form:

$$\mathfrak{p}_{\mathbb{C}} = \mathbb{C}X_{13} \oplus \mathbb{C}X_{23} \oplus \mathbb{C}X_{31} \oplus \mathbb{C}X_{32} \,. \tag{4.17}$$

We can decompose $\mathfrak{p}_{\mathbb{C}}$ into two subspaces which are invariant under the $\mathfrak{k}_C$ transformation $[\mathfrak{k}_{\mathbb{C}}, \cdot]$.

$$\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_{+} \oplus \mathfrak{p}_{-} \,, \tag{4.18}$$

where

$$\mathfrak{p}_{+} = \mathbb{C}X_{13} \oplus \mathbb{C}X_{23}, \quad \mathfrak{p}_{-} = \mathbb{C}X_{31} \oplus \mathbb{C}X_{32}. \tag{4.19}$$

The complexification of the compact part is:

$$\mathfrak{k} \cong \mathfrak{su}(2) \oplus \mathfrak{u}(1) \quad \longmapsto \quad \mathfrak{k}_{\mathbb{C}} \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathbb{C} \,. \tag{4.20}$$

From equation 4.10 we see that $\mathfrak{p}_{+}$ and $\mathfrak{p}_{-}$ transform as separate copies of the fundamental representation of $\mathfrak{sl}(2,\mathbb{C})$. Let us draw the root system of $\mathfrak{sl}(3,\mathbb{C})$. Choose the Cartan subalgebra to be compact and spanned by:

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{4.21}$$

Notice how h_1 is the Cartan generator of the $\mathfrak{sl}(2,\mathbb{C})$ part and h_2 is the Cartan generator of the $\mathbb{C}$ part in equation 4.20. For each off-diagonal element $X_{\mu\nu}$ we can associate a root $\beta_{\mu\nu}$:

$$[h, X_{\mu\nu}] = \beta_{\mu\nu}(h) X_{\mu\nu}, \tag{4.22}$$

here $h = a_1 h_1 + a_2 h_2$ where $a_1, a_2 \in \mathbb{C}$. We have the following commutation relations:

$$\begin{aligned} &\beta_{12} = (2,0), \quad \beta_{23} = (-1,3), \; \beta_{13} = (1,3) \\ &\beta_{21} = (-2,0), \; \beta_{32} = (1,-3), \; \beta_{31} = (-1,-3). \end{aligned} \tag{4.23}$$

The metric on $\mathfrak{h}_\mathbb{C}$ is induced by the Killing form, and is up to a constant equal to:

$$\gamma_{ab} = \mathrm{tr}(h_a h_b) = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}_{ab}, \tag{4.24}$$

where $a, b = 1, 2$. And the metric on the dual space $\mathfrak{h}^*_\mathbb{C}$ is:

$$\gamma^{ab} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{6} \end{pmatrix}^{ab}. \tag{4.25}$$

The root system is shown in figure 4.1. Next consider possible choices for the positive roots. In order to fulfil the second condition in definition 10, the 3 positive roots must be adjacent. This implies that there are 6 possible choices of positive roots. And there is always 1 positive root, which is also compact: either β_{12} or β_{21}.

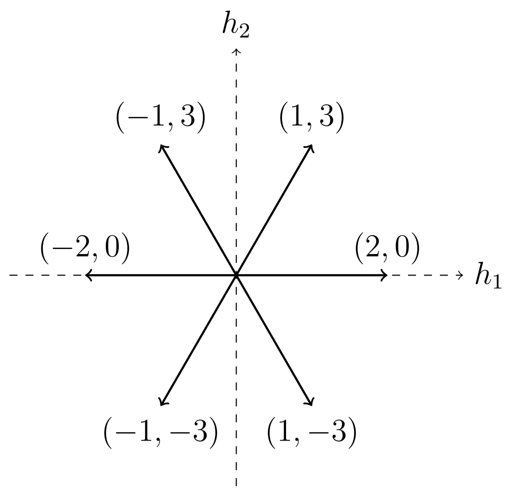

Figure 4.1: Root system of $\mathfrak{sl}(3,\mathbb{C})$. The Cartan subalgebra is defined in equation 4.21.

4.3 Basis for representations of $\mathfrak{k}_\mathbb{C}$

As a preparation for Chapter 5, let us find a basis for irreducible representation of

$$\mathfrak{k}_\mathbb{C} \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathbb{C} \cong \big(\mathbb{C}X_{21} \oplus \mathbb{C}h_1 \oplus \mathbb{C}X_{12}\big) \oplus \mathbb{C}h_2. \tag{4.26}$$

Consider an irreducible representation (Π, V_λ) with highest weight $\lambda = (N, M) \in \mathbb{Z}^2$. Where N is the highest weight of the $\mathfrak{sl}(2,\mathbb{C})$ part and M is the weight of the $\mathbb{C}$ part in equation 4.26. The module V_λ is spanned by $N + 1$ weight vectors

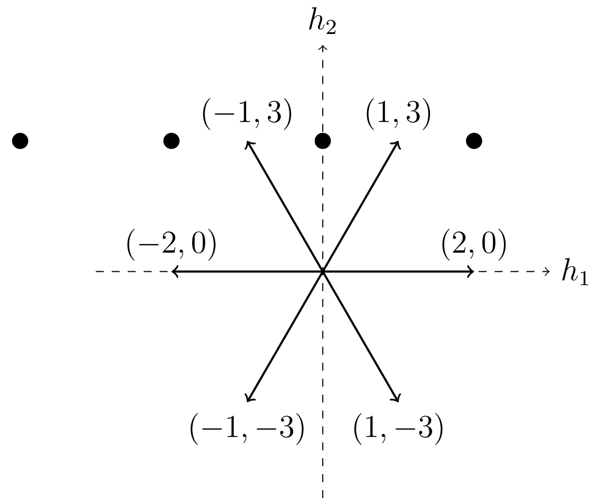

Figure 4.2: Example of representation module of $\mathfrak{k}_{\mathbb{C}}$, shown with black circles. Where the highest weight is $\lambda = (4,3)$.

with different weights with respect to the Cartan generator h_1. An example of V_λ is shown geometrically in figure 4.2.

