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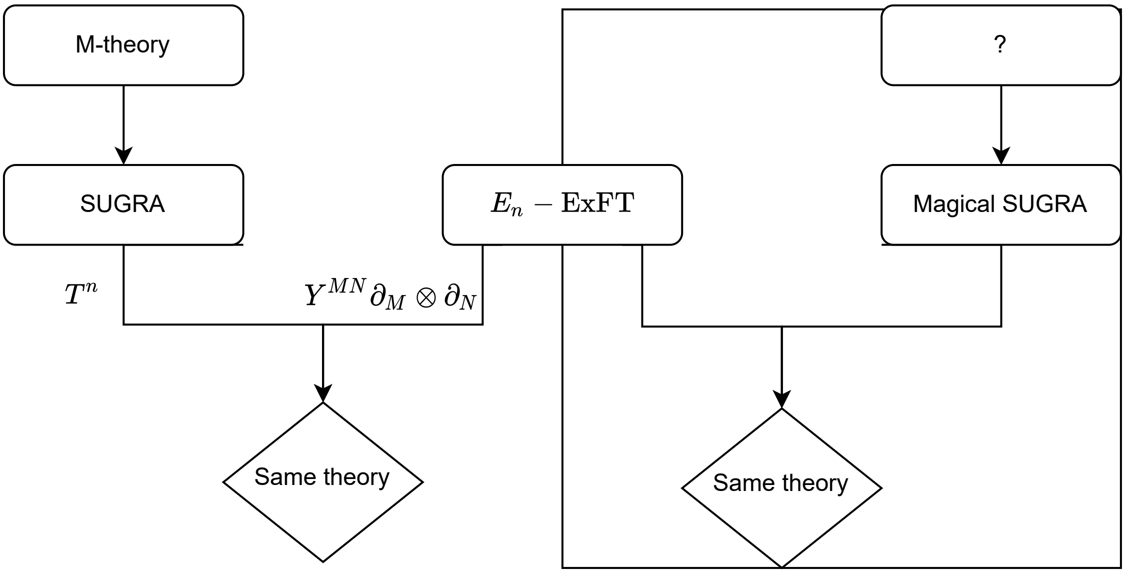

Extended Geometry and Magical Supergravities

Master's thesis in Physics (MPPHS)

LINUS SUNDBERG

DEPARTMENT OF PHYSICS

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Extended Geometry and Magical Supergravities

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Department of Physics
Division of Subatomic, High Energy and Plasma Physics
Chalmers University of Technology
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Cover: The connections between string theory, supergravity, exceptional field theory and magical supergravities. The box illustrate the subjects and their connections of which this thesis has been devoted to investigate.

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Abstract

During the last decades, double and exceptional field theory have been introduced in an attempt to understand the dualities of compactified supergravities from a geometrical perspective. There has been much work done on this subject where the exceptional field theories have been intended to understand the symmetries of M-theory. However, there exists another class of supergravities, which has deep ties to the Lie groups found in the magical square of Lie algebras, and which has yet to be understood to full extent. The aim of this thesis is to understand and develop a formalism using extended geometry for the non-gauged magical supergravities, more specifically the bosonic sector, in various external dimensions. This includes a systematic investigation of solutions of the section constraint for general real form of the internal structure algebra.

Keywords: Magical supergravities, Extended geometry, Section constraint, general real forms, Satake diagrams.

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Linus Sundberg, Gothenburg, June 2022

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1

Introduction

String theory is an attempt to find a theory which unifies the four known fundamental forces into a single unified theory, and provides a consistent framework for quantum gravity. There are many different examples of string theories, such as bosonic string theory, heterotic strings and superstring theories. One of the most studied cases is the superstring theory. Superstring theory can only be formulated in $D = 10$ [1, 2], however physics that we are used to is formulated in $D = 4$. Thus one needs to compactify these theories on manifolds. Besides the case with $D = 10$ one could also construct a supergravity theory in $D = 11$ [3]. The two examples are different limits of a single theory known as M-theory. M-theory is a presumed non-perturbative theory that includes string theory. M-theory also has a vacuum where the low energy excitations are described by $D = 11$ supergravity.

When the string theories were compactified on a torus, it was found that the theories were related to each others by what is known as dualities. There has during the last two decades been much work done in order to understand the dualities which are found from compactifying string theory to lower dimensions [4–7]. These dualities are found to be discrete, however their appearance suggests a geometrical origin. Much work has been done to develop the geometrical framework for these types of theories. During the late 2000's Hull [8, 9] constructed the formalism which introduced the geometrical formulation. In the extended geometry formalism the U-duality groups are incorporated before compactification in order to make them "manifest". The incorporation is done by replacing the structure group of diffeomorphism $GL(d)$ by the enlarged group $E_{d(d)} \times \mathbb{R}$ and letting the coordinates transform in a module of it where d is the number of compactified directions, however to find the correct number of physical coordinates the section constraint is employed [2].

The extended geometry [10–16] formalism has been explored for many of the cases found from the compactification of $D = 11$ supergravity and type II supergravities. In these cases the real form of the structure algebras has been chosen to be the split real form [12, 16–19]. One would then like to be able to do the same for the general cases of real forms.

Besides the split real form there exist other real forms of the algebras which are of interest. How one solves the section constraint for these has not been clear. Therefore a systematic investigation into how the section constraint could be solved for other real forms of Lie algebras is imperative in order to find the physical coordinates of the exceptional field theories (ExFT's) for general real forms encountered.

These are especially important in the case of magical supergravities since their scalar manifold is not given in terms of the split real form of the structure algebras.

Magical supergravities are classes of supergavities which symmetries are given by the symmetry groups found in Tits, Fruedenthal and Rozenfeld's magical square of Lie algebras [20]. The matter content in these theories then also takes values in modules of the structure group. The most unique feature of these theories are that their parent-theory is formulated in $D = 6$ and upon compactification of these theories one then find the exceptional groups as in the case with supergravities, however with other real forms. There has to the author's knowledge not been a study conducted on the construction of an ExFT for magical supergravities. Because of the way in which the magical square is constructed one would like to find a formulation which treats the groups found in the same dimension in a unified manner.

The process of solving the section constraint requires that the involutions that correspond to the real forms of Lie algebras have to be taken into account. These involutions are encoded in a Dynkin-diagram-like figure, known as a Satake diagram, through coloured nodes and nodes containing arrows [21]. A goal of this work is then to present the relevant knowledge in a compact and simple fashion to make it, hopefully, more accessible to a greater audience.

To illustrate the position this thesis aims to fill, in relation to what has previously been researched, one could see figure 1.1. From the figure one can clearly see the relations between the supergravities and ExFT that was mentioned previously and has been studied previously. The figure also shows the aim and focus of this thesis as the subjects which are enclosed by the box.

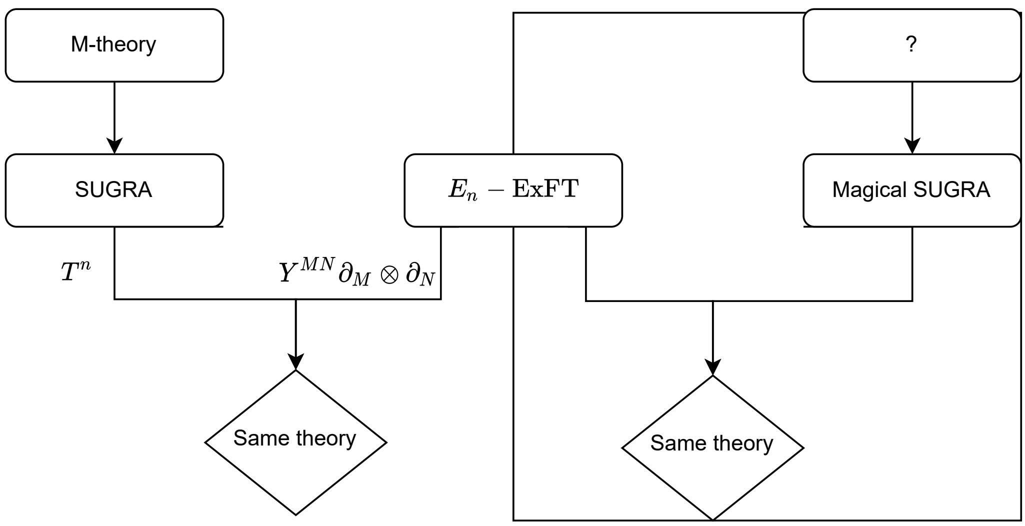


Figure 1.1: The figure illustrates the relations between compactification of supergravity theories, denoted SUGRA, on a torus T^n and E_n-exceptional field theories, abbreviated as ExFT, with the solution to the section constraint. The figure also shows the subjects which this thesis aims to investigate inside the box.

1.1 Outline

The thesis is split into six chapters. The second chapter contains the necessary information needed in order to understand Lie algebras to the extent used in this work. The second chapter also contain the description of Satake diagrams and how they encode the involution which is equivalent to determining the real form [21]. That chapter then provides the vital knowledge for understanding how the section constraint could be solved in a later chapter. The third chapter constitutes of a brief review of string theory and supergravity as well as the dualities which appears from compactification. The fourth chapter contain some original work and is devoted to magical supergravity and its content together with an example of how the fields combine into the modules upon compactification.

In the fifth chapter, the topic of extended geometry is introduced and explained. Within the mentioned chapter, as an example of an exceptional field theory, E_6-ExFT is worked through in detail. The chapter also contains a section with original work that generalise the E_6-ExFT example further to create an ExFT including the groups of the $D = 5$ of the magical supergravities. The chapter then continues to generalise the known ExFT's to accommodate the remaining two dimensions $D = 4$ and 3 of the magical supergravities. Finally the two remaining section of the chapter are devoted to solving the section constraint, first in the previously known cases, and secondly to the cases with general real forms. As examples of these, the sections which solve the section constraint for the magical supergravities are then derived.

The final sections is devoted to a discussion of the results as well as an outlook of what remains to be explored.

2

Lie Algebra

The purpose of this chapter is to briefly summarise the concept of Lie algebras and how they are classified with help of Dynkin diagrams. These concepts are then used to introduce the Satake diagrams, a generalisation of the Dynkin diagrams which encodes different real forms of a Lie algebra. The Satake diagrams are an essential tool in the later sections where they will be applied in the context of extended geometries.

2.1 Lie algebra

This section aims to introduce the concept of Lie algebras and establish some notation that will be used later in this thesis. For a more more thorough review of Lie algebras see e.g. [22, 23].

A Lie algebra $\mathfrak{g}$ is defined as a vector space which has an skew-symmetric product $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which has to satisfy an identity known as the Jacobi identity. The Jacobi identity is

$$3[x_{[1},[x_2,x_{3]}]] := [x_1,[x_2,x_3]] + [x_2,[x_3,x_1]] + [x_3,[x_1,x_2]] = 0, \tag{2.1}$$

where $x_1, x_2, x_3 \in \mathfrak{g}$. The product $[\cdot,\cdot]$ is also known as the Lie bracket. The basis elements for the Lie algebra are referred to as generators of the Lie algebra, and is commonly denoted as T^α where $\alpha = 1, ..., \dim(\mathfrak{g})$. As the generators are a basis for the algebra one can then express the result of the product $[,]$ as a linear combination of basis elements as

$$[T^\alpha, T^\beta] = f^{\alpha\beta}{}_\gamma T^\gamma, \tag{2.2}$$

where $f^{\alpha\beta}{}_\gamma$ is called the structure constants and is determined of the Lie algebra $\mathfrak{g}$. The complete list of finite dimensional simple Lie algebras is

$$A_n = \mathfrak{sl}(n+1), \tag{2.3}$$
$$B_n = \mathfrak{so}(2n+1), \tag{2.4}$$
$$C_n = \mathfrak{sp}(2n), \tag{2.5}$$
$$D_n = \mathfrak{so}(2n), \tag{2.6}$$
$$\mathfrak{g}_2,\ \mathfrak{f}_4,\ \mathfrak{e}_6,\ \mathfrak{e}_7 \quad \text{and} \quad \mathfrak{e}_8 \tag{2.7}$$

where the first four are families of matrix algebras and the last five are exceptional Lie algebras. One can also form another Lie algebra from two algebras through the direct sum $\oplus$. Thus one can take $\mathfrak{g}_1$ and $\mathfrak{g}_2$ to be two lie algebras, then

$$\mathfrak{g}_1 \oplus \mathfrak{g}_2, \tag{2.8}$$

is a Lie algebra where the vector space is a direct sum of the vector spaces and the Lie bracket is given by

$$[(g_1, g_2), (g_1', g_2')] = ([g_1, g_1'], [g_2, g_2']), \tag{2.9}$$

where $g_1, g_1' \in \mathfrak{g}_1$ and $g_2, g_2' \in \mathfrak{g}_2$.

For a Lie algebra one can construct a representation of the algebra using matrices that satisfies the same structure as the Lie algebra elements in the bracket. Through out this thesis a representation will be denoted with $R(\lambda)$ where λ is the highest weight of the representation. There are further representation, they could be created through the direct sum of two representation or tensor product of them. The tensor product representation could generally be broken into smaller representations that are called irreducible. An irreducible representation is a representation that cannot be found by addition of other representations. One has then that

$$R(\lambda) \otimes R(\lambda) = \bigoplus_i R_i, \tag{2.10}$$

where R_i denotes irreducible representations [2].

2.2 Jordan algebras and construction of magical square

In this section we shall define what a Jordan algebra is and then construct the magical square of Lie algebras from them. In order to accomplish this the concept of division algebras is needed and therefore we start with discussing them.

2.2.1 Division algebras

A division algebra is a composite algebra with a positive definite norm. Composite algebras $\mathbb{F}$ over $\mathbb{R}$ have a quadratic form, $|\cdot|^2$, which is non-degenerate and satisfies

$$|xy|^2 = |x|^2|y|^2, \tag{2.11}$$

where $x, y \in \mathbb{F}$. From existence of a identity element one finds that $\mathbb{R}$ is embedded in the algebra [24]. Thus one can define the conjugate by the map which maps the real part to itself and everything else to minus itself and satisfies

$$\overline{xy} = \overline{y}\,\overline{x}, \quad x\overline{x} = |x|^2. \tag{2.12}$$

This quadratic form induces an inner product which is given by

$$\langle x, y \rangle = \mathrm{Re}\,(x\overline{y}) := \frac{1}{2}(x\overline{y} + \overline{x}y), \tag{2.13}$$

with the special case

$$\langle x, x\rangle = \text{Re}\,(x\overline{x}) = |x|^2. \tag{2.14}$$

By Hurwitz's theorem there exist only four normed composite algebras, namely, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ [25, 26]. All the maps, for example left (right) multiplication L_u (R_u) with $u \in \mathbb{F}$, $|u|^2 = 1$, which leaves the quadratic form invariant form a rotation group called $SO(\mathbb{F})$. The automorphism group then forms a subgroup of the rotation group. The corresponding algebra, the derivation algebra, is given by the set of linear maps $D : \mathbb{F} \to \mathbb{F}$ such that

$$D(xy) = D(x)y + xD(y), \tag{2.15}$$

where $x, y \in \mathbb{F}$. The maps of the form $D(x) = D_a(x) := [a, x]$ with $a \in \mathbb{F}$ are derivations, this shows that

$$\text{Der}(\mathbb{R}) = 0 = \text{Der}(\mathbb{C}), \tag{2.16}$$

as they are commutative. For the non-commutative cases one has that

$$\text{Der}(\mathbb{H}) = C(\text{Im}(H)) \quad \text{and} \quad \text{Der}(\mathbb{O}) = G_2, \tag{2.17}$$

where $C(\text{Im}(H))$ denotes the set of all commutator maps between imaginary elements in $\text{Im}(H)$. In the case of $\mathbb{O}$, $D(x) = D_{(a,b)}(x) := [a, b, x] + \frac{1}{3}[[a, b], x]$ with $a, b \in \mathbb{F}$ are also derivations, where $[a, b, x]$ denotes the associator defined as $[a, b, c] = (ab)c - a(bc)$.