Introduce a basis for V_λ, and call it the standard basis.

Definition 19 (Standard basis)**.** Define a basis $\{\boldsymbol{v}_n^\lambda\}_{n=0}^N$ for (Π, V_λ) by the action of the $\mathfrak{k}_{\mathbb{C}}$ generators:

$$\begin{cases} \Pi(h_2)\boldsymbol{v}_n^\lambda &= M\boldsymbol{v}_n^\lambda \\ \Pi(h_1)\boldsymbol{v}_n^\lambda &= (2n-N)\boldsymbol{v}_n^\lambda \\ \Pi(X_{12})\boldsymbol{v}_n^\lambda &= (n+1)\boldsymbol{v}_{n+1}^\lambda \\ \Pi(X_{21})\boldsymbol{v}_n^\lambda &= (N-(n-1))\boldsymbol{v}_{n-1}^\lambda \end{cases} \tag{4.27}$$

The explanation for the specific coefficients in the right-hand-side of equation 4.27 is explained by the following proposition.

Proposition 9. *The Standard basis* $\{\boldsymbol{v}_n^\lambda\}_{n=0}^N$ *is well-defined.*

Proof. We have to check consistency with all bracket relations in $\mathfrak{k}_{\mathbb{C}}$, and that $\boldsymbol{v}_{-1}^\lambda = \boldsymbol{v}_{N+1}^\lambda = 0$. First note that $\Pi(h_2)$ is independent of n and thus it follows immediately that $\Pi(h_2)$ commutes with all other generators, as expected. The raising operator $\Pi(X_{12})$ and lowering $\Pi(X_{21})$ have the correct bracket relation with $\Pi(h_1)$ since the weight of $\Pi(h_1)$ is increased by steps of 2. Check the bracket relation $[X_{12}, X_{21}] = h_1$:

$$[\Pi(X_{12}), \Pi(X_{21})]\boldsymbol{v}_n^\lambda = n\big(N-(n-1)\big)\boldsymbol{v}_n^\lambda - (N-n)(n+1)\boldsymbol{v}_n^\lambda = \Pi(h_1)\boldsymbol{v}_n^\lambda.$$

Finally we see that $\boldsymbol{v}^{\lambda}_{-1} = \boldsymbol{v}^{\lambda}_{N+1} = 0$ from the the fact that:

$$\Pi(X_{12})\boldsymbol{v}^{\lambda}_{-1} = 0, \quad \Pi(X_{21})\boldsymbol{v}^{\lambda}_{N+1} = 0. \tag{4.28}$$

□

Chapter 5

Schmid operator

In this chapter, the Schmid operator is introduced and then analysed for $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2,1)$. The Schmid operator is constructed in two steps. First we define a basis independent differential operator $\tilde{\mathbf{D}}$ with first order derivatives on the quotient space G/K. $\tilde{\mathbf{D}}$ transforms under the maximal compact subgroup K. This differential operator $\tilde{\mathbf{D}}$ is then projected to a certain K-invariant subspace.

5.1 Basis independent differential operator

The point of the Schmid operator is to find discrete series representations, just like equation 3.42 defines holomorphic discrete series for $\mathrm{SL}(2,\mathbb{R})$:

$$-2i\mathrm{e}^{-2i\theta}\big(y\partial_{\bar{z}} - \frac{1}{4}\partial_\theta\big)\varphi = 0. \tag{5.1}$$

We are looking for a generalisation of this equation for some general semisimple Lie group G, with maximal compact subgroup K. If G is compact, we automatically get discrete series and there is no need to impose any extra condition. For a non-compact group G, we only need to impose a condition on the non compact part G/K. It will then be useful to write out the K-dependence on the function φ explicitly. Moreover since K is compact it is sufficient to consider finite dimensional irreducible representations. We consider functions $\varphi : G \to V$, where (π, V) is a finite dimensional irreducible representation of K. We assume that φ transforms under K as:

$$\varphi(g) \overset{k}{\longmapsto} \varphi(gk) = \pi(k^{-1})\varphi(g). \tag{5.2}$$

Because of this transformation law φ is fully determined by its values on the quotient space G/K. Derivatives on the quotient space is precisely the non-compact part $\mathfrak{p}$ of the Cartan decomposition:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}. \tag{5.3}$$

Let X_i be a basis of $\mathfrak{p}$. When generalising equation 5.1, we should impose some first order differential condition on the quotient space. As a first ansatz consider a linear combination of first order derivatives:

$$\sum_i c_i X_i \varphi(g) = \sum_i c_i \frac{\mathrm{d}}{\mathrm{d}t} \varphi(g\mathrm{e}^{tX_i})\big|_{t=0}\,, \tag{5.4}$$

where c_i are some coefficients, and X_i acts on φ in the right regular representation. The summation index i runs over all basis elements X_i of $\mathfrak{p}$. This is a step in the right direction, but we would want an operator which is basis independent. One way to accomplish that is with the construction in the following definition.

Definition 20. Define a differential operator $\tilde{\mathbf{D}}$ by:

$$\tilde{\mathbf{D}}\varphi := \sum_i X_i \varphi \otimes X^i, \tag{5.5}$$

where X^i is a basis of the dual space $\mathfrak{p}^*$.

Remark. In equation 5.5, X_i is interpreted as a derivative acting on the function φ, while X^i is seen as an abstract vector in the Lie algebra. The dual basis is defined with respect to the Killing form $K(X,Y)$ by the relation:

$$K(X_i, X^j) = \delta_i^j, \tag{5.6}$$

where δ_i^j is the Kronecker delta.