2.2.2 Jordan algebras

In this section the division algebras discussed in the previous section will be used to define a Jordan algebra. Consider a set of $n \times n$ hermitian, with conjugate defined as in the specific division algebra under consideration, matrices $H_n(\mathbb{F})$ over a division algebra $\mathbb{F}$. By using the product $\circ : H_n(\mathbb{F}) \otimes H_n(\mathbb{F}) \to H_n(\mathbb{F})$ defined as

$$X \circ Y = \frac{1}{2}\{X, Y\} = \frac{1}{2}(XY + YX), \tag{2.18}$$

one finds what is known as a Jordan algebra. The product satisfies the relation

$$X^2 \circ (X \circ Y) = X \circ (X^2 \circ Y), \tag{2.19}$$

for $n < 4$ for $\mathbb{F}$ up to $\mathbb{O}$, and any n when division algebra is restricted to the three lowest division algebras [27]. The mapping

$$D_A(X) = [A, X], \tag{2.20}$$

where A is an anti-hermitian, and in the case of $n = 3$ also traceless, $n \times n$ matrix, forms a derivation of the Jordan algebras $H_n(\mathbb{F})$ with $n \leq 3$. This can be seen as

$$D_A(X \circ Y) = D_A(X) \circ Y + X \circ D_A(Y), \tag{2.21}$$

as in those cases the identity

$$[A,\{X,Y\}] = \{[A,X],Y\} + \{X,[A,Y]\} \tag{2.22}$$

holds [24].

The structure group is then generated by

$$\mathrm{Str}' H_3(\mathbb{F}) = L_3'(H_3(\mathbb{F})) \oplus \mathrm{Der}\,\mathbb{F} = \mathfrak{sl}(3,\mathbb{F}), \tag{2.23}$$

where $L_3(H_3(\mathbb{F}))$ denotes the set of all left multiplications by the elements of $H_3(\mathbb{F})$. This shows that the representation module of $\mathfrak{sl}(3,\mathbb{F})$ are hermitian 3×3-matrices. The structure groups of $H_3(\mathbb{F})$ can then be shown to be isomorphic to other groups [27, 28] and the result is displayed in table 2.1.

Table 2.1: The table shows the Lie algebras that corresponds to the structure algebras found for the Jordan algebras over the different division algebras.

$\mathbb{F}$	$\mathrm{Str}'(H_3(\mathbb{F}))$
$\mathbb{R}$	$\mathfrak{sl}(3,\mathbb{R})$
$\mathbb{C}$	$\mathfrak{sl}(3,\mathbb{C})$
$\mathbb{H}$	$\mathfrak{sl}(3,\mathbb{H}) \simeq SU(6)$
$\mathbb{O}$	$\mathfrak{e}_6$

In the case with $H_3(\mathbb{F})$ Jordan algebras there exists two products which maps $H_3(\mathbb{F}) \otimes H_3(\mathbb{F}) \to H_3(\mathbb{F})$, the first is the Jordan product defined earlier and the second one is the Freudenthal product [29]:

$$X \times Y = X \circ Y - \frac{1}{4}\mathrm{tr}(X)Y - \frac{1}{4}\mathrm{tr}(Y)X + \frac{1}{4}(\mathrm{tr}(X)\mathrm{tr}(Y) - \mathrm{tr}(X\circ Y))\mathbf{1}. \tag{2.24}$$

The Freudenthal product then maps the entries from the module $R(\lambda)$ of $\mathfrak{sl}(3,\mathbb{F})$ to the conjugate module $\overline{R(\lambda)}$ and thus presents an excellent candidate for construction of a triple product. One can then define the triple product $H_3(\mathbb{F})\otimes H_3(\mathbb{F})\otimes H_3(\mathbb{F}) \to \mathbb{R}$ as

$$(X,Y,Z) = \frac{1}{6}\mathrm{tr}(X\circ(Y\times Z)). \tag{2.25}$$

By inserting the definition of the Fruedenthal product and using the linearity of the trace one finds the following expression

$$(X,Y,Z) = \frac{1}{6}\mathrm{tr}(X\circ(Y\circ Z)) - \frac{1}{24}\left[\mathrm{tr}(X)\mathrm{tr}(Y\circ Z) + \mathrm{tr}(Y)\mathrm{tr}(X\circ Z) + \mathrm{tr}(Z)\mathrm{tr}(X\circ Y)\right] + \frac{1}{24}\mathrm{tr}(X)\mathrm{tr}(Y)\mathrm{tr}(Z), \tag{2.26}$$

from which it is clear that $(X,Y,Z) = (X,Z,Y)$, since the Jordan product fulfil $X\circ Y = Y\circ X$ and the remaining expression is symmetric in Y and Z. Note also that the result is real since the diagonal of any matrix in $H_3(\mathbb{F})$ is real and the two products maps to another element in $H_3(\mathbb{F})$ thus the trace picks out the real diagonal. Next one can show by direct computation that

$$\mathrm{tr}(X \circ (Y \circ Z)) = \mathrm{tr}(Y \circ (Z \circ Y)) = \mathrm{tr}(Z \circ (Y \circ X)). \tag{2.27}$$

This implies that (X, Y, Z) is fully symmetric in the inputs, which will be used in later chapters.

Apart from the structure group discussed earlier in the section there also exists the conformal group which is defined as

$$\mathrm{Con}(H_3(\mathbb{F})) = \mathrm{Str}(H_3(\mathbb{F})) \oplus H_3(\mathbb{F})^2, \tag{2.28}$$

which resulting Lie algebras for the different division algebras are shown in table 2.2.

Table 2.2: The table shows the Lie algebras that corresponds to the conformal algebras found for the Jordan algebras over the different division algebras.

$\mathbb{F}$	$\mathrm{Con}'(H_3(\mathbb{F}))$
$\mathbb{R}$	$\mathfrak{sl}(6, \mathbb{R})$
$\mathbb{C}$	$\mathfrak{su}(3, 3)$
$\mathbb{H}$	$\mathfrak{sp}(6, \mathbb{H}) \simeq SO(12)$
$\mathbb{O}$	$\mathfrak{e}_7$

The conformal and structure algebras can be identified from a more general structure which gives rise to what is known as the magical square of Lie algebras, or Tits' magical square, which is found from

$$L_3(\mathbb{F}_1, \mathbb{F}_2) := \mathrm{Der}(H_3(\mathbb{F}_1)) \oplus H_3(\mathbb{F}_1) \otimes \mathrm{Im}(\mathbb{F}_2) \oplus \mathrm{Der}\, \mathbb{F}_2. \tag{2.29}$$

The structure group is then $\mathrm{Str}(H_3(\mathbb{F})) \simeq L_3(\mathbb{F}, \tilde{\mathbb{C}})$ and conformal group is identified $\mathrm{Con}(H_3(\mathbb{F})) \simeq L_3(\mathbb{F}, \tilde{\mathbb{H}})$, where $\tilde{\mathbb{C}}$ and $\tilde{\mathbb{H}}$ are the split form of the division algebras. One can then also extend this another step to consider the result of $\mathbb{F}_2 = \tilde{\mathbb{O}}$ which gives the last column in the magical square. This is displayed in table 2.3.

Table 2.3: This table is constitute what is known as the magical square of Lie algebras.

$\mathbb{F}_1 \vert \mathbb{F}_2$	$\mathbb{R}$	$\tilde{\mathbb{C}}$	$\tilde{\mathbb{H}}$	$\tilde{\mathbb{O}}$
$\mathbb{R}$	$\mathfrak{so}(3)$	$\mathfrak{sl}(3, \mathbb{R})$	$\mathfrak{sl}(6, \mathbb{R})$	$\mathfrak{f}_4$
$\mathbb{C}$	$\mathfrak{su}(3)$	$\mathfrak{sl}(3, \mathbb{C})$	$\mathfrak{su}(3, 3)$	$\mathfrak{e}_6$
$\mathbb{H}$	$\mathfrak{sq}(3)$	$\mathfrak{sl}(3, \mathbb{H}) \simeq SU(6)$	$\mathfrak{sp}(6, \mathbb{H}) \simeq SO(12)$	$\mathfrak{e}_7$
$\mathbb{O}$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

The entries of table 2.3 is not unique as there exist isomorphism from some of them to other Lie algebras and therefore could be exchanged.

2.3 Dynkin diagrams and highest weight representation

In previous sections it was discussed that the generators form a basis for the vector space corresponding to the Lie algebra. There exist a peculiarly useful basis which will often be used through out this thesis, the Chevalley-Serre basis. This basis is constructed from e_i, f_i, h_i where e_i acts as a creation operator and f_i an annihilator operator when acting on a state, more on this later. The basis is particularly interesting as for every i it acts as an $\mathfrak{sl}(2)$ subalgebra of the algebra under consideration. The elements h_i forms a subalgebra which is called the Cartan subalgebra which consists of mutually commuting generators. By studying the result of the Cartan-Killing form defined as

$$\eta^{\alpha\beta} = K(T^\alpha, T^\beta) = \mathrm{tr}(T^\alpha T^\beta), \tag{2.30}$$

one finds a metric $\eta^{\alpha\beta}$ for the generators and by restriction to the Cartan subalgebra one finds what is known as the Cartan matrix A. When $\mathfrak{g}$ is a semisimple Lie algebra $\eta^{\alpha\beta}$ is non-degenerate. This matrix encodes the structure of the algebra under consideration as relations between the different $\mathfrak{sl}(2)$ subalgebras using the relations

$$[e_i, f_j] = \delta_{ij} h_j, \tag{2.31}$$
$$[h_i, e_j] = A_{ij} e_j, \tag{2.32}$$
$$[h_i, f_j] = -A_{ij} f_j. \tag{2.33}$$

One can then decompose the algebra in terms of the eigenvalues of subspaces under the Cartan algebra. The vector α corresponding to a given eigenvalue is called a root and thus we have the decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha, \tag{2.34}$$

where

$$\mathfrak{g}_\alpha = \{ g \in \mathfrak{g} | [h, g] = \langle h, \alpha \rangle g, \forall h \in \mathfrak{h} \}, \tag{2.35}$$

where $\langle \cdot, \cdot \rangle$ is a scalar product. This shows that roots lie in the dual space to $\mathfrak{h}$, however since the Killing form is a metric on the Cartan subalgebra one can use them to map a root into $\mathfrak{h}$ and use the scalar product for the generators $(\cdot, \cdot) \mapsto \mathbb{R}$. Then one can show that

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}, \tag{2.36}$$

where the longest root, α_i, satisfies $(\alpha_i, \alpha_i) = 2$. For later convinience one can define coroots as $\alpha_i^\vee = 2\alpha_i/(\alpha_i, \alpha_i)$, which then implies that $A_{ij} = (\alpha_i^\vee, \alpha_j)$. These matrices can also be encoded in what is known as Dynkin-diagrams which are figures with nodes, correpsonding to roots of the algebra, and lines connecting the nodes describes the connection between the $\mathfrak{sl}(2)$-subalgebras, via the Serre relations. As the off-diagonal elements of the Cartan matrix indicates how many lines there should be connecting the nodes. Thus one can from a Dynkin-diagram write down the Cartan matrix for the Lie algebra. The former mentioned Serre relations is given by

$$(\text{ad } e_i)^{1-A_{ij}} \, e_j = 0, \tag{2.37}$$

where

$$(\text{ad } e_i) \, e_j := [e_i, e_j]. \tag{2.38}$$

The relation gives that acting via ad e_i on another creation generator can give a new generator iff $-A_{ij} > 0$. The value of $-A_{ij}$ shows how many times this acting can be done to produce a new generator. The same result holds for the annihilator generator as well. As a note on the notation, another common way to denote the creation and annihilation generators are e_α and $e_{-\alpha}$ respectively and will be occasionally used throughout this thesis.

As examples of how the Cartan matrix can be written down one could take the case of $\mathfrak{sl}(3)$. The diagram

describes this algebra as soon will be shown. From the rules one have that

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \tag{2.39}$$

is the Cartan matrix corresponding to the diagram as the nodes are connected with a single line. From this one can see that the generators are e_1, f_1, h_1 and e_2, f_2, h_2, however as the -1 indicates there also exist a generator $[e_1, e_2]$ and correspondingly for $[f_1, f_2]$. These behaves as the generators of this algebra that can be derived from the properties of the algebra only. This algebra requires only that the matrices has to be traceless 3×3 matrices. Therefore one can easily construct such a basis of such matrices as above which shows that they describe the same thing. From the Serre-realtion requirement one can see that A_{ij} has to be an integer which then can be used to classify the different Lie algebras. However this will not be further discussed here.

Another example would be the example with

which then gives the Cartan matrix

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \tag{2.40}$$

as there is no line connecting the nodes. From the diagram and Cartan matrix one sees that there exists two $\mathfrak{sl}(2)$ subalgebras that do not interact, thus one have that this diagram represent $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$. These two examples illustrate how the algebras can be encoded into the diagram.

Finally having discussed the Dynkin diagrams and the notation and construction of the algebra through roots one can discuss highest weight representations. If one considers a representation which is finite-dimensional one finds that there exists a state with the highest weight. This state is then labelled by the eigenvalue of each generator in the Cartan subalgebra as

$$h_i |\lambda\rangle = (\alpha_i^\vee, \lambda) |\lambda\rangle . \tag{2.41}$$

Because the representation is finite dimensional one then finds that this state can only be lowered using f_i finitely many times and one then have

$$f_i^{p_i+1} |\lambda\rangle = 0, \tag{2.42}$$

which gives the relation for the Dynkin labels $p_i = (\alpha_i^\vee, \lambda)$. The representation is then given by the highest weight state as well as the ones created by acting with lowering operators sequentially until the final state is reached when the maximum number of lowering operators allowed has been applied. I.e. before the action of any of the lowering operators will annihilate the state.

What is known as the fundamental weights Λ_i could be introduced such that they fulfil $(\alpha_i^\vee, \Lambda_j) = \delta_{ij}$ which then can be used to describe the vector λ as

$$\lambda = \sum_i p_i \Lambda_i. \tag{2.43}$$

The fundamental weights then form a basis for a dual space to the coroot space known as the weight space. The scalar product on the weight space uses the inverted Cartan matrix as the metric. Thus if one have two vectors in the weight space μ and λ one write $(\mu, \lambda) = \sum_{ij} A^{ij} \mu_j \lambda_j$, where A^{ij} is the inverse Cartan matrix. The representations corresponding to the highest weight state, in that representation, is now denoted as a vector over the fundamental weights via giving the coefficients for the fundamental weights as $\lambda = [a, b, c, ...]$, which will be used in later chapters to denote the representations. The existence of a highest weight representation $R(\lambda)$ suggest that there exist a dual representation $R(-\lambda)$ that has the lowest weight $-\lambda$.

2.4 Satake diagrams

Throughout this section the notation of Araki [21] will be employed. In section 2.3 Dynkin diagrams were introduced. The Satake diagrams are Dynkin diagrams which include different coloured nodes and arrows connecting the nodes. The diagrams are used to classify the real forms of a Lie algebra.

Given a complex Lie algebra $\mathfrak{g}_\mathbb{C}$ and a real subalgebra $\mathfrak{g}$, then $\mathfrak{g}$ is a real form if its complexification $\mathbb{C} \otimes \mathfrak{g}$ can be identified with $\mathfrak{g}_\mathbb{C}$ by [21]:

$$\alpha \otimes X \mapsto \alpha X, \quad \alpha \in \mathbb{C},\ X \in \mathfrak{g}. \tag{2.44}$$

The conjugation with respect to a real form is then defined as

$$\sigma(\alpha X) = \bar{\alpha} X, \tag{2.45}$$

and thus forms an anti-linear involution. From this one can see that the existence of an anti-linear involution gives rise to a real form of a complex Lie algebra by finding the fixed elements of this involution.

The nodes • indicates a compact node, that means that the corresponding root α_i transform as

$$\alpha_i' = \sigma(\alpha_i) = -\alpha_i, \tag{2.46}$$

under the involution σ. A trivial example of this type is A_1, which corresponds to $\mathfrak{su}(2)$, one knows that $(\alpha, \alpha) = 2$ has to apply for this algebra. Thus the same has to hold for the transformed roots, i.e. $(\alpha', \alpha') = 2$, by substituting in the expression for the roots one finds

$$(\alpha', \alpha') = (-\alpha, -\alpha) = (\alpha, \alpha) = 2. \tag{2.47}$$

As for the generators one finds that $\sigma(e_{\alpha_i}) = -e_{-\alpha_i}$ as well as $h_i \mapsto -h_i$. The involution is then complex conjugation of the generators. This suggest that the generators are of the form found from $\mathfrak{su}(2)$.