In order for $\tilde{\mathbf{D}}$ to truly be a differential operator on the quotient space, it must transform under K. Suppose that we guess that the generalisation of equation 5.1 is given by $\tilde{\mathbf{D}}\varphi = 0$. Then the condition on φ should be K-independent. Equation 2.12 gives a suggestion of how X_i should transform under K:

$$X_i \overset{k}{\longmapsto} k^{-1} X_i k \in \mathfrak{p}. \tag{5.7}$$

Denote this representation by $(\sigma, \mathfrak{p})$ where σ is defined by:

$$\sigma(k^{-1})X_i := k^{-1} X_i k, \tag{5.8}$$

where the multiplication with group and algebra elements is expressed in the fundamental matrix representation. Given this transformation law, let us show the following proposition.

Proposition 10. $\tilde{\mathbf{D}}$ *is a covariant derivative, i.e. commutes with the group action of* K.

Proof.

$$\tilde{\mathbf{D}}\varphi = \sum_i X_i\varphi(g)\otimes X^i \overset{k}{\longmapsto} \sum_i [k^{-1}X_ik]\varphi(gk)\otimes k^{-1}X^ik. \tag{5.9}$$

Write the derivative explicitly and move the k factor in the argument of φ.

$$[k^{-1}X_ik]\varphi(gk) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi(gk\mathrm{e}^{tk^{-1}X_ik})\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\varphi(g\mathrm{e}^{tX_i}k)\Big|_{t=0} = \pi(k^{-1})X_i\varphi(g). \tag{5.10}$$

This shows that $\tilde{\mathbf{D}}$ is a covariant derivative. □

5.2 Projection operator

From proposition 10, we see that the condition $\tilde{\mathbf{D}}\varphi = 0$ is $K-$invariant. But we know that in general there is more than one discrete series and so we need more than one condition. Suppose $\tilde{\mathbf{D}}\varphi$ decompose into some irreducible representations of K, then setting any of them to zero, imposes a condition on φ which is K-invariant. Let us look more closely at how $\tilde{\mathbf{D}}\varphi$ transforms under K. The representation of $\tilde{\mathbf{D}}\varphi$ under K is denoted $(\pi, V)\otimes(\sigma, \mathfrak{p})$, and acts in the following way:

$$\tilde{\mathbf{D}}\varphi \overset{k}{\longmapsto} \sum_i \pi(k^{-1})X_i\varphi(g)\otimes\sigma(k^{-1})X^i. \tag{5.11}$$

Let Π and Σ denote the Lie algebra versions π and σ respectively. The pairs are related by $\pi(\mathrm{e}^Y) = \mathrm{e}^{\Pi(Y)}$ and $\sigma(\mathrm{e}^Y) = \mathrm{e}^{\Sigma(Y)}$. The explicit expression for Σ is:

$$\Sigma(Y)X_i = \frac{\mathrm{d}}{\mathrm{d}t}\sigma(\mathrm{e}^{tY})X_i\Big|_{t=0} = [Y, X_i]. \tag{5.12}$$

At the algebra level, the transformation law under $Y \in \mathfrak{k}$ is:

$$\tilde{\mathbf{D}}\varphi \overset{Y}{\longmapsto} \sum_i \Pi(-Y)X_i\varphi(g)\otimes X^i + \sum_i X_i\varphi(g)\otimes[X^i, Y]. \tag{5.13}$$

Let us investigate how $\tilde{\mathbf{D}}\varphi$ decomposes into subrepresentations of K. We will follow the procedure Wilfried Schmid did in his original article [3]. By theorem 8, we get discrete series precisely when $\operatorname{rank} G = \operatorname{rank} K$. Assuming this condition, the Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{k}$ can be chosen to be compact. In order to get a 1-1 correspondence between roots and Lie algebra elements, we consider the complexification of the vector spaces $\mathfrak{g}, \mathfrak{k}, \mathfrak{p}, \mathfrak{h}$:

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}, \quad \mathfrak{h}_{\mathbb{C}} \subseteq \mathfrak{k}_{\mathbb{C}}. \tag{5.14}$$

The Killing form induces a metric on $\mathfrak{h}_{\mathbb{C}}$ and its dual space $\mathfrak{h}^*_{\mathbb{C}}$. This metric defines a scalar product $\langle\cdot,\cdot\rangle$ on $\mathfrak{h}^*_{\mathbb{C}}$. Let Δ denote the set of non-zero roots. Partition the non-zero roots $\Delta = \Delta_K \cup \Delta_P$, where Δ_K and Δ_P are sets of compact and non-compact roots respectively. From equation 5.12 we see that the non-compact roots Δ_P are the weights of the $(\Sigma, \mathfrak{p}_\mathbb{C})$ representation. The root decomposition takes the form:

$$\mathfrak{g}_\mathbb{C} = \mathfrak{h}_\mathbb{C} \oplus \bigoplus_{\alpha \in \Delta_K} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Delta_P} \mathfrak{g}_\alpha \,. \tag{5.15}$$

Choose a set of positive roots Δ^+. And partition the positive roots into a compact Δ_K^+ and a non-compact part Δ_P^+:

$$\Delta^+ = \Delta_K^+ \cup \Delta_P^+. \tag{5.16}$$

Denote the weight lattice by Λ. Define Λ^+ to be the set of dominant weights with respect to compact roots.

$$\Lambda^+ = \{\lambda \in \Lambda | \langle \lambda, \alpha \rangle \geq 0, \, \forall \alpha \in \Delta_K^+\}. \tag{5.17}$$