Another type of diagram that could appear is one where arrows appear. The arrows which appear on the non-compact nodes imply that the nodes are interchanged. I.e. if the node i and j has an arrow between them, that means that the roots transform as

$$\alpha_i' = \sigma(\alpha_i) = \alpha_j, \tag{2.48}$$
$$\alpha_j' = \sigma(\alpha_j) = \alpha_i, \tag{2.49}$$
$$\tag{2.50}$$

under the involution. An example of such an algebra is the $\mathfrak{sl}(2, \mathbb{C})$ seen as a real algebra which has the diagram:

$$\underset{1}{\circ} \leftrightarrow \underset{2}{\circ}$$

This shows that if by mapping the set of generators to the other by complex conjugation one finds the real algebra.

Next one would like to know how the non-compact nodes transform under the involution. This is not always easily given as with the case of the compact nodes and depends on what other nodes the node is connected to. As the roots form a basis for the root space one knows that the resulting root after the involution can be written in terms of the roots thus one can express a non-compact root as

$$\alpha_i' = \sigma_i^{\ j} \alpha_j, \tag{2.51}$$

where $\sigma_i^{\ j}$ is a matrix. The form of the involution for a root corresponding to a non-compact node α_i is of the form

$$\sigma(\alpha_i) = \alpha_{S(i)} + \sum_{r \in \Delta_0} c_r \alpha_r, \ c_r \in \mathbb{Z}, \tag{2.52}$$

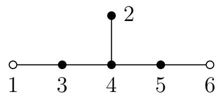
where Δ_0 is the set of fundamental roots corresponding to the compact nodes and $S(i)$ is a mapping to a index of a fundamental root that does not lie in Δ_0 [21]. As this transformation should be an automorphism one knows that it should preserve the scalar product between two roots, (α_i, α_j), i.e. the Cartan matrix. One thus find the requirement that

$$A_{ij} = \sigma_i{}^k A_{kl} \sigma_j{}^l, \tag{2.53}$$

or on matrix format

$$A = \sigma A \sigma^T. \tag{2.54}$$

As an example one could look at the involution corresponding to $E_{6(-26)}$, the Satake diagram is:



which means that one could then write down the transformation matrix $\sigma_i{}^j$ as

$$\sigma = \begin{pmatrix} 1 & a & b & c & d & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & e & f & g & h & 1 \end{pmatrix}. \tag{2.55}$$

Equations (2.54) could then be solved using some symbolic handling software, e.g. Mathematica, to find that

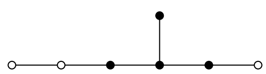
$$\sigma = \begin{pmatrix} 1 & 1 & 2 & 2 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 2 & 2 & 1 \end{pmatrix}. \tag{2.56}$$

This means that one have

$$\alpha_1' = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5, \tag{2.57}$$

$$\alpha_6' = \alpha_6 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5. \tag{2.58}$$

This example illuminates that the non-compact nodes connected to a compact node involve the compact nodes in the involution. One should also note that the non-compact node doesn't contribute to the involution. Next considering the case of $E_{7(-25)}$ which have the diagram



where the left-most node is denoted as α_0. One should now consider

$$\alpha_0' = \alpha_0 + C^i \alpha_i, \tag{2.59}$$

where $C^1 = 0 = C^6$ since they corresponds to non-compact nodes and thus doesn't enter into the involution according to eq. (2.52). One can then calculate the case that

$$0 = (\alpha_0', \alpha_2') = -(\alpha_0, \alpha_2) - C^i(\alpha_i, \alpha_2) \tag{2.60}$$

the first term on the right-hand-side is trivially zero while the second one gives an equation for two coefficients (C^2 and C^4). By then proceeding to the same for the remaining compact nodes one finds the following equations for the coefficients

$$2C^2 - C^4 = 0, \tag{2.61}$$
$$2C^3 - C^4 = 0, \tag{2.62}$$
$$2C^4 - C^2 - C^3 - C^5 = 0, \tag{2.63}$$
$$2C^5 - C^4 = 0, \tag{2.64}$$

from which one have the solution $C^i = 0$. By direct computation one finds that the same involution is found for α_1 and α_6 as in the $\mathrm{E}_{6(-26)}$ case.

From previous example one can see the more general the result that a white node (non-compact node) with no arrows connected to another white node is not involved in the involution in a non-trivial way. In other words, the white nodes which are only connected to other white nodes behaves as with the case of the split form.

Finally collecting the results form this section one has that

- White nodes corresponds to non-compact nodes,
- Black nodes corresponds to compact nodes,
- An arrow means that the roots on which the arrow points are interchanged under the involution,
- Compact nodes imply that the root is multiplied by (-1),
- A white node which only have neighbouring white nodes transform trivially.
- Non-compact nodes transform as in eq. (2.52),

However there is no way to determine the coefficients appearing in that equation by direct inspection of the diagram in general. Using these rules and a computer one could determine the coefficients for general real forms.

The knowledge of the roots action under the involution induces an involution on the dual space to the roots and consequently on the generators. The generators in the compact set then map as

$$\sigma e_\alpha = \rho_\alpha e_{\sigma^* \alpha}, \tag{2.65}$$

where $\rho_\alpha = \pm 1$ depending on the root. The above relation holds for any node however there exist constraints for ρ_α for all roots, which can be seen in [21], that specially reduce to the previously stated when considering the compact nodes. However the purely real generators need to have the trivial involution.

3

String Theory, Super Gravity and Dualities

3.1 String theory

Sting theory is, as a first quantised theory, a two dimensional quantum field theory in which the fundamental object is a string which is described by two parameters, (τ, σ), which corresponds to the time and location along the string. The string however exists in a space which is D-dimensional. In the case of superstring theory that has been shown to be $D = 10$ [1], for which one have coordinates $X^i(\tau, \sigma)$ which describes the string's location in the space in which it is embedded. The strings themselves can oscillate which then can be quantised and leads to the quantum field theory. There exist two types of configurations for a string, open or closed, which leads to different boundary conditions of the string. In the case of closed superstrings one find two different theories, type IIA and IIB. Apart from these two examples of string theories one have three more which are the type I superstring theory and the two heterotic string theories $\mathrm{E}_8 \times \mathrm{E}_8$ and $SO(32)$. The heterotic string theories are formulated in $D = 10$, however their formulation will not be discussed here. The low energy limit of string theory, i.e. inclusion of only massless states, leads to what is known as supergravity theories.

3.2 $D = 11$ supergravity

In 1978 Cremmer, Julia and Scherk formulated a $D = 11$ supergravity theory [3] which has an action of the form

$$S = \int \mathrm{d}^{11}x\sqrt{g}\left(\mathcal{R} - \frac{1}{2}F_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma}\right) - \frac{1}{6}\int A \wedge F \wedge F + S_F, \tag{3.1}$$

where S_F is the fermionic terms. This form of supergravity has been greatly studied and compactified to various number of dimensions before. From the field content presented above one have

$$D = 11\text{-SUGRA} = \begin{cases} g_{\mu\nu} \ - \ \text{graviton} \\ A_{\mu\nu\rho} \ - \ \text{3-form} \end{cases} \tag{3.2}$$

as the bosonic fields of interest. Upon compactification on a n-torus, i.e. making the split $\mathcal{M}_{11} = \mathcal{M}_{11-n} \times T^n$, one finds that these fields splits into

$$\text{Field content} = \begin{cases} g_{\mu\nu} & - \text{ graviton} \\ g_{\mu m} & - \text{ graviphoton} \\ g_{mn} & - \text{ scalars} \\ A_{\mu\nu\rho} & - \text{ 3-form} \\ A_{\mu\nu m} & - \text{ 2-forms} \\ A_{\mu mn} & - \text{ 1-forms} \\ A_{mnk} & - \text{ scalars} \end{cases} \tag{3.3}$$

which is seen from the external perspective with μ, ν, ρ is the spacetime indices for $11 - n$ coordinates, and m, n, k are the indices for T^n. However in the cases with $n > 3$ the three form can be dualised into a lower forms and similarly for higher n the two form can also be dualised to lower forms. Thus we can count the number of fields of each type, taking the case of $n = 6$, one have

$$\begin{cases} \text{Scalars } 21 + 20 + 1 = 42, \\ \text{1-forms } 6 + 6 + 15 = 27, \\ \text{Metric } 1, \end{cases} \tag{3.4}$$

where one have the extra 1 in the scalars from the dualised $A_{\mu\nu\rho}$. Similarly in the number of 1-forms one have dualised the 2-forms to 1-forms and thus finds another 6. This concludes the number of fields one can have in the case of $n = 6$ as the 2-forms are (Hodge-)dual to 1-forms. The appearance of 27 and 42 here is the motivation to use the E_6-ExFT with the scalar coset $E_6/USp(8)$. The dimension of the scalar coset is found to be $78 - 36 = 42$, and 27 is the dimension of the fundamental representation of E_6.

3.3 Dualities

Dualities are a common occurance in physics and in [1] some examples of these are given. Not unlike the other areas of physics string theory also have dualities, examples includes T-duality, S-duality and U-duality. The latter of which is generated by T- and S-dualities [7]. To answer the question of what T-duality is one has to discuss the action of compactification of a dimension. Starting from the sting theory mass spectrum for a closed string compactified on a T^1 with radius R is given by [1]:

$$M^2 = \frac{n^2}{R^2} + \frac{m^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N^\perp + \bar{N}^\perp - 2), \tag{3.5}$$

with $m, n \in \mathbb{Z}$. One can see that if one takes the transformation $R \mapsto \frac{\alpha'}{R}$ all the same massive states are found and thus one have a duality. The same principle holds if more than one direction is compactified. However one not only have the inverted radius transformation for each of them one could exchange radii thus finds a larger symmetry of T-duality. These duality group are however quite clearly discrete.

4

Magical Supergravities

In Chapter 3 the most known form of supergravity theory, $D = 11$ supergravity, was discussed. However this is not the only type of supergravity theory. There exists a special class of supergravity theories that are constructed in symmetrical spaces in dimension $D = 3, 4, 5$ and 6 [20]. This class of supergravities when reduced to $D = 5$, form supergravity theories with $N = 2$. The name of the class comes from the field content living inside representations of the algebras of the Tits, Fruendenthal and Rozenfeld magical square of Lie algebras. This square were discussed in Chapter 2 and in [24]. The bosonic field content of which these $D = 6$ theories consist is given by the following

$$g_{\mu\nu}, \quad \text{metric,} \tag{4.1}$$

$$B^I_{\mu\nu}, \quad \text{Self dual 2-form potential,} \tag{4.2}$$

$$L^I, \quad \text{Scalar parametrisation of coset,} \tag{4.3}$$

$$A^A_\mu, \quad \text{gauge fields,} \tag{4.4}$$

$$\phi^X, \quad \text{Hypermultiplet scalars,} \tag{4.5}$$

where the Greek indices take values in the 6-dimensional spacetime and the capital latin letters I and A takes values in the vector and chiral spinor representation of the $SO(1, \nu + 1)$ respectively. There also exists the possibility to add additional fields known as hypermultiplets in order to cancel anomalies. These hypermultiplet scalar fields span a quaternionic manifold, however we will not consider them as they are not affected by compactification.

Upon compactification of the external dimensions one finds that, seen from the external perspective, the number of scalar fields grows and thus lie in a larger quotient space. The scalar fields of these theories then span the quotient space found in table 4.1 for the various dimensions of the external space under consideration.

Table 4.1: The table represents the scalar target space for the magical supergravity theory in the respective dimension. The rows correpsonds to increasing dimension of the division algebra, i.e. $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$.

$\mathbb{F}$	$D=6$	$D=5$	$D=4$	$D=3$
$\mathbb{R}$	$\frac{SO(1,2)}{SO(2)}$	$\frac{SL(3,\mathbb{R})}{SO(3)}$	$\frac{Sp(6,\mathbb{R})}{U(3)}$	$\frac{F_{4(4)}}{USp(6)\times USp(2)}$
$\mathbb{C}$	$\frac{SO(1,3)}{SO(3)}$	$\frac{SL(3,\mathbb{C})}{SU(3)}$	$\frac{SU(3,3)}{SU(3)\times SU(3)\times U(1)}$	$\frac{E_{6(2)}}{SU(6)\times SU(2)}$
$\mathbb{H}$	$\frac{SO(1,5)}{SO(5)} \times \frac{SU(2)}{SU(2)}$	$\frac{SU^*(6)}{USp(6)}$	$\frac{SO^*(12)}{U(6)}$	$\frac{E_{7(-5)}}{SO(12)\times SU(2)}$
$\mathbb{O}$	$\frac{SO(1,9)}{SO(9)}$	$\frac{E_{6(-26)}}{F_4}$	$\frac{E_{7(-25)}}{E_6\times SO(2)}$	$\frac{E_{8(-24)}}{E_7\times SU(2)}$

To illustrate how the vectors combine into a larger representation upon compactification, the next section is therefore devoted to studying the case with compactification from parent-theories to the $D = 5$ theories.

4.1 Dimensional reduction D=6 to D=5

Previously the action of compactification has been discussed and it was stated that the scalar manifold for the different division algebras should be given by 4.1. However how the fields combine into this result is very important as well as to know how the vector fields combine upon compactification to $D = 5$. Therefore we start with the field content and use the split $\hat{\mu} = (\mu, 5)$ to find that each type of field becomes,

$$g_{\hat{\mu}\hat{\nu}} \mapsto (g_{\mu\nu}, g_{\mu 5}, g_{55}), \tag{4.6}$$
$$B^I_{\hat{\mu}\hat{\nu}} \mapsto (B^I_{\mu\nu}, B^I_{\mu 5}, B^I_{55}), \tag{4.7}$$
$$L^I \mapsto L^I, \tag{4.8}$$
$$A^A_{\hat{\mu}} \mapsto (A^A_\mu, A^A_5). \tag{4.9}$$

One should note that the self duality and anti-self duality found in $D = 6$ is removed upon compactification and replaced by the dualisation between 2-forms and 1-forms, therefore $B^I_{\mu\nu}$ does not contribute to any new information. In order to count the number of fields one has to know the amount of values the indices I and A can take. The dimension of the vector representation of $SO(1, \nu + 1)$ is $\nu + 1$ and the dimension of the spinor representation of $SO(1, 1+\nu)$ is n_V. As the vector multiplet should transform in the spinor representation one have that $A = 1, ..., n_V$ and the tensor multiplet in the vector representation $I = \nu + 2$ [20]. Thus one finds that the scalar manifolds combines as

$$L^I,\ g_{55},\ A^A_5 : \quad (\nu + 1) + 1 + n_V, \tag{4.10}$$

where one has noted that there are $\nu + 1$ scalars parametericing the coset $SO(1, \nu + 1)/SO(\nu + 1)$. The number of tensor multiplets are then explicitly $2, 3, 5, 9$. While the number of vector multiplets are found to be $n_V = 2, 4, 8, 16$ seen as real. Next we see that the number of vectors also increase when compactified so that they become

$$A^A_\mu,\ B^I_{\mu 5},\ g_{\mu 5} : \quad n_V + (\nu + 2) + 1. \tag{4.11}$$

Summarising these results into table 4.2 and comparing the number of scalars found to that of the expected results from the quotient spaces found in table 4.1.

Table 4.2: The table summarises the resulting number of scalars and vector fields after compactification to $D = 5$ for the different divisions algebras. The table has also included a column for the expected number of scalars that are found from counting the degrees of freedom for the quotient spaces shown in tabel 4.1.

Division algebra	# scalars	# vectors	coset degrees of freedom
$\mathbb{R}$	5	6	5
$\mathbb{C}$	8	9	8
$\mathbb{H}$	14	15	14
$\mathbb{O}$	26	27	26

The table confirms that just from these fields one finds enough scalar field to parametrise the quotient space. Thus motivating that the hypermultiplet scalars remains separate.