Assume that the irreducible representation π is defined by a highest weight $\lambda \in \Lambda^+$, and denote the module V_λ. The tensor product $(\pi, V_\lambda) \otimes (\sigma, \mathfrak{p}_\mathbb{C})$ decompose into irreducible components with highest weights on the form $\lambda + \alpha$ for $\alpha \in \Delta_P$.

$$V_\lambda \otimes \mathfrak{p}_\mathbb{C} = V_\lambda \otimes \bigoplus_{\alpha \in \Delta_P} \mathfrak{g}_\alpha = \bigoplus_{\alpha \in \Delta_P} V_{\lambda + \alpha} \,. \tag{5.18}$$

Definition 21. Define a projection operator **pr** assoicated to the Schmid operator by:

$$\mathbf{pr}[V_\lambda \otimes \mathfrak{p}_\mathbb{C}] := \bigoplus_{\alpha \in \Delta_P^+} V_{\lambda - \alpha} \,. \tag{5.19}$$

Remark. The projection operator picks out the components with the minimal highest weight. This projection operator is an essential ingredient to defining the Schmid operator. Notice that the projection is dependent on the choice of positive roots. This is expected since there are in general different discrete series for a Lie group, and different choices of positive roots pick out different discrete series.

We have defined a basis independent differential operator $\tilde{\mathbf{D}}$ and a projection operator **pr**. We are now ready to define the Schmid operator.

Definition 22 (Schmid operator)**.** Let G be a semisimple Lie group, with maximal compact subgroup K. Assume $\operatorname{rank} K = \operatorname{rank} G$. Consider functions $\varphi : G \to V_\lambda$, where (π, V_λ) is a finite irreducible representation of K labelled by its highest weight λ. Choose a set of positive roots Δ^+ with respect to a compact Cartan subalgebra $\mathfrak{h}_\mathbb{C} \subseteq \mathfrak{k}_\mathbb{C}$. This choice defines the projection operator **pr**. Define the Schmid operator $\mathbf{D}$ to be:

$$\mathbf{D}\varphi := \mathbf{pr}[\tilde{\mathbf{D}}\varphi]. \tag{5.20}$$

5.3 Special linear group $\mathrm{SL}(2,\mathbb{R})$

Let us consider the Schmid operator for the easiest group possible, $G = \mathrm{SL}(2,\mathbb{R})$. The maximal compact subgroup is $K = \mathrm{SO}(2)$. Here both G and K have rank 1, so the Schmid operator is well-defined. Use the Chevalley-Serre basis h, e, f. Recall the Cartan decomposition from section 3:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathfrak{k} = \mathbb{R}(e-f), \quad \mathfrak{p} = \mathbb{R}h \oplus \mathbb{R}(e+f). \tag{5.21}$$

The Cartan subalgebra $\mathfrak{h}$ is chosen to be compact $\mathfrak{h} = \mathfrak{k}$. Notice that the matrix:

$$e - f = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{5.22}$$

has eigenvalues $\pm i$ and cannot be diagonalised over the real numbers. It is therefore necessary to introduce a complex basis:

$$H = -i(e-f), \quad E = \frac{1}{2}(h + i(e+f)), \quad F = \frac{1}{2}(h - i(e+f)), \tag{5.23}$$

with the following commutation relations:

$$[H,E] = 2E, \quad [H,F] = -2F, \quad [E,F] = H. \tag{5.24}$$

The inverse relations are:

$$e - f = iH, \quad h = E + F, \quad e + f = i(F - E). \tag{5.25}$$

Let the basis of $\mathfrak{p}$ be $X_i = (h, e+f)_i$ for $i = 1, 2$. The Killing form is:

$$K(X_i, X_j) = \mathrm{tr}(\mathrm{ad}\, X_i \circ \mathrm{ad}\, X_j) = 8\delta_{ij}, \tag{5.26}$$

which gives the dual basis $X^i = \frac{1}{8}(h, e+f)^i$. Write down the $\tilde{\mathbf{D}}$ operator:

$$\tilde{\mathbf{D}}\varphi = \sum_i X_i\varphi \otimes X^i = \frac{1}{8}\big(h\varphi \otimes h + (e+f)\varphi \otimes (e+f)\big). \tag{5.27}$$

Express this in the complex basis:

$$\begin{aligned}\tilde{\mathbf{D}}\varphi &= \frac{1}{8}\big((E+F)\varphi \otimes (E+F) + i(F-E)\varphi \otimes i(F-E)\big) = \\ &= \frac{1}{4}\big(E\varphi \otimes F + F\varphi \otimes E\big).\end{aligned} \tag{5.28}$$

Now consider the projection operator. The irreducible representations of $K \cong \mathrm{U}(1)$ are, up to isomorphism, 1 dimensional. Thus the function φ will be a complex scalar and will just follow along on the ride. There are two roots $\Delta = \{2, -2\}$, corresponding to E and F. It is instructive to see the transformation $[H,E]=2E$ and $[H,F]=-2F$ at group level with $k\in K$:

$$E \longmapsto kEk^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \mathrm{e}^{2i\theta} E,$$

and

$$F \longmapsto kFk^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \mathrm{e}^{-2i\theta} F.$$

There are two possible choices for positive roots. First assume E to correspond to the positive root. Then the projection operator picks out the F term:

$$\mathbf{D}\varphi = \mathrm{pr}\,\big[\tilde{\mathbf{D}}\varphi\big] = \mathrm{pr}\,\Big[\frac{1}{4}\big(E\varphi\otimes F + F\varphi\otimes E\big)\Big] = \frac{1}{4}E\varphi\otimes F. \tag{5.29}$$

Setting $\mathbf{D}\varphi = 0$ implies that $E\varphi = 0$. Write this in terms of the differential operators obtained in section 3.1:

$$2i\mathrm{e}^{2i\theta}\big(y\partial_z - \frac{1}{4}\partial_\theta\big)\varphi = 0. \tag{5.30}$$

This is exactly equation 3.43, which defines the anti-holomorphic discrete series. If we instead assume the positive root to correspond to F. Then the projection operator picks out the E term:

$$\mathbf{D}\varphi = \mathrm{pr}\,\big[\tilde{\mathbf{D}}\varphi\big] = \mathrm{pr}\,\Big[\frac{1}{4}\big(E\varphi\otimes F + F\varphi\otimes E\big)\Big] = \frac{1}{4}F\varphi\otimes E. \tag{5.31}$$

Setting $\mathbf{D}\varphi = 0$ implies that $F\varphi = 0$. Write this in terms of the differential operators obtained in section 3.1:

$$-2i\mathrm{e}^{-2i\theta}\big(y\partial_{\bar{z}} - \frac{1}{4}\partial_\theta\big)\varphi = 0. \tag{5.32}$$

This is exactly equation (3.42), which defines the holomorphic discrete series.

5.4 Special unitary group SU$(2,1)$

Now consider the Schmid operator for the Lie group SU(2,1). We will follow the procedure described in [12]. First write down $\tilde{\mathbf{D}}\varphi$ using the basis $\{Y_i\}_{i=1}^4$ for $\mathfrak{p}$ from equation 4.14.

$$\begin{aligned}\tilde{\mathbf{D}}\varphi =& \frac{1}{2}\Big((X_{13}+X_{31})\varphi\otimes(X_{13}+X_{31})+i(X_{13}-X_{31})\varphi\otimes i(X_{13}-X_{31})+\\&+(X_{23}+X_{32})\varphi\otimes(X_{23}+X_{32})+i(X_{23}-X_{32})\varphi\otimes i(X_{23}-X_{32})\Big)\\=&X_{13}\varphi\otimes X_{31}+X_{31}\varphi\otimes X_{13}+X_{23}\varphi\otimes X_{32}+X_{32}\varphi\otimes X_{23}\end{aligned} \tag{5.33}$$

Denote the representation of φ under $\mathfrak{k}_{\mathbb{C}}$ by (Π, V_λ), with highest weight $\lambda = (N, M) \in \mathbb{Z}^2$. Where N is the highest weight of $\mathfrak{sl}(2,\mathbb{C}) \subset \mathfrak{k}_{\mathbb{C}}$. Assume that $N \neq 0$, i.e. that φ is a non-trivial representation of $\mathfrak{sl}(2,\mathbb{C})$. Without loss of generality, we can assume $N \geq 1$. Write φ in the standard basis discussed in section 4.3:

$$\varphi = \sum_{n=0}^{N} \varphi_n \boldsymbol{v}_n^\lambda, \tag{5.34}$$

where φ_n are functions on the group $G = \mathrm{SU}(2,1)$. Since $N \neq 0$, $\tilde{\mathbf{D}}\varphi$ is a sum of four irreducible representations of $\mathfrak{k}_{\mathbb{C}}$:

$$V_\lambda \otimes \mathfrak{p}_{\mathbb{C}} = V_{\lambda-\beta_{13}} \oplus V_{\lambda-\beta_{31}} \oplus V_{\lambda-\beta_{32}} \oplus V_{\lambda-\beta_{23}}. \tag{5.35}$$

As was mentioned in section 4.2, there exists 6 possible choices of positive roots. However in the definition of the projection operator, the highest weight λ is dominant with respect to the compact roots. This implies that β_{12} is positive and β_{21} is negative. With this fixed choice, it remains three possible choices of positive roots. Corresponding to the quaternionic-, holomorphic, anti-holomorphic discrete series respectively.

Quaternionic discrete series

Start by choosing the positive roots to be:

$$\Delta^+ = \{\beta_{12}, \beta_{13}, \beta_{32}\} = \{(2,0),(1,3),(1,-3)\}. \tag{5.36}$$

Notice that both non-compact roots have equal weights with respect to the $\mathfrak{sl}(2,\mathbb{C})$ generator h_1. This choice implies that the projection is to the minimal highest weight of $\mathfrak{sl}(2,\mathbb{C}) \subset \mathfrak{k}_{\mathbb{C}}$. And therefore this corresponds to the quaternionic discrete series. With this choice the projection operator is defined as:

$$\mathbf{pr}[V_\lambda \otimes \mathfrak{p}_{\mathbb{C}}] = V_{\lambda-\beta_{13}} \oplus V_{\lambda-\beta_{32}}. \tag{5.37}$$

This relation can be divided into two cases depending on the weight with respect to the second Cartan generator h_2:

$$\begin{cases} \mathbf{pr}[V_\lambda \otimes (\mathbb{C}X_{31} \oplus \mathbb{C}X_{32})] & = V_{\lambda-\beta_{13}} \\ \mathbf{pr}[V_\lambda \otimes (\mathbb{C}X_{13} \oplus \mathbb{C}X_{23})] & = V_{\lambda-\beta_{32}} \end{cases} \tag{5.38}$$

The following proposition makes this statement more precise.

Proposition 11. *The projection operator acts on* $\tilde{\mathbf{D}}\varphi$ *in the following way:*

$$\begin{cases} \mathbf{pr}[\boldsymbol{v}_n^\lambda \otimes X_{31}] & = \boldsymbol{v}_{n-1}^{\lambda-\beta_{13}} \\ \mathbf{pr}[\boldsymbol{v}_n^\lambda \otimes X_{32}] & = \boldsymbol{v}_{n}^{\lambda-\beta_{13}} \\ \mathbf{pr}[\boldsymbol{v}_n^\lambda \otimes X_{23}] & = \boldsymbol{v}_{n-1}^{\lambda-\beta_{32}} \\ \mathbf{pr}[\boldsymbol{v}_n^\lambda \otimes X_{13}] & = -\boldsymbol{v}_{n}^{\lambda-\beta_{32}} \end{cases} \tag{5.39}$$