When considering the representation for the vector fields one would like them to live in the coordinate representation of the groups found in table 4.1. However as we know from previous results the representation has to be decomposed into modules of the smaller Lie algebra. To confirm our previous results one could break down the coordinate representation of the $D = 5$ group into modules of the group corresponding to $D = 6$ which gives the result

$$SL(3,\mathbb{R}) : \mathbf{6} = \mathbf{3} \oplus \mathbf{1} \oplus \mathbf{2}, \qquad \text{as SO(1,2) module,} \tag{4.12}$$
$$SL(3,\mathbb{C}) : \mathbf{9} = \mathbf{4} \oplus \mathbf{1} \oplus \mathbf{4}, \qquad \text{as SO(1,3) module,} \tag{4.13}$$
$$SU(6) : \mathbf{15} = \mathbf{6} \oplus \mathbf{1} \oplus \mathbf{8}, \qquad \text{as SO(1,5) module,} \tag{4.14}$$
$$E_6 : \mathbf{27} = \mathbf{10} \oplus \mathbf{1} \oplus \mathbf{16}, \qquad \text{as SO(1,9) module,} \tag{4.15}$$

from which in all cases one finds that this have the schematic form

$$\text{vector} \oplus \text{scalar} \oplus \text{spinor}. \tag{4.16}$$

This confirms that in all the cases considered before when compactified the fields combines into a vector representation of the symmetry group of $D = 5$ column. Next one needs to break down the adjoint representation of the target group as well. One finds then that the results:

$$SL(3,\mathbb{R}) : \mathbf{8} = \mathbf{2} \oplus (\mathbf{3} \oplus \mathbf{1}) \oplus \mathbf{2}, \qquad \text{as SO(1,2) module,} \tag{4.17}$$
$$SL(3,\mathbb{C}) : \mathbf{16} = \bar{\mathbf{4}} \oplus (\mathbf{6} \oplus \mathbf{1} \oplus \mathbf{1}) \oplus \mathbf{4}, \qquad \text{as SO(1,3) module,} \tag{4.18}$$
$$SU(6) : \mathbf{35} = \bar{\mathbf{8}} \oplus (\mathbf{15} \oplus \mathbf{1} \oplus \mathbf{3}) \oplus \mathbf{8}, \qquad \text{as SO(1,5) module,} \tag{4.19}$$
$$E_6 : \mathbf{78} = \bar{\mathbf{16}} \oplus (\mathbf{45} \oplus \mathbf{1}) \oplus \mathbf{16}, \qquad \text{as SO(1,9) module.} \tag{4.20}$$

From which one sees that the decomposition follow, for the first three cases, the general decomposition

$$\overline{\text{spinor}} \oplus (\text{adjoint} \oplus \mathbf{1}_{\text{grading}} \oplus \text{Im } \mathbb{F}) \oplus \text{spinor}, \tag{4.21}$$

with the exception of the octonions since right multiplication by unit octonions does not commute with left multiplication.

As was known from the construction of the magical square of Lie algebras one has that the vector representation of these target groups are formed by 3×3 hermitian matrices over the different division algebras. Thus it should be possible to write such a matrix for each group where the vector fields should be entries. Such a matrix is of the form

$$J = \begin{pmatrix} x & \xi & \eta \\ \bar{\xi} & y & \zeta \\ \bar{\eta} & \bar{\zeta} & z \end{pmatrix}, \tag{4.22}$$

with $x, y, z \in \mathbb{R}$ and $\xi, \eta, \zeta \in \mathbb{F}$. By simply counting the real degrees of freedom one finds that they are $3(\nu + 1)$. Now by starting with the case of $\mathbb{F} = \mathbb{R}$ one have that

$$B_{\mu 5}^{0,1,2}, A_{\mu}^{1,2}, g_{\mu 5}, \tag{4.23}$$

thus one realises that the $g_{\mu 5}$ will look the same for all the division algebras and is a single real scalar thus one sets it as $z = g_{\mu 5}$. Now one imposes the ansatz,

$$J = \begin{pmatrix} B_{\mu 5}^0 + B_{\mu 5}^1 & B_{\mu 5}^2 & A_{\mu}^1 \\ B_{\mu 5}^2 & B_{\mu 5}^0 - B_{\mu 5}^1 & A_{\mu}^2 \\ A_{\mu}^1 & A_{\mu}^2 & \sqrt{2} g_{\mu 5} \end{pmatrix}, \tag{4.24}$$

which then becomes in the complex case

$$J = \begin{pmatrix} B_{\mu 5}^0 + B_{\mu 5}^1 & B_{\mu 5}^2 + iB_{\mu 5}^3 & A_{\mu}^1 + iA_{\mu}^2 \\ B_{\mu 5}^2 - iB_{\mu 5}^3 & B_{\mu 5}^0 - B_{\mu 5}^1 & A_{\mu}^3 + iA_{\mu}^4 \\ A_{\mu}^1 - iA_{\mu}^2 & A_{\mu}^3 - iA_{\mu}^4 & \sqrt{2} g_{\mu 5} \end{pmatrix}, \tag{4.25}$$

which covers all the available components. This construction seems to work in every division algebra since $A = 2, 4, 8, 16$ all are 2ν, i.e. double the degrees of the algebra, and $I = \nu + 2$, thus another number of degree of the algebra. Schematically this becomes

$$J = \begin{pmatrix} \sqrt{2} B^+ & \mathbf{B}^\perp & \mathbf{A}^1 \\ \overline{\mathbf{B}^\perp} & \sqrt{2} B^- & \mathbf{A}^2 \\ \overline{\mathbf{A}^1} & \overline{\mathbf{A}^2} & \sqrt{2} g \end{pmatrix}, \tag{4.26}$$

where the bold font indicates that the components have been made into an element in $\mathbb{F}$. This is an element in the Jordan algebra. We know from Chapter 2 that there exists a triple product for the algebra elements to $\mathbb{R}$ by (X, Y, Z) but we also know that the matrix represents a vector which can then be written in index notation as X^M. Thus we have a tensor d_{MNK} such that

$$(X, Y, Z) = d_{MNK} X^M Y^N Z^K. \tag{4.27}$$

It follows from the properties of the triple product that the d tensor must be fully symmetric and it exists in all cases.

Up to this point the action has not explicitly been given. However in order to proceed the $D = 6$ action has to be given. The bosonic part of the action is

$$\mathcal{L} = e\mathcal{R} - \frac{1}{12}eg_{IJ}G^I_{\mu\nu\rho}G^{\mu\nu\rho J} - \frac{1}{4}em_{AB}F^A_{\mu\nu}F^{\mu\nu B} - e\frac{1}{8}\varepsilon^{\mu\nu\rho\sigma\lambda\tau}\Gamma_{IAB}B^I_{\mu\nu}F^A_{\rho\sigma}F^B_{\lambda\tau} \\ - \frac{1}{4}eg^{\mu\nu}L^{Ia}\partial_\mu L_I L^{Ja}\partial_\nu L_J, \quad (4.28)$$

where $g_{IJ} = L_I L_J + L^a_I L_{Ja}$ is the scalar metric for the vectors and m_{AB} for the spinors and

$$G^I_{\mu\nu\rho} = 3\partial_{[\mu}B^I_{\nu\rho]} + 3\Gamma^I_{AB}F^A_{[\mu\nu}A^B_{\rho]}. \quad (4.29)$$

From this it is clear that the action contains both 2-forms and 1-forms, and since the action is in $D = 6$ these are not related by any dualisation. However there is a self-duality relation for the 2-forms which has to be imposed by hand, which reads $*G_{I\mu\nu\rho} = g_{IJ}G^J_{\mu\nu\rho}$. The number of self-dual and anti-self-dual fields is given by the signature of g_{IJ}.

When these fields are compactified to $D = 5$ it is well known that 1-forms are dual to 2-forms and hence one could dualise all 2-forms into 1-forms. This gives that the field strength should be a 2-form which is expected as vector fields in $D = 5$ should consist of the same fields as was discussed earlier. One will also find that the terms that contains scalar terms seen from the exterior perspective which combines into the scalar metrics. One therefore finds that the action becomes schematically

$$\mathcal{L} = e\mathcal{R} - C_1\, e\mathscr{M}_{MN}\mathscr{F}^M \wedge *\mathscr{F}^N - C_2\, eC_{MNK}\mathscr{F}^M \wedge \mathscr{F}^N \wedge \mathscr{A}^K \\ - C_3 e\mathcal{D}_\mu \mathcal{M}_{MN}\mathcal{D}^\mu \mathcal{M}^{MN} + C_4 eV(\mathcal{M}, g), \quad (4.30)$$

where the last term only consists of interior derivatives and scalars.

5

Extended Geometry

Extended geometry is a term used to describe theories which introduce an extension of a d-dimensional space to form some module for a structure group G. In these theories the group $GL(d)$ is found to be a subgroup of $G \times R^+$ locally and is selected through the solving of what is known as the section constraint. The section constraint is a condition on the allowed momenta and chooses which coordinates are physical [16]. Some examples of extended geometry includes Double Field Theory (abbreviated as DFT), see [5] for a review, and the series of Exceptional Field Theories, abbreviated ExFT, with examples the E_d-series found in [6, 17–19].

This chapter starts off with a review of the results found for extended geometries with a Kac-Moody algebra with a symmetrisable and indecomposable Cartan matrix, to then explicitly derive the E_6-ExFT as an example to illustrate the framework. This example is then followed by generalisation to algebras which behaviour is similar to that of E_6. The subsequent section is devoted to do the same for the cases E_7 and E_8. This concludes the construction of ExFT-theories for the magical supergravities. The last section of this chapter is devoted to solving the section constraint. That includes a review for the cases encountered for the split real forms of the structure algebras as well as the extension to the cases of general real forms, with the example of the algebras found in the magical supergravities.

5.1 Generalised diffeomorphisms

The extended geometries include both a set of coordinates which are referred to as exterior coordinates, which are the coordinates that would remain unchanged under a compactification, and internal coordinates. The internal coordinates take values in the module $R(\lambda)$ of the structure group G whose algebra is $\mathfrak{g}$. This yields that there can exist two types of diffeomorphisms, the internal diffeomorphisms and the external diffeomorphisms. In the case of extended geometries the internal space consist of more than just the physical compactified coordinates, and there exists some gauge transformations between the internal coordinates. Thus one can discuss those transformation as special cases of a single type of diffeomorphism which is known as generalised diffeomorhpism. The generalised diffeomorphisms are given by

$$\mathscr{L}_\xi V^M = \xi^N \partial_N V^M - V^N \partial_N \xi^M + Y^{MN}{}_{PQ}\, \partial_N \xi^P V^Q, \tag{5.1}$$

where the indices M takes values between 0 and $\dim(R(\lambda))$ [16]. The Y-tensor is then given on the general form

$$Y^{MN}{}_{PQ} = -k\eta_{\alpha\beta}T^{\alpha N}{}_{P}T^{\beta M}{}_{Q} + \beta\delta^N_P\delta^M_Q + \delta^M_P\delta^N_Q, \tag{5.2}$$

for some values of the real constants k and β. The expression for the generalised diffeomorphism can then be rewritten using the definition of Y to become

$$\begin{aligned}\mathscr{L}_\xi V^M = \xi^N\partial_N V^M - V^N\partial_N\xi^M - k\eta_{\alpha\beta}T^{\alpha N}{}_{P}T^{\beta M}{}_{Q}\partial_N\xi^P V^Q + \beta\delta^N_P\delta^M_Q\partial_N\xi^P V^Q\\ + \delta^M_P\delta^N_Q\partial_N\xi^P V^Q\\ = \xi^N\partial_N V^M - k\eta_{\alpha\beta}\partial_N(T^\alpha\xi)^N(T^\beta V)^M + \beta\partial_N\xi^N V^M.\end{aligned} \tag{5.3}$$

The form found on the last line presents itself a bit easier when rederiving the identity given by equation (2.5) in [16]. In order to minimise the unnecessary clutter we take the special case $k = 1$ which corresponds to the case when $\mathfrak{g}$ is simply laced, however the result can be derived using general k as stated in the reference mentioned. By taking another diffeomorphism one find that this becomes

$$\begin{aligned}\mathscr{L}_\xi\mathscr{L}_\eta V^M &= \mathscr{L}_\xi(\eta^N\partial_N V^M - \eta_{\alpha\beta}\partial_N(T^\alpha\eta)^N(T^\beta V)^M + \beta\partial_N\eta^N V^M)\\ &= \xi^P\partial_P(\eta^N\partial_N V^M - \eta_{\alpha\beta}\partial_N(T^\alpha\eta)^N(T^\beta V)^M + \beta\partial_N\eta^N V^M)\\ &\quad - \eta_{\gamma\delta}\partial_P(T^\gamma\xi)^P(T^\delta\eta^N\partial_N V - \eta_{\alpha\beta}\partial_N(T^\alpha\eta)^N(T^\beta V) + \beta\partial_N\eta^N V)^M\\ &\quad + \partial_P\xi^P(\eta^N\partial_N V^M - \eta_{\alpha\beta}\partial_N(T^\alpha\eta)^N(T^\beta V)^M + \beta\partial_N\eta^N V^M).\end{aligned} \tag{5.4}$$

By introducing the notation $(\partial_A A) := \partial_N A^N$ to signify what the derivative is acting on and that the indices are contracted, one finds that this becomes

$$\begin{aligned}\mathscr{L}_\xi\mathscr{L}_\eta V^M &= \xi^P\partial_P\eta^N\partial_N V^M + \xi^P\eta^N\partial_p\partial_N V^M - \xi^P\partial_P\partial_N(T^\alpha\eta)^N(T_\alpha V)^M\\ &\quad - \xi^P(\partial_\eta T^\alpha\eta)\partial_P(T_\alpha V)^M + \beta\xi^P\partial_P\partial_N\eta^N V^M + \beta\xi^P(\partial_\eta\eta)\partial_P V^M\\ &\quad - (\partial_\xi T^\beta\xi)(T_\beta\eta^N\partial_N V)^M + (\partial_\xi T^\beta\xi)(\partial_\eta T^\alpha\eta)(T_\beta T_\alpha V)^M\\ &\quad - \beta(\partial_\xi T^\beta\xi)(\partial_\eta\eta)(T_\beta V)^M + \beta(\partial_\xi\xi)\eta^N\partial_N V^M\\ &\quad - \beta(\partial_\xi\xi)(\partial_\eta T^\alpha\eta)(T_\alpha V)^M + \beta^2(\partial_\xi\xi)(\partial_\eta\eta)V^M.\end{aligned} \tag{5.5}$$

From this result one can see that the second and last term is symmetric in ξ and η, thus they will cancel in the commutator, hence they will be disregarded. Similarly the first term in the second row and the first term in the third row are on the same form but with $\xi \leftrightarrow \eta$ and with the same sign, hence they too will cancel in the commutator. Exactly the same thing is found for the third term in the second row and the second term in the fourth row. Finally the first term in row four and first term in the last row also exhibits this behaviour. Therefore the commutator becomes

$$\begin{aligned}[\mathscr{L}_\xi, \mathscr{L}_\eta]V^M = \xi^P\partial_P\eta^N\partial_N V^M - \xi^P\partial_P\partial_N(T^\alpha\eta)^N(T_\alpha V)^M + \beta\xi^P\partial_P\partial_N\eta^N V^M\\ + \frac{1}{2}(\partial_\xi T^\beta\xi)(\partial_\eta T^\alpha\eta)f_{\beta\alpha}{}^\gamma(T_\gamma V)^M - (\xi \longleftrightarrow \eta),\end{aligned} \tag{5.6}$$

where the structure constant $f_{\beta\alpha}{}^{\gamma}$ appear from the need to swap order of the T's when anti-symmetrising in ξ and η, and the factor 1/2 from not double counting the term when anti-symmetrising the result above.

Next one has that

$$\mathscr{L}_{\xi}\eta^M = \xi^N\partial_N\eta^M - (\partial_\xi T^\alpha\xi)(T_\alpha\eta)^M + \beta(\partial_\xi\xi)\eta^M, \tag{5.7}$$

which then is used to find

$$\mathscr{L}_{\frac{1}{2}\mathscr{L}_\xi\eta}V^M. \tag{5.8}$$

Using the result in the last equation one finds that this becomes

$$\begin{aligned}
\mathscr{L}_{\frac{1}{2}\mathscr{L}_\xi\eta}V^M = &\frac{1}{2}(\xi^N\partial_N\eta^P - (\partial_\xi T^\alpha\xi)(T_\alpha\eta)^P + \beta(\partial_\xi\xi)\eta^P)\partial_P V^M\\
&- \frac{1}{2}(\partial_P(T^\beta\xi^N\partial_N\eta - T^\beta(\partial_\xi T^\alpha\xi)(T_\alpha\eta) + \beta(\partial_\xi\xi)T^\beta\eta)^P)(T_\beta V)^M\\
&+ \frac{\beta}{2}(\partial_P(\xi^N\partial_N\eta^P - (\partial_\xi T^\alpha\xi)(T_\alpha\eta)^P + \beta(\partial_\xi\xi)\eta^P))V^M\\
= &\frac{1}{2}\xi^N\partial_N\eta^P\partial_P V^M - \frac{1}{2}(\partial_\xi T^\alpha\xi)(T_\alpha\eta)^P\partial_P V^M + \frac{\beta}{2}(\partial_\xi\xi)\eta^P\partial_P V^M\\
&- \frac{1}{2}T^{\beta N}{}_Q T_\beta{}^M{}_S\,\partial_N\xi^P\partial_P\eta^Q V^S - \frac{1}{2}\xi^P\partial_P\partial_N(T^\beta\eta)^N(T_\beta V)^M\\
&+ \frac{1}{2}\partial_N\partial_P(T^\alpha\xi)^P(T_\beta T_\alpha\eta)^N(T^\beta V)^M + \frac{1}{2}(\partial_\xi T^\alpha\xi)(\partial_\eta T_\beta T_\alpha\eta)(T^\beta V)^M\\
&- \frac{\beta}{2}\partial_N\partial_P\xi^P(T_\beta\eta)^N(T^\beta V)^M - \frac{\beta}{2}(\partial_\xi\xi)(\partial_\eta T_\beta\eta)(T^\beta V)^M\\
&+ \frac{\beta}{2}\partial_N\xi^P\partial_P\eta^N V^M + \frac{\beta}{2}\xi^N\partial_N(\partial_\eta\eta)V^M - \frac{\beta}{2}\partial_N\partial_P(T^\alpha\xi)^P(T_\alpha\eta)^N V^M\\
&- \frac{\beta}{2}(\partial_\xi T^\alpha\xi)(\partial_\eta T_\alpha\eta)V^M + \frac{\beta^2}{2}\partial_N\partial_P\xi^P\eta^N V^M + \frac{\beta^2}{2}(\partial_\xi\xi)(\partial_\eta\eta)V^M.
\end{aligned} \tag{5.9}$$

To reduce the expression consider the last term in the first line after the equality sign and the first term in the second line, they can be combined into

$$\begin{aligned}
\frac{1}{2}(-T^{\alpha N}{}_Q T_\alpha{}^P{}_S\,\partial_N\xi^Q\eta^S + \beta\delta^N_Q\delta^P_S\partial_N\xi^Q\eta^S)\partial_P V^M = &\frac{1}{2}Y^{PN}{}_{QS}\,\partial_N\xi^Q\eta^S\partial_P V^M\\
&- \frac{1}{2}\eta^N\partial_N\xi^P\partial_P V^M,
\end{aligned} \tag{5.10}$$

which is the definition of the Y-tensor. If the section constraint is used

$$Y^{MN}{}_{PQ}\,(\partial_M\otimes\partial_N), \tag{5.11}$$

one finds that the Y term drops out and we are left with the last term only. If one use that and then removes every term which is symmetric in ξ and η from the equation one finds

$$\begin{aligned}
\mathscr{L}_{\frac{1}{2}\mathscr{L}_\xi\eta}V^M &= \frac{1}{2}(\xi^N\partial_N\eta^P\partial_P V^M - \eta^N\partial_N\xi^P\partial_P V^M) - \frac{1}{2}\partial_N\xi^P\partial_P(T^\beta\eta)^N(T_\beta V)^M \\
&- \frac{1}{2}\xi^P\partial_P\partial_N(T^\beta\eta)^N(T_\beta V)^M + \frac{1}{2}(T_\beta T_\alpha\eta)^N\partial_N\partial_P(T^\alpha\xi)^P(T^\beta V)^M \\
&+ \frac{1}{2}(\partial_\xi T^\alpha\xi)(\partial_\eta T_\beta T_\alpha\eta)(T^\beta V)^M - \frac{\beta}{2}(T^\beta\eta)^N\partial_N\partial_P\xi^P(T_\beta V)^M \\
&- \frac{\beta}{2}(\partial_\xi\xi)(\partial_\eta T_\beta\eta)(T^\beta V)^M + \frac{\beta}{2}\xi^N\partial_N(\partial_\eta\eta)V^M \\
&- \frac{\beta}{2}(T_\alpha\eta)^N\partial_N\partial_P(T^\alpha\xi)^P V^M + \frac{\beta^2}{2}\eta^N\partial_N\partial_P\xi^P V^M. \quad (5.12)
\end{aligned}$$

The first term above leads to the term $\xi^N\partial_N\eta^P\partial_P V^M - (\xi \longleftrightarrow \eta)$ when anti-symmetrised in ξ and η, which then cancels the term in the commutator. Next let us study the terms which contain mixed derivatives separately

$$\begin{aligned}
&- \frac{1}{2}\partial_N\xi^P\partial_P(T^\beta\eta)^N(T_\beta V)^M + \frac{1}{2}(\partial_\xi T^\alpha\xi)(\partial_\eta T_\beta T_\alpha\eta)(T^\beta V)^M \\
&\qquad - \frac{\beta}{2}(\partial_\xi\xi)(\partial_\eta T_\beta\eta)(T^\beta V)^M \\
&= \frac{1}{2}(\partial_\xi T^\alpha\xi)(\partial_\eta(f_{\beta\alpha}{}^\gamma T_\gamma + T_\alpha T_\beta)\eta)(T^\beta V)^M - \frac{\beta}{2}(\partial_\xi\xi)(\partial_\eta T_\beta\eta)(T^\beta V)^M \\
&\qquad - \frac{1}{2}\partial_N\xi^P\partial_P(T^\beta\eta)^N(T_\beta V)^M \\
&= \frac{1}{2}(\partial_\xi T^\alpha\xi)(\partial_\eta T_\gamma\eta)f_{\beta\alpha}{}^\gamma(T^\beta V)^M + \frac{1}{2}(T^{\alpha N}{}_P T_\alpha{}^S{}_Q\,\partial_N\xi^P\partial_S(T_\beta\eta)^Q(T^\beta V)^M) \\
&\qquad - \frac{\beta}{2}\delta^N_P\delta^S_Q\partial_N\xi^P\partial_S(T_\beta\eta)^Q(T^\beta V)^M - \frac{1}{2}\partial_N\xi^P\partial_P(T_\beta\eta)^N(T^\beta V)^M \\
&= \frac{1}{2}(\partial_\xi T^\alpha\xi)(\partial_\eta T_\gamma\eta)f_{\beta\alpha}{}^\gamma(T^\beta V)^M - \frac{1}{2}(Y^{SN}{}_{PQ} - \delta^S_P\delta^N_Q)\partial_N\xi^P\partial_S(T_\beta\eta)^Q(T^\beta V)^M \\
&\qquad - \frac{1}{2}\partial_N\xi^P\partial_P(T_\beta\eta)^N(T^\beta V)^M = \frac{1}{2}(\partial_\xi T^\alpha\xi)(\partial_\eta T_\gamma\eta)f_{\beta\alpha}{}^\gamma(T^\beta V)^M, \quad (5.13)
\end{aligned}$$

where the section constraint has again been used. Thus the resulting mixed derivative terms in the closure becomes

$$\begin{aligned}
&([\mathscr{L}_\xi, \mathscr{L}_\eta] - \mathscr{L}_{\frac{1}{2}(\mathscr{L}_\xi\eta - \mathscr{L}_\eta\xi)})V^M = \frac{1}{2}(\partial_\xi T^\beta\xi)(\partial_\eta T^\alpha\eta)f_{\beta\alpha}{}^\gamma(T_\gamma V)^M \\
&- \frac{1}{2}(\partial_\xi T^\alpha\xi)(\partial_\eta T_\gamma\eta)f_{\beta\alpha}{}^\gamma(T^\beta V)^M - (\xi \longleftrightarrow \eta) + (\text{Double derivative terms}). \quad (5.14)
\end{aligned}$$

Since all indices are summed over in the structure constant and the two terms are just permutations the two terms cancels. Now let's study the terms which contain double derivatives, we find that they are

$$
\begin{aligned}
([\mathscr{L}_\xi,\mathscr{L}_\eta]-\mathscr{L}_{\frac{1}{2}(\mathscr{L}_\xi\eta-\mathscr{L}_\eta\xi)})V^M &= -\xi^P\partial_P(\partial_\eta T^\alpha\eta)(T_\alpha V)^M+\beta\xi^P\partial_P(\partial_\eta\eta)V^M \\
+\frac{1}{2}\xi^P\partial_P(\partial_\eta T^\alpha\eta)(T_\alpha V)^M-\frac{1}{2}(T_\beta T_\alpha\eta)^N\partial_N(\partial_\xi T^\alpha\xi)(T^\beta V)^M&+\frac{\beta}{2}(T^\beta\eta)^N\partial_N(\partial_\xi\xi)(T_\beta V)^M \\
-\frac{\beta}{2}\xi^P\partial_P(\partial_\eta\eta)V^M+\frac{\beta}{2}(T_\alpha\eta)^N\partial_N(\partial_\xi T^\alpha\xi)V^M&-\frac{\beta^2}{2}\eta^N\partial_N(\partial_\xi\xi)V^M-(\xi\longleftrightarrow\eta).
\end{aligned}
\tag{5.15}
$$

In some terms the role of ξ and η needs to be swapped in order for them to be combined, thus one finds

$$
\begin{aligned}
([\mathscr{L}_\xi,\mathscr{L}_\eta]-\mathscr{L}_{\frac{1}{2}(\mathscr{L}_\xi\eta-\mathscr{L}_\eta\xi)})V^M &= \frac{1}{2}(T^\alpha T^\beta\xi)^P\partial_P(\partial_\eta T_\beta\eta)(T_\alpha V)^M-\frac{\beta}{2}(T^\alpha\xi)^P\partial_P(\partial_\eta\eta)(T_\alpha V)^M \\
-\frac{1}{2}\xi^R\partial_R(\partial_\eta T^\alpha\eta)(T_\alpha V)^M&-\frac{\beta}{2}(T^\beta\xi)^P\partial_P(\partial_\eta T_\beta\eta)V^M+\frac{\beta^2}{2}\xi^P\partial_P(\partial_\eta\eta)V^M+\frac{\beta}{2}\xi^R\partial_R(\partial_\eta\eta)V^M \\
&=\frac{1}{2}(-T^{\alpha P}{}_Q T_\alpha{}^M{}_N+\beta\delta^P_Q\delta^M_N)(-(T^\beta\xi)^Q\partial_P(\partial_\eta T_\beta\eta)V^N+\beta\xi^Q\partial_P(\partial_\eta\eta)V^N+\xi^R\partial_R\partial_P\eta^Q V^N) \\
&=\frac{1}{2}Z^{MP}{}_{QN}(-T_\beta{}^R{}_S T^{\beta Q}{}_T+\beta\delta^R_S\delta^Q_T+\delta^Q_S\delta^R_T)\xi^T\partial_P\partial_R\eta^S V^N=\frac{1}{2}Z^{MP}{}_{QN}Y^{QR}{}_{ST}\,\xi^T\partial_P\partial_R\eta^S V^N.
\end{aligned}
\tag{5.16}
$$

Hence we have found that

$$
([\mathscr{L}_\xi,\mathscr{L}_\eta]-\mathscr{L}_{\frac{1}{2}(\mathscr{L}_\xi\eta-\mathscr{L}_\eta\xi)})V^M=\frac{1}{2}Z^{MP}{}_{QN}Y^{QR}{}_{ST}\,\xi^T\partial_P\partial_R\eta^S V^N-(\xi\longleftrightarrow\eta), \tag{5.17}
$$

which agrees with the result found in [16]. This shows that the generalised diffeomorphisms can fail to close depending upon the given structure groups. This failure to close indicates the existence of extra transformations, so called ancillary transformations [16]. However in the cases of $E_{d(d)}$ with $d \leq 7$ one has that they close.

5.2 E$_6$-exceptional field theory

A special example of an extended geometry is the E$_6$-exceptional field theory. This case has been studied previously and the formalism is known. The following section is devoted to recomputed the results found in [17] as to show how the theory works and also to see if the following computations were applicable for the general results.

In the special case of E$_6$ one has that the dimension of the fundamental representation is **27**, which capital Latin letters labels, and it has a dual that is $\overline{\mathbf{27}}$. The group E$_6$ also have two fully-symmetric invariant tensors d_{MNK} and d^{MNK}, which appears in the projection operator that projects onto the adjoint representation

$$
\mathbb{P}^M{}_N{}^K{}_L=\frac{1}{6}t^{\alpha M}{}_N t_\alpha{}^K{}_L=\frac{1}{18}\delta^M_N\delta^K_L+\frac{1}{6}\delta^K_N\delta^M_L-\frac{5}{3}d^{MKR}d_{NLR}=\mathbb{P}^K{}_L{}^M{}_N. \tag{5.18}
$$

Using the above expression the generalised Lie derivative takes the form

$$\mathscr{L}_\Lambda V^M = \Lambda^K \partial_K V^M - 6\mathbb{P}^M{}_N{}^K{}_L \partial_K \Lambda^L V^N + \lambda \partial_K \Lambda^K V^M, \tag{5.19}$$

$$\mathscr{L}_\Lambda W_M = \Lambda^K \partial_K W_M + 6\mathbb{P}^N{}_M{}^K{}_L \partial_K \Lambda^L W_N + \lambda' \partial_K \Lambda^K W_M, \tag{5.20}$$

and where λ and λ' is the weight of the vector and co-vector respective. The generalised Lie derivative generalises in the same way as the ordinary Lie derivative to tensors, i.e. by the Leibniz rule. It follows from how the Lie derivative acts on covectors and contravariant vectors that

$$\mathscr{L}_\Lambda (V^M W_M) = \Lambda^K \partial_K (V^M W_M) + (\lambda + \lambda')\partial_K \Lambda^K (V^M W_M), \tag{5.21}$$

which is an important property of the transformation.