Proof. Focus on the first relation in equation 5.38. The space $V_{\lambda-\beta_{23}}$ has the weights:

$$(-N-1, M-3), (-N+1, M-3), \ldots, (N+1, M-3),$$

and is generated by the lowest weight vector $\boldsymbol{v}_0^\lambda \otimes X_{31}$ with weight $(-N-1, M-3)$.

$$\begin{aligned} V_{\lambda-\beta_{23}} &= \operatorname{span}\Big\{(X_{12})^n \cdot \big(\boldsymbol{v}_0^\lambda \otimes X_{31}\big)\Big\}_{n=0}^{N+1} = \\ &= \operatorname{span}\Big\{n!\big(\boldsymbol{v}_n^\lambda \otimes X_{31} - \boldsymbol{v}_{n-1}^\lambda \otimes X_{32}\big)\Big\}_{n=0}^{N+1} \end{aligned}$$

The projection operator annihilates $V_{\lambda-\beta_{23}}$, thus:

$$\mathbf{pr}\big(\boldsymbol{v}_n^\lambda \otimes X_{31} - \boldsymbol{v}_{n-1}^\lambda \otimes X_{32}\big) = 0. \tag{5.40}$$

Next consider the space $V_{\lambda-\beta_{13}}$, which has the weights:

$$(-N+1, M-3), (-N+3, M-3), \ldots, (N-1, M-3),$$

and is generated by a lowest weight vector:

$$(N+1)\boldsymbol{v}_0^{\lambda-\beta_{13}} = N\boldsymbol{v}_0^\lambda \otimes X_{32} + \boldsymbol{v}_1^\lambda \otimes X_{31}, \tag{5.41}$$

with weight $(-N+1, M-3)$ and $\Pi(X_{21})\boldsymbol{v}_0^{\lambda-\beta_{13}} = 0$. Act with $(X_{12})^n$ on both sides of equation 5.41 and divide by $n!$:

$$(N+1)\boldsymbol{v}_n^{\lambda-\beta_{13}} = (N-n)\boldsymbol{v}_n^\lambda \otimes X_{32} + (n+1)\boldsymbol{v}_{n+1}^\lambda \otimes X_{31}. \tag{5.42}$$

Combining equation 5.40 and 5.42 yields:

$$\begin{cases} \mathbf{pr}[\boldsymbol{v}_n^\lambda \otimes X_{31}] = \boldsymbol{v}_{n-1}^{\lambda-\beta_{13}} \\ \mathbf{pr}[\boldsymbol{v}_n^\lambda \otimes X_{32}] = \boldsymbol{v}_{n}^{\lambda-\beta_{13}} \end{cases} \tag{5.43}$$

The other two relations in equation 5.39 are shown in the same manner. □

This proposition is now used to write down the Schmid operator explicitly. Write $\tilde{\mathbf{D}}$ from equation 5.33 in the basis given in 5.34.

$$\tilde{\mathbf{D}}\varphi = \sum_{n=0}^{N} \Big[X_{13}\varphi_n \boldsymbol{v}_n^{\lambda} \otimes X_{31} + X_{31}\varphi_n \boldsymbol{v}_n^{\lambda} \otimes X_{13} + \\ + X_{23}\varphi_n \boldsymbol{v}_n^{\lambda} \otimes X_{32} + X_{32}\varphi_n \boldsymbol{v}_n^{\lambda} \otimes X_{23} \Big]. \tag{5.44}$$

The Schmid operator is:

$$\mathbf{D}\varphi = \sum_{n=0}^{N} \Big[X_{13}\varphi_n \boldsymbol{v}_{n-1}^{\lambda-\beta_{13}} - X_{31}\varphi_n \boldsymbol{v}_n^{\lambda-\beta_{32}} + \\ + X_{23}\varphi_n \boldsymbol{v}_n^{\lambda-\beta_{13}} + X_{32}\varphi_n \boldsymbol{v}_{n-1}^{\lambda-\beta_{32}} \Big]. \tag{5.45}$$

Setting $\mathbf{D}\varphi = 0$ yields a system of $2N$ differential equations, one equation for each basis elements $\{\boldsymbol{v}_n^{\lambda-\beta_{13}}\}_{n=0}^{N-1}$ and $\{\boldsymbol{v}_n^{\lambda-\beta_{32}}\}_{n=0}^{N-1}$. The system of differential equations is:

$$\begin{cases} X_{13}\varphi_{n+1} + X_{23}\varphi_n &= 0 \\ -X_{31}\varphi_n + X_{32}\varphi_{n+1} &= 0 \end{cases} \tag{5.46}$$

for $n = 0, 1, \ldots, N-1$.

Holomorphic and anti-holomorphic discrete series

There are two other choices of positive roots:

$$\Delta^+ = \{\beta_{12}, \beta_{13}, \beta_{23}\} = \{(2,0), (1,3), (-1,3)\},$$

or

$$\Delta^+ = \{\beta_{12}, \beta_{31}, \beta_{32}\} = \{(2,0), (-1,-3), (1,-3)\}.$$

Notice that the non-compact roots have equal weights with respect to the Cartan generator h_2 corresponding to $\mathfrak{u}(1) \subset \mathfrak{k}$. Setting the Schmid operator to zero will yield the holomorphic and anti-holomorphic discrete series respectively. Recall from section 4.2, that $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$. The projection operator in the two cases are defined:

$$\mathbf{pr}[V_\lambda \otimes \mathfrak{p}_{\mathbb{C}}] = V_\lambda \otimes \mathfrak{p}_-, \tag{5.47}$$

or

$$\mathbf{pr}[V_\lambda \otimes \mathfrak{p}_{\mathbb{C}}] = V_\lambda \otimes \mathfrak{p}_+. \tag{5.48}$$

Focus on the first case given by equation 5.47. Then the Schmid operator is:

$$\mathbf{D}\varphi = \sum_{n=0}^{N} \Big[X_{13}\varphi_n \boldsymbol{v}_n^{\lambda} \otimes X_{31} + X_{23}\varphi_n \boldsymbol{v}_n^{\lambda} \otimes X_{32} \Big]. \tag{5.49}$$

Setting $\mathbf{D}\varphi = 0$ yields a system of $2(N+1)$ differential equations, one equation for each basis element $\{\boldsymbol{v}_n^\lambda \otimes X_{31}\}_{n=0}^N$ and $\{\boldsymbol{v}_n^\lambda \otimes X_{32}\}_{n=0}^N$. The system of differential equations is:

$$X_{13}\varphi_n = X_{23}\varphi_n = 0, \tag{5.50}$$

for $n = 0, 1, \ldots, N$. Similarly for the other case given by equation 5.48, then the system of differential equations is:

$$X_{31}\varphi_n = X_{32}\varphi_n = 0, \tag{5.51}$$

for $n = 0, 1, \ldots, N$. Notice that in equation 5.50 and 5.51 the systems of differential equations are decoupled. This means that the specific $\mathrm{SU}(2) \subset K$ representation is irrelevant. Notice also the resemblance with the $\mathrm{SL}(2, \mathbb{R})$ case. Where we had either $E\varphi = 0$ or $F\varphi = 0$.

5.5 Tensorial formalism

So far we have seen how the Schmid operator is constructed and applied. The Schmid operator is a differential operator acting on the quotient space G/K. G/K is a symmetric space and is therefore interesting to study from the perspective of differential geometry. In this chapter we will briefly discuss this alternative perspective. More specifically we will consider the tensorial formalism that is presented in the following article [6]. First we define a covariant differential operator on the symmetric space G/K and then show how it is related to the Schmid operator.

Covariant derivative on the symmetric space G/K

First let us set the stage. Let G be a Lie group G with a maximal compact subgroup K and with Iwasawa decomposition $G = NAK$. Parameterise NA with a coordinate $\Phi = (\sigma, \phi)$ in the following way: $g = n(\sigma)a(\phi)k \in G$. The coordinate Φ is vector valued with components Φ^μ. The quotient space G/K can be parameterised by the representative $\mathcal{V}(\Phi) = n(\sigma)a(\phi)k$. The choice of k is arbitrary and one can even allow k to depend on Φ. As a start we will keep things as general as possible and allow dependence: $k = k(\Phi)$. There is a natural group action of K on the quotient space:

$$\mathcal{V} \overset{k}{\longmapsto} \mathcal{V}k. \tag{5.52}$$

The Maurer Cartan form of the representative $\mathcal{V}$ is:

$$\mathcal{V}^{-1}\partial_\mu \mathcal{V} \mathrm{d}\Phi^\mu. \tag{5.53}$$

Here we use Einstein summation convention, where repeated indices μ are summed over. Consider the Cartan decomposition:

$$\mathcal{V}^{-1}\partial_\mu \mathcal{V} = P_\mu + Q_\mu, \tag{5.54}$$

where P_μ and Q_μ are bases for $\mathfrak{p}$ and $\mathfrak{k}$ respectively, and they are expressed in terms of the Cartan involution θ:

$$\begin{cases} P_\mu := \frac{1}{2}\big(\mathcal{V}^{-1}\partial_\mu \mathcal{V} - \theta[\mathcal{V}^{-1}\partial_\mu \mathcal{V}]\big) \\ Q_\mu := \frac{1}{2}\big(\mathcal{V}^{-1}\partial_\mu \mathcal{V} + \theta[\mathcal{V}^{-1}\partial_\mu \mathcal{V}]\big) \end{cases} \tag{5.55}$$

Proposition 12. *P_μ and Q_μ transform under K in the following way:*

$$\begin{cases} P_\mu \overset{k}{\longmapsto} k^{-1}P_\mu k \\ Q_\mu \overset{k}{\longmapsto} k^{-1}Q_\mu k + k^{-1}\partial_\mu k \end{cases} \tag{5.56}$$

Proof. Use the facts that $\theta[k] = -k^{-1}$ and $\partial_\mu(k^{-1}k) = 0$.

$$\begin{aligned} P_\mu &\overset{k}{\longmapsto} \frac{1}{2}\big(k^{-1}\mathcal{V}^{-1}\partial_\mu \mathcal{V}k - \theta[k^{-1}\mathcal{V}^{-1}\partial_\mu \mathcal{V}k]\big) \\ &= k^{-1}P_\mu k + \frac{1}{2}\big(k^{-1}\partial_\mu k - \theta[k^{-1}\partial_\mu k]\big) = k^{-1}P_\mu k. \end{aligned}$$

And for Q_μ:

$$\begin{aligned} Q_\mu &\overset{k}{\longmapsto} \frac{1}{2}\big(k^{-1}\mathcal{V}^{-1}\partial_\mu \mathcal{V}k + \theta[k^{-1}\mathcal{V}^{-1}\partial_\mu \mathcal{V}k]\big) \\ &= k^{-1}P_\mu k + \frac{1}{2}\big(k^{-1}\partial_\mu k + \theta[k^{-1}\partial_\mu k]\big) = k^{-1}Q_\mu k + k^{-1}\partial_\mu k. \end{aligned}$$

□

The Killing form induces a metric $G_{\mu\nu}$ on the symmetric space, and a basis $V^\mu = G^{\mu\nu}P_\nu$ for the dual space. Here $G^{\mu\nu}$ denotes the inverse metric. Since the metric is independent of K, V^μ transforms the same way as P_μ. Next let us consider functions $f_{R_K} : G/K \to R_K$ defined on the symmetric space and where (π_{R_K}, R_K) is a representation of K. The function f_{R_K} transforms as:

$$f_{R_K} \overset{k}{\longmapsto} \pi_{R_k}(k^{-1})f_{R_K}. \tag{5.57}$$

Define a differential operator $\mathcal{D}$ on the symmetric space G/K:

$$\mathcal{D}f_{R_K}(\Phi) := V^\mu \otimes (\partial_\mu + \pi_{R_K}(Q_\mu))f_{R_K}(\Phi). \tag{5.58}$$

Notice that with this notation, π_{R_K} denotes both a group and algebra representation depending on the argument.