One should also note that the d-tensors have the following properties:

$$d_{S(MN}d_{KL)T}d^{STR} = \frac{2}{15}\delta^R_{(M}d_{NKL)}, \tag{5.22}$$

$$d^{S(MN}d^{KL)T}d_{STR} = \frac{2}{15}\delta^{(M}_R d^{NKL)}, \tag{5.23}$$

and that they have been chosen such that $d_{MKL}d^{NKL} = \delta^N_M$, in accordance with [17, 30]. By demanding the invariance of the d-symbol one can determine the weight of it. This is done by computation of $\mathscr{L}_\Lambda d_{MNK}$ and demanding that it should be zero. One finds that

$$\begin{aligned}
\mathscr{L}_\Lambda d_{MNK} &= \Lambda^L \partial_L d_{MNK} + 3 \cdot 6\mathbb{P}^T{}_L{}^P{}_{(M} d_{NK)P} \partial_T \Lambda^L + \lambda \partial_L \Lambda^L d_{MNK} \\
&= \Lambda^L \partial_L d_{MNK} + 3(\frac{1}{3}\delta^T_L \delta^P_{(M} + \delta^P_L \delta^T_{(M} - 10 d^{TPR} d_{RL(M}) d_{NK)P} \partial_T \Lambda^L + \lambda \partial_L \Lambda^L d_{MNK} \\
&= \Lambda^L \partial_L d_{MNK} + d_{(NKM)} \partial_L \Lambda^L + 3\partial_{(M}\Lambda^L d_{NK)L} - 30 d^{TPR} d_{RL(M} d_{NK)P} \partial_T \Lambda^L \\
&\qquad + \lambda \partial_L \Lambda^L d_{MNK}.
\end{aligned} \tag{5.24}$$

Let's focus on the "d^3" term and write it out explicitly

$$\begin{aligned}
&3d^{TPR} d_{RL(M} d_{NK)P} \partial_T \Lambda^L \\
&\quad = (d^{TPR} d_{RLM} d_{NKP} + d^{TPR} d_{RLN} d_{KMP} + d^{TPR} d_{RLK} d_{MNP}) \partial_T \Lambda^L \\
&\quad = 3 d^{TPR} d_{R(LM} d_{NK)P} \partial_T \Lambda^L,
\end{aligned} \tag{5.25}$$

where the factor 3 on the last line comes from knowing that there are 8 combination which leads to the same term and the prefactor has to compensate for the division by 24 to find the terms before the equal sign. Now using the cubic identity one finds that this becomes

$$3 d^{TPR} d_{R(LM} d_{NK)P} \partial_T \Lambda^L = \frac{6}{15} \delta^T_{(L} d_{MNK)} \partial_T \Lambda^L, \tag{5.26}$$

which leads to

$$\begin{aligned}
\mathscr{L}_\Lambda d_{MNK} &= \Lambda^L \partial_L d_{MNK} + 4\partial_{(M}\Lambda^L d_{NKL)} - \frac{60}{15}\delta^T_{(L} d_{MNK)} \partial_T \Lambda^L + \lambda \partial_L \Lambda^L d_{MNK} \\
&= \Lambda^L \partial_L d_{MNK} + \lambda \partial_L \Lambda^L d_{MNK},
\end{aligned} \tag{5.27}$$

and since the d is constants one finds that

$$\mathscr{L}_\Lambda d_{MNK} = \lambda \partial_L \Lambda^L d_{MNK}, \tag{5.28}$$

which shows that d_{MNK} must carry weight $\lambda = 0$ in order to be invariant.
Next consider the trivial gauge parameters $\Lambda^M = d^{MNK}\partial_N \chi_K$, and what the Lie derivative of a vector is

$$\begin{aligned} \mathscr{L}_\Lambda V^M = &\, d^{LNK}\partial_N \chi_K \partial_L V^M - V^K \partial_K (d^{MNP}\partial_N \chi_P) \\ &+ 10 d_{NLR} d^{MKR} \partial_K (d^{LPQ}\partial_P \chi_Q) V^N + (\lambda - \frac{1}{3}) \partial_K (d^{KPQ}\partial_P \chi_Q) V^M, \end{aligned} \tag{5.29}$$

which when the section constraint $d^{MNK}\ \partial_N \otimes \partial_K = 0$ is employed one finds that this becomes

$$\mathscr{L}_\Lambda V^M = -V^K d^{MNP} \partial_K \partial_N \chi_P + 10 d_{NRL} d^{RMK} d^{PQL} \partial_K \partial_P \chi_Q V^N. \tag{5.30}$$

Now using the same result as in (5.25) for the cubic d tensor one then finds that

$$\begin{aligned} 3 d_{NRL} d^{R(MK} d^{PQ)L} \partial_K \partial_P \chi_Q = &\, (d_{NRL} d^{RMK} d^{PQL} + d_{NRL} d^{RMP} d^{KQL} \\ &+ d_{NRL} d^{RMQ} d^{PKL}) \partial_K \partial_P \chi_Q = 2 d_{NRL} d^{RMK} d^{PQL} \partial_K \partial_P \chi_Q, \end{aligned} \tag{5.31}$$

where the last equality is found after relabeling of K and P as well as interchanging of derivatives together with the use of the section constraint on the last term. Thus one finds that the generalised Lie derivative becomes

$$\begin{aligned} \mathscr{L}_\Lambda V^M &= -V^K d^{MNP} \partial_K \partial_N \chi_P + 10 \frac{3}{2} \frac{2}{15} \delta_N^{(M} d^{KPQ)} \partial_K \partial_P \chi_Q V^N \\ &= -V^K d^{MNP} \partial_K \partial_N \chi_P + \frac{2}{4} (\delta_N^K d^{PQM} + \delta_N^P d^{QMK}) \partial_K \partial_P \chi_Q V^N = 0, \end{aligned} \tag{5.32}$$

where the section constraint has been used on the other terms in the symmetrization. From this one find that parameters on this form do not generate any transformation.

The Lie derivative could also be rewritten in terms of the Y-tensor for E_6. From the definition found in [16] that was used earlier

$$Y^{MN}{}_{PQ} = -k T^{\alpha N}{}_P T_\alpha{}^M{}_Q + \beta \delta_P^N \delta_Q^M + \delta_P^M \delta_Q^N, \tag{5.33}$$

where $\beta = k(\lambda, \lambda) - 1 = 1/3$ and $k = 1$ for E_6 as it is simply laced. By inserting the expression found in the projection operator one finds that this becomes

$$\begin{aligned} Y^{MN}{}_{PQ} = &\, -(\frac{1}{3}\delta_P^N \delta_Q^M + \delta_P^M \delta_Q^N - 10 d^{MNR} d_{RPQ}) + \frac{1}{3}\delta_P^N \delta_Q^M + \delta_P^M \delta_Q^N \\ &= 10 d^{MNR} d_{RPQ}. \end{aligned} \tag{5.34}$$

From [16] and section 5.1 we know that the generalised Lie derivatives close in the case of E_6. Hence under the assumption that the gauge parameters Λ_i carry weight $\lambda = 1/3$ one finds that

$$[\mathscr{L}_{\Lambda_1}, \mathscr{L}_{\Lambda_2}] = \mathscr{L}_{[\Lambda_1, \Lambda_2]_E}, \tag{5.35}$$

where

$$[\Lambda_1, \Lambda_2]_E^M = 2\Lambda_{[1}^K \partial_K \Lambda_{2]}^M - 10 d^{MNR} d_{KLR} \Lambda_{[1}^K \partial_N \Lambda_{2]}^L. \tag{5.36}$$

Now for the remaining part of this section the construction of the action for the E_6 exceptional field theory will be discussed as well as its gauge invariance and invariance under external diffeomorphisms.

5.2.1 E_6 action and gauge invariance

Since the exterior derivatives are not gauge covariant under E_6 we need to covariantize them and therefore introduce the gauge connection A_μ^M which takes values in the fundamental representation. The covariant derivative is then of the form

$$\mathcal{D}_\mu = \partial_\mu - \mathscr{L}_{A_\mu}. \tag{5.37}$$

From this one can determine the transformation of the gauge connection by requiring that

$$\delta(\mathcal{D}_\mu V^M) = \mathscr{L}_\Lambda(\mathcal{D}_\mu V^M). \tag{5.38}$$

Now by computing the L.H.S. one finds that

$$\begin{aligned}\delta(\mathcal{D}_\mu V^M) &= -\mathscr{L}_{\delta A_\mu} V^M + \partial_\mu(\mathscr{L}_\Lambda V^M) - \mathscr{L}_{A_\mu}\mathscr{L}_\Lambda V^M \\ &= (\mathscr{L}_\Lambda \partial_\mu V^M + \mathscr{L}_{\partial_\mu \Lambda} V^M) - \mathscr{L}_{\delta A_\mu} V^M - (\mathscr{L}_{[A_\mu, \Lambda]_E} V^M + \mathscr{L}_\Lambda \mathscr{L}_{A_\mu} V^M) \\ &= \mathscr{L}_\Lambda(\mathcal{D}_\mu V^M) + \mathscr{L}_{-\delta A_\mu + \partial_\mu \Lambda - [A_\mu, \Lambda]_E} V^M. \end{aligned} \tag{5.39}$$

Hence we have the following requirement of the transformation of the gauge connection

$$\begin{aligned}\delta A_\mu^M &= \partial_\mu \Lambda^M - [A_\mu, \Lambda]_E^M = \partial_\mu \Lambda^M - A_\mu^K \partial_K \Lambda^M + \Lambda^K \partial_K A_\mu^M \\ &\quad + 5 d^{MNR} d_{KLR}(A_\mu^K \partial_N \Lambda^L - \Lambda^K \partial_N A_\mu^L) = \partial_\mu \Lambda^M - A_\mu^K \partial_K \Lambda^M \\ &\quad + \Lambda^K \partial_K A_\mu^M - 10 d^{MNR} d_{KLR} \Lambda^K \partial_N A_\mu^L + 5 d^{MNR} d_{KLR} \partial_N (A_\mu^K \Lambda^L). \end{aligned} \tag{5.40}$$

Which can be simplified to

$$\delta A_\mu^M = \mathcal{D}_\mu \Lambda^M + 5 d^{MNR} d_{KLR} \partial_N (A_\mu^K \Lambda^L), \tag{5.41}$$

as long as Λ carries weight $\lambda = 1/3$. Now one should remember that the requirement is that this parameter generate a zero transformation and the last term is of the form $d^{MNR}\partial_N \chi_R$ which was recognized to give a trivial gauge transformation, hence can be dropped. Therefore one finds that

$$\delta A_\mu^M = \mathcal{D}_\mu \Lambda^M. \tag{5.42}$$

The corresponding field strength is given by

$$F_{\mu\nu}{}^M = 2\partial_{[\mu} A_{\nu]}^M - [A_\mu, A_\nu]_E^M, \tag{5.43}$$

and like in geometry one find that

$$\begin{aligned}[\mathcal{D}_\mu, \mathcal{D}_\nu] &= [\partial_\mu - \mathscr{L}_{A_\mu}, \partial_\nu - \mathscr{L}_{A_\nu}] = -[\mathscr{L}_{A_\mu}, \partial_\nu] - [\partial_\mu, \mathscr{L}_{A_\nu}] + [\mathscr{L}_{A_\mu}, \mathscr{L}_{A_\nu}] \\ &= \mathscr{L}_{\partial_\nu A_\mu} + \mathscr{L}_{\partial_\mu A_\nu} - \mathscr{L}_{[A_\mu, A_\nu]_E} = -\mathscr{L}_{2\partial_{[\mu} A_{\nu]} - [A_\mu, A_\nu]_E} = -\mathscr{L}_{F_{\mu\nu}}. \end{aligned} \tag{5.44}$$

where we have used $[\partial_\mu, \mathscr{L}_{A_\nu}] = \mathscr{L}_{\partial_\mu A_\nu}$ which follows from a direct computation. The variation of the field strengths is found to be

$$\begin{aligned}\delta F_{\mu\nu}{}^M &= 2\partial_{[\mu}\delta A^M_{\nu]} - 2\delta A^K_{[\mu}\partial_K A^M_{\nu]} - 2A^K_{[\mu}\partial_K\delta A^M_{\nu]} + 10d^{MKR}d_{NLR}\delta A^N_{[\mu}\partial_K A^L_{\nu]} \\ &+ 10d^{MKR}d_{NLR}A^N_{[\mu}\partial_K\delta A^L_{\nu]} = 2(\partial_{[\mu}\delta A^M_{\nu]} - A^K_{[\mu}\partial_K\delta A^M_{\nu]} + \delta A^K_{[\nu}\partial_K A^M_{\mu]} \\ &- 5d^{MKR}d_{NLR}\delta A^N_{[\nu}\partial_K A^L_{\mu]}) + 10d^{MKR}d_{NLR}A^N_{[\mu}\partial_K\delta A^L_{\nu]} \\ &= 2\mathcal{D}_{[\mu}\delta A^M_{\nu]} + 10d^{MKR}d_{NLR}A^N_{[\mu}\partial_K\delta A^L_{\nu]} + 10d^{MKR}d_{NLR}\delta A^N_{[\nu}\partial_K A^L_{\mu]} \\ &= 2\mathcal{D}_{[\mu}\delta A^M_{\nu]} + 10d^{MKR}d_{NLR}\partial_K(A^N_{[\mu}\delta A^L_{\nu]}).\end{aligned} \quad (5.45)$$

From this one can see that this is not a covariant transformation and hence one introduces

$$\mathcal{F}_{\mu\nu}{}^M = F_{\mu\nu}{}^M + 10d^{MNK}\partial_K B_{\mu\nu N}, \quad (5.46)$$

where $B_{\mu\nu N}$ is defined such that

$$\delta\mathcal{F}_{\mu\nu}{}^M = 2\mathcal{D}_{[\mu}\delta A^M_{\nu]} + 10d^{MNK}\partial_K\Delta B_{\mu\nu N}, \quad (5.47)$$

in which

$$\Delta B_{\mu\nu N} = \delta B_{\mu\nu N} + d_{NPQ}A^P_{[\mu}\delta A^Q_{\nu]}. \quad (5.48)$$

Using that the second term in $\mathcal{F}^M_{\mu\nu}$ is of the form $d^{MNK}\partial_N\chi_K$ one immediately finds that

$$\mathscr{L}_{F_{\mu\nu}} = \mathscr{L}_{\mathcal{F}_{\mu\nu}}. \quad (5.49)$$

The two-form potential $B_{\mu\nu M}$ introduced here has its own gauge field $\Xi_{\mu M}$ which then gives the total gauge transformation

$$\begin{aligned}\delta A^M_\mu &= \mathcal{D}_\mu\Lambda^M - 10d^{MNK}\partial_N\Xi_{\mu K} \\ \Delta B_{\mu\nu M} &= 2\mathcal{D}_{[\mu}\Xi_{\nu]M} + d_{MNK}\Lambda^N\mathcal{F}^K_{\mu\nu} + \mathcal{O}_{\mu\nu M},\end{aligned} \quad (5.50)$$

where $\Xi_{\mu M}$ carries weight $\lambda = 2/3$ and $\mathcal{O}_{\mu\nu M}$ are terms which are zero under $d^{MNK}\partial_N\mathcal{O}_{\mu\nu K}$. These terms can be shown to generate a gauge transformation by simply inserting them into the variation of the covariant field strength. This gives

$$\begin{aligned}\delta\mathcal{F}^M_{\mu\nu} &= 2\mathcal{D}_{[\mu}(\mathcal{D}_{\nu]}\Lambda^M - 10d^{MNK}\partial_N\Xi_{\nu]K}) + 10d^{MNK}\partial_K(2\mathcal{D}_{[\mu}\Xi_{\nu]N} + d_{NLR}\Lambda^L\mathcal{F}^R_{\mu\nu} + \mathcal{O}_{\mu\nu N}) \\ &= -\mathscr{L}_{\mathcal{F}_{\mu\nu}}\Lambda^M - 20\mathcal{D}_{[\mu}(d^{MNK}\partial_N\Xi_{\nu]K}) + 20d^{MNK}\partial_N(\mathcal{D}_{[\mu}\Xi_{\nu]K}) + 10d^{MNK}\partial_N(d_{KLR}\Lambda^L\mathcal{F}^R_{\mu\nu}).\end{aligned} \quad (5.51)$$

By taking the first and last term and studying them separately one finds that

$$\begin{aligned}&-\mathscr{L}_{\mathcal{F}_{\mu\nu}}\Lambda^M + 10d^{MNR}d_{LKR}\partial_N(\Lambda^L\mathcal{F}^K_{\mu\nu}) = -\mathcal{F}^K_{\mu\nu}\partial_K\Lambda^M + \Lambda^K\partial_K\mathcal{F}^M_{\mu\nu} \\ &\quad - 10d^{MNR}d_{KLR}\partial_N\mathcal{F}^K_{\mu\nu}\Lambda^L + 10d^{MNR}d_{LKR}\partial_N(\Lambda^L\mathcal{F}^K_{\mu\nu}) = \mathscr{L}_\Lambda\mathcal{F}^M_{\mu\nu},\end{aligned} \quad (5.52)$$

which is what was wanted and now one sees that this is a vector of weight $\lambda = 1/3$. Now to check that the two middle terms cancel out we note that the exterior and interior derivatives commute. One then has to study the Lie derivatives inside the covariant derivatives, given that $\Xi_{\mu M}$ carries weight $\lambda = 2/3$ one can use the relation

$$\mathcal{D}_\mu(d^{MNK}\partial_N V_K) = d^{MNK}\partial_N(\mathcal{D}_\mu V_K), \quad (5.53)$$

which then gives that the two middle terms cancels each other, proof of this identity consists of a direct computation and comparison between the two sides. Thus one has found that

$$\delta\mathcal{F}^M_{\mu\nu} = \mathscr{L}_\Lambda \mathcal{F}^M_{\mu\nu}, \tag{5.54}$$

under the gauge transformation.

The introduced 2-form $B_{\mu\nu M}$ has also a field strength associated with it that will be denoted $\mathcal{H}_{\mu\nu\rho M}$. This field strength can be defined such that it appears in the Bianchi identity for the field strength $\mathcal{F}^M_{\mu\nu}$. The Bianchi identity for the field strength $\mathcal{F}^M_{\mu\nu}$ is given by

$$3\mathcal{D}_{[\mu}\mathcal{F}^M_{\nu\rho]} = 10 d^{MNK}\partial_K \mathcal{H}_{\mu\nu\rho N}, \tag{5.55}$$

where $\mathcal{H}_{\mu\nu\rho N}$ has been defined to be the terms which results from a straight-forward computation of the left hand side. The exact form of $\mathcal{H}_{\mu\nu\rho N}$ can be found in [31]. The field strength $\mathcal{H}$ carries the weight of $\lambda = 2/3$ under general diffeomorphisms. The Bianchi identity for $\mathcal{H}$ then gives the relation

$$4\mathcal{D}_{[\mu}\mathcal{H}_{\nu\rho\sigma]M} = -3 d_{MPQ}\mathcal{F}^P_{[\mu\nu}\mathcal{F}^Q_{\rho\sigma]}, \tag{5.56}$$

up to terms which vanishes from action by $d^{MNK}\partial_K$. The non-triviality of the Bianchi identities follows from the existence of a Chern-Simons like topological term that then requires that the Bianchi identites for the field has to have a non-zero right-hand-side and take this form. One then find that the because the external space is 5-dimensional one can dualise the 2-form field strength into a 3-form which should appear in the Bianchi identity because of the self-duality.

5.2.1.1 Einstein-Hilbert and kinetic terms

If one wants to write an action which is fully E_6 covariant and then the ordinary derivatives have to be changed to E_6-covariant derivatives which means that the Einstein term in the theory has to be modified. This leads to the term

$$\mathcal{L}_{EH} = e\hat{\mathcal{R}} = e e_a{}^\mu e_b{}^\nu \hat{\mathcal{R}}_{\mu\nu}{}^{ab}, \tag{5.57}$$

with the inverse fünfbeins $e_a{}^\mu$ as well as their determinant e, where

$$\hat{\mathcal{R}}_{\mu\nu}{}^{ab} = \mathcal{R}_{\mu\nu}{}^{ab} + \mathcal{F}^M_{\mu\nu} e^{a\rho}\partial_M e_\rho{}^b, \tag{5.58}$$

and the Riemann tensor has been created using the usual vielbein formalism however with the replacement

$$\partial_\mu \to \mathcal{D}_\mu. \tag{5.59}$$

As was shown before the contracted term gives only contributions via their weights which then allows us to find the weight of the Riemann tensor to be $\lambda = 1/3 - 1/3 - 1/3 + 1/3 = 0$ since the fünfbein is a scalar density that carries weight $\lambda = 1/3$. Thus the fünfbein determinant has to carry weight $\lambda = 5/3$ in order for the whole term to carry weight $\lambda = 1$ as the inverse fünfbeins carry weight $\lambda = -1/3$. Thus the whole term transform into a total derivative under a gauge transformation. This shows that the whole term transform covariantly under the internal diffeomorphisms and could be included into the action.

The next terms which needs to be introduced is the kinetic terms for the scalars spanned by the quotient space and the Yang-Mills like field strength. One then has to introduce the metric over the scalar manifold, $\mathcal{M}_{MN}$, with its inverse $\mathcal{M}^{MN}$. A kinetic term one can write down is

$$\mathcal{L}_{SC} = C_{SC}\; eg^{\mu\nu}\mathcal{D}_\mu\mathcal{M}_{MN}\mathcal{D}_\nu\mathcal{M}^{MN}, \tag{5.60}$$

where C_{SC} is a coefficient that is to be determined later. One should note that the derivatives transform covariantly which was the basis for the introduction of the gauge field and as long as $\mathcal{M}_{MN}$ transforms as a covariant rank 2 tensor $\mathcal{D}_\mu\mathcal{M}_{MN}$ transforms covariantly. From this expression one can see that for the weights from the fünfbein metric and the inverse metric adds up to $\lambda = 1$ thus the weights for the metric and its inverse has to cancel each other out. Next one would like to have a Yang-Mills term which can be constructed as

$$\mathcal{L}_{YM} = C_{YM}\; e\mathcal{M}_{MN}\mathcal{F}^{\mu\nu M}\mathcal{F}^{N}_{\mu\nu}, \tag{5.61}$$

where C_{YM} is another coefficient that is left to be determined until later. As was shown earlier under gauge transformations the $\mathcal{F}^{M}_{\mu\nu}$ transforms as a contravariant vector of weight $\lambda = 1/3$ and thus the whole term transform covariantly under gauge transformations. Adding up the weights for this term one finds that it becomes

$$\frac{5}{3} + \lambda(\mathcal{M}) - \frac{2}{3} - \frac{2}{3} + \frac{1}{3} + \frac{1}{3} = 1 + \lambda(\mathcal{M}), \tag{5.62}$$

where the two $-2/3$ comes from the two exterior inverse metrics. Since this term should carry weight $\lambda = 1$ in order to transform into a total derivative the scalar metric has to carry weight $\lambda(\mathcal{M}) = 0$, and so does its inverse.

5.2.1.2 Topological term

Next one could introduce the topological terms which then is written as

$$\mathcal{L}_{top} = C_{top}\kappa \int_{\mathcal{M}_6} d_{MNK}\mathcal{F}^M \wedge \mathcal{F}^N \wedge \mathcal{F}^K - 40 d^{MNK}\mathcal{H}_M \wedge \partial_N \mathcal{H}_K, \tag{5.63}$$

which can be seen to carry the right weight and both is an exact form on a six-dimensional manifold. However in order to see that this term is gauge invariant one has to vary and find that [17]:

$$\delta\mathcal{L}_{top} = C_{top}\kappa\varepsilon^{\mu\nu\rho\sigma\tau}\left(\frac{3}{4}d_{MNK}\mathcal{F}^M_{\mu\nu}\mathcal{F}^N_{\rho\sigma}\delta A^K_\tau + 5d^{MNK}\partial_N\mathcal{H}_{\mu\nu\rho M}\Delta B_{\sigma\tau K}\right). \tag{5.64}$$

Inserting the gauge transformation one finds that this becomes

$$\begin{aligned}\delta\mathcal{L}_{top} = C_{top}\kappa\varepsilon^{\mu\nu\rho\sigma\tau}\Bigg(&\frac{3}{4}d_{MNK}\mathcal{F}^M_{\mu\nu}\mathcal{F}^N_{\rho\sigma}(\mathcal{D}_\tau\Lambda^K - 10d^{KPQ}\partial_Q\Xi_{\tau P}) \\ &+ 5d^{MNK}\partial_N\mathcal{H}_{\mu\nu\rho M}(2\mathcal{D}_{[\sigma}\Xi_{\tau]K} + d_{KPQ}\Lambda^P\mathcal{F}^Q_{\sigma\tau} + \mathcal{O}_{\sigma\tau K})\Bigg)\end{aligned} \tag{5.65}$$

Now using the Bianchi identity for the field strength on the second term in the last line one finds

$$5d^{MNK}\partial_N\mathcal{H}_{\mu\nu\rho M}d_{KPQ}\Lambda^P\mathcal{F}^Q_{\sigma\tau}=\frac{3}{2}\mathcal{D}_{[\mu}\mathcal{F}^K_{\nu\rho]}d_{KPQ}\Lambda^P\mathcal{F}^Q_{\sigma\tau},\tag{5.66}$$

which then combines with the first term into a total derivative like

$$\begin{aligned}C_{top}\kappa\varepsilon^{\mu\nu\rho\sigma\tau}(\frac{3}{4}d_{MNK}\mathcal{F}^M_{\mu\nu}\mathcal{F}^N_{\rho\sigma}\mathcal{D}_\tau\Lambda+\frac{3}{2}d_{MNK}\mathcal{D}_\mu\mathcal{F}^M_{\nu\rho}\mathcal{F}^N_{\sigma\tau}\Lambda^Q)&\\=\frac{3}{4}C_{top}\kappa\varepsilon^{\mu\nu\rho\sigma\tau}\mathcal{D}_\mu(\mathcal{F}^M_{\nu\rho}\mathcal{F}^N_{\sigma\tau}\Lambda^Q),&\end{aligned}\tag{5.67}$$

and since the ε is just a symbol one finds that it is a total derivative. One can also see that the last term can be rewritten as a total derivative. Thus one find that

$$\begin{aligned}\delta\mathcal{L}_{top}=\mathcal{D}_\mu(\frac{3}{4}C_{top}\kappa\varepsilon^{\mu\nu\rho\sigma\tau}\mathcal{F}^M_{\nu\rho}\mathcal{F}^N_{\sigma\tau}\Lambda^Q)+C_{top}\kappa\varepsilon^{\mu\nu\rho\sigma\tau}(\frac{-15}{2}d_{MNK}\mathcal{F}^M_{\mu\nu}\mathcal{F}^N_{\rho\sigma}d^{KPQ}\partial_Q\Xi_{\tau P}&\\+10d^{MNK}\partial_N\mathcal{H}_{\mu\nu\rho M}\mathcal{D}_\rho\Xi_{\tau K})+\partial_N(5C_{top}\kappa\varepsilon^{\mu\nu\rho\sigma\tau}d^{MNK}\mathcal{H}_{\mu\nu\rho M}\mathcal{O}_{\sigma\tau K}).&\end{aligned}\tag{5.68}$$

Now focus on the third term in the expression

$$\begin{aligned}10d^{MNK}\partial_N\mathcal{H}_{\mu\nu\rho M}\mathcal{D}_\sigma\Xi_{\tau K}&=10\partial_N(d^{MNK}\mathcal{H}_{\mu\nu\rho M}\mathcal{D}_\sigma\Xi_{\tau K})-10d^{MNK}\mathcal{H}_{\mu\nu\rho M}\partial_N\mathcal{D}_\sigma\Xi_{\tau K}\\&=10\partial_N(d^{MNK}\mathcal{H}_{\mu\nu\rho M}\mathcal{D}_\sigma\Xi_{\tau K})-10\mathcal{H}_{\mu\nu\rho M}\mathcal{D}_\sigma(d^{MNK}\partial_N\Xi_{\tau K})\\&=10\partial_N(d^{MNK}\mathcal{H}_{\mu\nu\rho M}\mathcal{D}_\sigma\Xi_{\tau K})-10\mathcal{D}_\sigma(d^{MNK}\mathcal{H}_{\mu\nu\rho M}\partial_N\Xi_{\tau K})+10\mathcal{D}_\sigma\mathcal{H}_{\mu\nu\rho M}d^{MNK}\partial_N\Xi_{\tau K},\end{aligned}\tag{5.69}$$

which has two total derivatives. Let's separate the non-total derivative term and since the contraction with the Levi-Civita symbol is contracted with this term one find a factor of -1 when changing position of the derivative index and since the anti-symmetry property one can use the Bianchi identity for $\mathcal{H}$ to find that

$$-10C_{top}\kappa\varepsilon^{\mu\nu\rho\sigma\tau}\mathcal{D}_\mu\mathcal{H}_{\nu\rho\sigma M}d^{MNK}\partial_N\Xi_{\tau K}=\frac{30}{4}C_{top}\kappa\varepsilon^{\mu\nu\rho\sigma\tau}d^{MNK}d_{MPQ}\mathcal{F}^P_{\mu\nu}\mathcal{F}^Q_{\rho\sigma}\partial_N\Xi_{\tau K}.\tag{5.70}$$

Now inserting this back into the expression one sees that this cancels the term with $-15/2$ coefficient perfectly and thus finds that $\mathcal{L}_{top}$ is gauge invariant up to some total derivatives.

5.2.1.3 The potential

The last type of term to be included into the action is the scalar potential as

$$\mathcal{L}_V=eV,\tag{5.71}$$

which takes the form

$$\begin{aligned}V=A\,\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_N\mathcal{M}_{KL}+B\,\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_L\mathcal{M}_NK&\\+C\,g^{-1}\partial_Mg\partial_N\mathcal{M}^{MN}+D\,\mathcal{M}^{MN}g^{-1}\partial_Mgg^{-1}\partial_Ng+E\,\mathcal{M}^{MN}\partial_Mg_{\mu\nu}\partial_Ng^{\mu\nu}.&\end{aligned}\tag{5.72}$$

Now one would like to verify that this term is gauge invariant. The scalar metric carries weight $\lambda = 0$ and transforms covariantly as a rank 2 tensor. How ever the terms which are not clear are the terms which contain a derivative with respect to the internal coordinate. Now one finds that

$$\delta_\Lambda(\partial_M \mathcal{M}^{KL}) = \partial_M \delta_\Lambda \mathcal{M}^{KL} = \partial_M(\Lambda^P \partial_P \mathcal{M}^{KL} - 12\mathbb{P}^P{}_Q{}^{(K}{}_R \mathcal{M}^{L)R} \partial_P \Lambda^Q), \quad (5.73)$$

while the Lie derivative of this term gives

$$\begin{aligned} \mathscr{L}_\Lambda \partial_M \mathcal{M}^{KL} = \Lambda^P \partial_P(\partial_M \mathcal{M}^{KL}) - 12\mathbb{P}^P{}_Q{}^{(K}{}_R \partial_M \mathcal{M}^{L)R} \partial_P \Lambda^Q \\ + 6\mathbb{P}^P{}_Q{}^R{}_M \partial_R \mathcal{M}^{KL} \partial_P \Lambda^Q + \lambda \partial_P \Lambda^P \partial_M \mathcal{M}^{KL}, \quad (5.74) \end{aligned}$$

where the weight λ has been introduced for the term $\partial\mathcal{M}$ as it does not have to carry the same as $\mathcal{M}$. From this one can see that the terms which are non covariant are

$$\begin{aligned} (\delta_\Lambda - \mathscr{L}_\Lambda)(\partial_M \mathcal{M}^{KL}) = \partial_M \Lambda^P \partial_P \mathcal{M}^{KL} - 12\mathbb{P}^P{}_Q{}^{(K}{}_R \mathcal{M}^{L)R} \partial_M \partial_P \Lambda^Q \\ - 6\mathbb{P}^P{}_Q{}^R{}_M \partial_R \mathcal{M}^{KL} \partial_P \Lambda^Q - \lambda \partial_P \Lambda^P \partial_M \mathcal{M}^{KL}. \quad (5.75) \end{aligned}$$

Now expanding the second projection operator with one finds that

$$\begin{aligned} 6\mathbb{P}^P{}_Q{}^R{}_M \partial_R \mathcal{M}^{KL} \partial_P \Lambda^Q = \frac{1}{3} \partial_M \mathcal{M}^{KL} \partial_P \Lambda^P + \partial_M \Lambda^P \partial_P \mathcal{M}^{KL} \\ - 10 d^{PRS} d_{SQM} \partial_R \mathcal{M}^{KL} \partial_P \Lambda^Q, \quad (5.76) \end{aligned}$$

however the last term is zero by the section constraint. Thus one finds that if $\lambda = -1/3$ the only non covariant term is

$$(\delta_\Lambda - \mathscr{L}_\Lambda)(\partial_M \mathcal{M}^{KL}) = -12\mathbb{P}^P{}_Q{}^{(K}{}_R \mathcal{M}^{L)R} \partial_M \partial_P \Lambda^Q. \quad (5.77)$$

From this calculation it is clear that the same mechanism works for $\partial_M \mathcal{M}_{\mathcal{KL}}$ and thus gives

$$(\delta_\Lambda - \mathscr{L}_\Lambda)(\partial_M \mathcal{M}_{KL}) = 12\mathbb{P}^P{}_Q{}^R{}_{(K} \mathcal{M}_{L)R} \partial_M \partial_P \Lambda^Q. \quad (5.78)$$