Proposition 13. *The differential operator $\mathcal{D}$ is covariant, transforming in the $\mathfrak{p} \otimes R_K$ representation and is independent of Lie algebra basis.*

Proof. $\mathcal{D}$ is independent of Lie algebra basis since the dual basis V_μ transforms with the inverse compared to ∂_μ and Q_μ. Next check the covariance. And to simplify notation we write $\pi_{R_K} = \pi$.

$$\mathcal{D}f_{R_K} \overset{k}{\longmapsto} k^{-1}V^\mu k \otimes \Big(\partial_\mu + \pi(k^{-1}Q_\mu k + k^{-1}\partial_\mu k)\Big)\pi(k^{-1})f_{R_K}. \tag{5.59}$$

Use the fact that $\pi(k^{-1}Q_\mu k) = \pi(k^{-1})\pi(Q_\mu)\pi(k)$ and $\pi(k^{-1}\partial_\mu k) = \pi(k^{-1})\partial_\mu \pi(k)$, then after simplifications:

$$\mathcal{D}f_{R_K} \overset{k}{\longmapsto} k^{-1}V^\mu k \otimes \pi(k^{-1})\big(\partial_\mu + \pi(Q_\mu)\big)f_{R_K}. \tag{5.60}$$

We see that $\mathcal{D}$ is a covariant derivative and the K-representation is $\mathfrak{p} \otimes R_K$. □

Relation to the Schmid operator

We have now discussed the tensorial formalism and it is time to see how the Schmid operator fit in that framework. Recall that the Schmid operator is defined in two steps: $\mathbf{D} = \mathbf{pr}[\tilde{\mathbf{D}}]$. The differential operator $\tilde{\mathbf{D}}$ can easily be written in the tensorial formalism, as shown by the following proposition.

Proposition 14. *The Schmid operator before the projection:*

$$\tilde{\mathbf{D}}\varphi = \sum_i X_i\varphi \otimes X^i, \tag{5.61}$$

is equivalent to the covariant derivative in the tensorial formalism:

$$\mathcal{D}f_{R_K}(\Phi) := V^\mu \otimes (\partial_\mu + \pi_{R_K}(Q_\mu))f_{R_K}(\Phi). \tag{5.62}$$

Proof. First relate the functions $\varphi : G \to R_K$ and $f_{R_K} : G/K \to R_K$. We can regard f_{R_K} as a function of the representative $\mathcal{V}$ rather than Φ. Then $f_{R_K}(\mathcal{V})$ transforms in the right-regular action under K and is equivalent to φ. Denote the right regular action by $*$. We have the following transformation laws of f_{R_K}:

$$\begin{cases} k * f_{R_K} & = \pi_{R_K}(k^{-1})f_{R_K}, \quad k \in K \\ Q_\mu * f_{R_K} & = \pi_{R_K}(-Q_\mu)f_{R_K}, \quad Q_\mu \in \mathfrak{k} \end{cases}. \tag{5.63}$$

Rewrite $\mathcal{D}$, by using the following relation:

$$\partial_\mu f_{R_K} = \mathcal{V}^{-1}\partial_\mu\mathcal{V} * f_{R_K} = P_\mu * f_{R_K} + Q_\mu * f_{R_K} = P_\mu * f_{R_K} + \pi_{R_K}(-Q_\mu)f_{R_K}.$$

Hence $\mathcal{D}$ is:

$$\mathcal{D}f_{R_K} := V^\mu \otimes P_\mu * f_{R_K}. \tag{5.64}$$

By choosing the basis $X_i = P_\mu$ in equation 5.61, we see that the two operators are equal, $\mathcal{D} = \tilde{\mathbf{D}}$. □

The next natural step would be to consider the projection operator **pr**. However the projection operator takes exactly the same form in both formulations given by equation 5.19.

Chapter 6

Outlook and discussion

The thesis has been a review of the Schmid operator. Both SL$(2, \mathbb{R})$ and SU(2,1) were interesting examples to study. In the case of SL$(2, \mathbb{R})$, the projection operator acts directly on $\mathfrak{p}$, giving a single differential equation that can be expressed as a Lie algebra element E or F. For SU(2,1) the situation was a bit more complicated, since φ was a $N + 1$-dimensional SU(2) representation. The projection operator acted on the tensor product $V_\lambda \otimes \mathfrak{p}$ and the setting $\mathbf{D}\varphi = 0$, led to a system of $2N$ differential equations. However, in the holomorphic and anti-holomorphic case the differential equations decoupled giving a similar result as in the SL$(2, \mathbb{R})$ case. In chapter 5.5 we saw that the Schmid operator can be expressed in a tensorial formalism. However since the projection operator takes the same form, there is no obvious advantage to do so.

There are many possible continuations of this project, both concrete and theoretical. A concrete continuations could be to continue with the analysis of SU(2,1). One could parameterise the group and solve the system of differential equations 5.46 and find quaternionic discrete series explicitly. This could then be used for automorphic forms. Another more theoretical continuation is to investigate if there is any connection between the Schmid operator and Schur's lemma. The Schmid operator is commuting with the group action, and if we require that φ and $\mathbf{D}\varphi$ are irreducible representations of K then by Schur's lemma $\mathbf{D}\varphi = 0$. So it would be interesting to see if there is a connection.

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