Now using theese two relations one can easily see that

$$\begin{aligned} &\delta(\mathcal{M}^{MN} \partial_M \mathcal{M}^{KL} \partial_N \mathcal{M}_{KL}) = \mathscr{L}_\Lambda(\mathcal{M}^{MN} \partial_M \mathcal{M}^{KL} \partial_N \mathcal{M}_{KL}) \\ &+ 12\mathbb{P}^P{}_Q{}^R{}_K (\partial_M \mathcal{M}^{-1} \mathcal{M})^K{}_R \mathcal{M}^{MN} \partial_N \partial_P \Lambda^Q - 12\mathbb{P}^P{}_Q{}^K{}_R (\mathcal{M}^{-1} \partial_N \mathcal{M})^R{}_K \mathcal{M}^{MN} \partial_M \partial_P \Lambda^Q \\ &= \mathscr{L}_\Lambda(\mathcal{M}^{MN} \partial_M \mathcal{M}^{KL} \partial_N \mathcal{M}_{KL}) + 24\mathbb{P}^P{}_Q{}^R{}_K (\partial_M \mathcal{M}^{-1} \mathcal{M})^K{}_R \mathcal{M}^{MN} \partial_N \partial_P \Lambda^Q. \quad (5.79) \end{aligned}$$

By using that $(\partial_M \mathcal{M}^{-1} \mathcal{M})$ is in the adjoint representation of E_6 and traceless thus the projection acts as the identity one then finds that

$$(\delta_\Lambda - \mathscr{L}_\Lambda)(\mathcal{M}^{MN} \partial_M \mathcal{M}^{KL} \partial_N \mathcal{M}_{KL}) = 24(\partial_M \mathcal{M}^{-1} \mathcal{M})^P{}_Q \mathcal{M}^{MN} \partial_N \partial_P \Lambda^Q. \quad (5.80)$$

Now by including the fünfbein determinant factor one finds that the first term leads to a total derivative while the second term has to be canceled by the other terms in the potential. Using the same method for the for the second term one has

$$\begin{aligned} (\delta_\Lambda - \mathscr{L}_\Lambda)(\mathcal{M}^{MN} \partial_M \mathcal{M}^{KL} \partial_L \mathcal{M}_{NK}) = -12\mathcal{M}^{MN} \mathbb{P}^P{}_Q{}^{(K}{}_R \mathcal{M}^{L)R} \partial_L \mathcal{M}_{NK} \partial_M \partial_P \Lambda^Q \\ + 12\mathcal{M}^{MN} \partial_M \mathcal{M}^{KL} \mathbb{P}^P{}_Q{}^R{}_{(N} \mathcal{M}_{K)R} \partial_L \partial_P \Lambda^Q, \quad (5.81) \end{aligned}$$

and by direct computations one finds that

$$12\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\mathbb{P}^P{}_Q{}^R{}_{(N}\mathcal{M}_{K)R}\partial_L\partial_P\Lambda^Q = \frac{2}{3}\partial_K\mathcal{M}^{KL}\partial_L\partial_P\Lambda^P + \partial_M\mathcal{M}^{KL}\partial_L\partial_K\Lambda^M + \mathcal{M}^{MN}(\partial_M\mathcal{M}^{-1}\mathcal{M})^L{}_Q\partial_L\partial_N\Lambda^Q - 20d^{PRS}d_{SQ(N}\mathcal{M}_{K)R}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_L\partial_P\Lambda^Q, \tag{5.82}$$

and

$$12\mathcal{M}^{MN}\mathbb{P}^P{}_Q{}^{(K}{}_R\mathcal{M}^{L)R}\partial_L\mathcal{M}_{NK}\partial_M\partial_P\Lambda^Q = -\frac{2}{3}\mathcal{M}^{ML}\partial_M\partial_P\Lambda^P - \partial_L\mathcal{M}^{MP}\partial_M\partial_P\Lambda^L - \mathcal{M}^{PL}(\partial_L\mathcal{M}^{-1}\mathcal{M})^M{}_K\partial_M\partial_P\Lambda^K + 20d_{RQS}d^{SP(K}\mathcal{M}^{L)R}(\partial_L\mathcal{M}^{-1}\mathcal{M})^M{}_K\partial_M\partial_P\Lambda^Q. \tag{5.83}$$

Thus we have

$$(\delta_\Lambda - \mathscr{L}_\Lambda)(\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_L\mathcal{M}_{NK}) = \frac{4}{3}\partial_K\mathcal{M}^{KL}\partial_L\partial_P\Lambda^P + 2\partial_M\mathcal{M}^{KL}\partial_L\partial_K\Lambda^M + 2\mathcal{M}^{MN}(\partial_M\mathcal{M}^{-1}\mathcal{M})^L{}_Q\partial_L\partial_N\Lambda^Q - 20d^{PRS}d_{SQN}\mathcal{M}^{MN}(\partial_M\mathcal{M}^{-1}\mathcal{M})^L{}_R\partial_L\partial_P\Lambda^Q. \tag{5.84}$$

Now from the form of the last term and that $\partial_M\mathcal{M}^{-1}\mathcal{M}$ is in the adjoint representation together with that d should be invariant tensor one has

$$-20d^{RS(P}(\partial_M\mathcal{M}^{-1}\mathcal{M})^{L)}{}_R\partial_L\partial_P\Lambda^Q\mathcal{M}^{MN}d_{SQN} = -10d^{RPL}(\partial_M\mathcal{M}^{-1}\mathcal{M})^S{}_R\partial_L\partial_P\Lambda^Q\mathcal{M}^{MN}d_{SQN} = 0, \tag{5.85}$$

by the section constraint. Thus the non-covariant terms are

$$(\delta_\Lambda - \mathscr{L}_\Lambda)(\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_L\mathcal{M}_{NK}) = \frac{4}{3}\partial_K\mathcal{M}^{KL}\partial_L\partial_P\Lambda^P + 2\partial_M\mathcal{M}^{KL}\partial_L\partial_K\Lambda^M + 2\mathcal{M}^{MN}(\partial_M\mathcal{M}^{-1}\mathcal{M})^L{}_Q\partial_L\partial_N\Lambda^Q. \tag{5.86}$$

From this one finds that in order for the potential to be gauge invariant one must have $A = -F/24$ and $B = F/2$. Next thing to consider is the terms which contains the exterior metric $g_{\mu\nu}$, since they transforms as densities of weight 2/3 under internal diffeomorphisms, similarly g carries weight $\lambda = 10/3$, as a result of the weights of the fünfbein and its determinant.

$$\delta_\Lambda(g^{-1}\partial_M g) = \Lambda^P\partial_P g^{-1}\partial_M g - \frac{10}{3}\partial_P\Lambda^P g^{-1}\partial_M g + g^{-1}\partial_M(\Lambda^P\partial_P g + \frac{10}{3}\partial_P\Lambda^P g) = \Lambda^P\partial_P(g^{-1}\partial_M g) + g^{-1}\partial_M\Lambda^P\partial_P g + \frac{10}{3}\partial_M\partial_P\Lambda^P, \tag{5.87}$$

while

$$\mathscr{L}_\Lambda(g^{-1}\partial_M g) = \Lambda^P\partial_P(g^{-1}\partial_M g) + \lambda\partial_P\Lambda^P g^{-1}\partial_M g + 6\mathbb{P}^P{}_Q{}^R{}_M g^{-1}\partial_R g\partial_P\Lambda^Q = \Lambda^P\partial_P(g^{-1}\partial_M g) + \lambda\partial_P\Lambda^P g^{-1}\partial_M g + \frac{1}{3}\partial_P\Lambda^P g^{-1}\partial_M g + g^{-1}\partial_Q g\partial_M\Lambda^Q - 10d^{PRS}d_{SQM}g^{-1}\partial_R g\partial_P\Lambda^Q, \tag{5.88}$$

where the last term is zero because of the section constraint. Thus one has

$$(\delta_\Lambda - \mathscr{L}_\Lambda)(g^{-1}\partial_M g) = \frac{10}{3}\partial_M\partial_P\Lambda^P, \tag{5.89}$$

as long as $g^{-1}\partial_M g$ carries weight $\lambda = 1/3$. From this calculation one sees that the same principle works for other scalar densities however with some small alterations of the final result and then finds

$$\begin{aligned}(\delta_\Lambda - \mathscr{L}_\Lambda)(\partial_M g_{\mu\nu}) =& \frac{2}{3}\partial_M\partial_P\Lambda^P g_{\mu\nu} &(5.90)\\ (\delta_\Lambda - \mathscr{L}_\Lambda)(\partial_M g^{\mu\nu}) =& -\frac{2}{3}\partial_M\partial_P\Lambda^P g^{\mu\nu} &(5.91)\end{aligned}$$

where the weight of the derivative has to be $-1/3$ together such that the derivative $\partial_M(S)$ has weight $\lambda = -1/3 + \lambda(S)$. Thus one finds

$$\begin{aligned}(\delta_\Lambda - \mathscr{L}_\Lambda)(g^{-1}\partial_M g\partial_N\mathcal{M}^{MN}) &= \frac{10}{3}\partial_M\partial_P\Lambda^P\partial_N\mathcal{M}^{MN} - 12g^{-1}\partial_M g\mathbb{P}^P{}_Q{}^{(M}{}_R\mathcal{M}^{N)R}\partial_N\partial_P\Lambda^Q\\ &= \frac{10}{3}\partial_M\partial_P\Lambda^P\partial_N\mathcal{M}^{MN} - 2g^{-1}\partial_M g(\frac{1}{3}\mathcal{M}^{MN}\partial_N\partial_P\Lambda^P + \partial_N\partial_P\Lambda^{(M}\mathcal{M}^{N)P}\\ &\quad - 10d_{QRS}d^{SP(M}\mathcal{M}^{N)R}\partial_N\partial_P\Lambda^Q), \end{aligned}\tag{5.92}$$

with the last term dropping out because of the section constraint. Thus one has

$$\begin{aligned}(\delta_\Lambda - \mathscr{L}_\Lambda)(g^{-1}\partial_M g\partial_N\mathcal{M}^{MN}) = &\frac{10}{3}\partial_M\partial_P\Lambda^P\partial_N\mathcal{M}^{MN} - \frac{5}{3}g^{-1}\partial_M g\mathcal{M}^{MN}\partial_N\partial_P\Lambda^P\\ &- g^{-1}\partial_M g\partial_N\partial_P\Lambda^M\mathcal{M}^{NP}.\end{aligned}\tag{5.93}$$

From straightforward computation one finds

$$(\delta_\Lambda - \mathscr{L}_\Lambda)(\mathcal{M}^{MN}g^{-1}\partial_M g g^{-1}\partial_N g) = \frac{20}{3}\mathcal{M}^{MN}g^{-1}\partial_M g\partial_N\partial_P\Lambda^P, \tag{5.94}$$

and finally

$$(\delta_\Lambda - \mathscr{L}_\Lambda)(\mathcal{M}^{MN}\partial_M g^{\mu\nu}\partial_N g_{\mu\nu}) = -\frac{4}{3}\mathcal{M}^{MN}\partial_M\partial_P\Lambda^P g^{\mu\nu}\partial_N g_{\mu\nu}. \tag{5.95}$$

However one knows that

$$\partial_M g = g g^{\mu\nu}\partial_M g_{\mu\nu}, \tag{5.96}$$

one finds that this becomes

$$(\delta_\Lambda - \mathscr{L}_\Lambda)(\mathcal{M}^{MN}\partial_M g^{\mu\nu}\partial_N g_{\mu\nu}) = -\frac{4}{3}\mathcal{M}^{MN}\partial_M\partial_P\Lambda^P g^{-1}\partial_N g. \tag{5.97}$$

With these results one finds that all the terms now combines to

$$
\begin{aligned}
(\delta_\Lambda - \mathscr{L}_\Lambda)\mathcal{L}_V &= \frac{2F}{3} e\partial_K \mathcal{M}^{KL}\partial_L\partial_P\Lambda^P + F\, e\partial_M\mathcal{M}^{KL}\partial_L\partial_K\Lambda^M \\
&+ \frac{10C}{3} e\partial_N\mathcal{M}^{MN}\partial_M\partial_P\Lambda^P - \frac{5C}{3} e\mathcal{M}^{MN}g^{-1}\partial_M g\partial_N\partial_P\Lambda^P - C\, e\mathcal{M}^{NP}g^{-1}\partial_M g\partial_N\partial_P\Lambda^M \\
&+ \frac{20D}{3} e\mathcal{M}^{MN}g^{-1}\partial_M g\partial_N\partial_P\Lambda^P - \frac{4E}{3} e\mathcal{M}^{MN}g^{-1}\partial_N g\partial_M\partial_P\Lambda^P \\
&= \left(\frac{2F}{3} + \frac{10C}{3}\right) e\partial_N\mathcal{M}^{MN}\partial_M\partial_P\Lambda^P + F\, e\partial_M\mathcal{M}^{KL}\partial_L\partial_K\Lambda^M \\
&- C\, e\mathcal{M}^{KL}g^{-1}\partial_M g\partial_K\partial_L\Lambda^M + \left(\frac{20D}{3} - \frac{5C}{3} - \frac{4E}{3}\right) e\mathcal{M}^{MN}g^{-1}\partial_M g\partial_N\partial_P\Lambda^P,
\end{aligned}
\tag{5.98}
$$

which can in turn be simplified with the use of $e^2 = g$ to find that

$$
\begin{aligned}
(\delta_\Lambda - \mathscr{L}_\Lambda)\mathcal{L}_V &= \left(\frac{2F}{3} + \frac{10C}{3}\right) e\partial_N\mathcal{M}^{MN}\partial_M\partial_P\Lambda^P + F\, e\partial_M\mathcal{M}^{KL}\partial_L\partial_K\Lambda^M \\
&- 2C\mathcal{M}^{KL}\partial_M e\partial_K\partial_L\Lambda^M + 2\left(\frac{20D}{3} - \frac{5C}{3} - \frac{4E}{3}\right)\mathcal{M}^{MN}\partial_M e\partial_N\partial_P\Lambda^P.
\end{aligned}
\tag{5.99}
$$

By rewriting the last two terms as a total derivatives one finds that

$$
\begin{aligned}
(\delta_\Lambda - \mathscr{L}_\Lambda)\mathcal{L}_V &= \left(\frac{2F}{3} + \frac{10C}{3}\right) e\partial_N\mathcal{M}^{MN}\partial_M\partial_P\Lambda^P + F\, e\partial_M\mathcal{M}^{KL}\partial_L\partial_K\Lambda^M \\
&- 2C\partial_M(e\mathcal{M}^{KL}\partial_K\partial_L\Lambda^M) + 2Ce\partial_M\mathcal{M}^{KL}\partial_K\partial_L\Lambda^M + 2C\mathcal{M}^{KL}\partial_M\partial_K\partial_L\Lambda^M \\
&+ 2\left(\frac{20D}{3} - \frac{5C}{3} - \frac{4E}{3}\right)(\partial_M(e\mathcal{M}^{MN}\partial_N\partial_P\Lambda^P) - e\partial_M\mathcal{M}^{MN}\partial_N\partial_P\Lambda^P \\
&- e\mathcal{M}^{MN}\partial_M\partial_N\partial_P\Lambda^P),
\end{aligned}
\tag{5.100}
$$

from which we see that

$$8C = 20D - 4E, \tag{5.101}$$

in order for the $\partial^3\Lambda$ terms to cancel. Next requirement is that the two terms in the first line should be canceled by the non-total derivative terms in the last two lines, this gives the two equations

$$F + 2C = 0 \tag{5.102}$$

$$F + 5C = 20D - 5C + 4E, \tag{5.103}$$

thus one has $C = -F/2$ and $5D - E = -F$. To find the same potential as in [17] one chooses $D = -F/4$. Thus the only non-covariant terms left are total derivatives which then doesn't give any contributions. Thus we have found that

$$
\begin{aligned}
\mathcal{L}_V = F\, e(&-\frac{1}{24}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_N\mathcal{M}_{KL} + \frac{1}{2}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_L\mathcal{M}_{NK} \\
&- \frac{1}{2}g^{-1}\partial_M g\partial_N\mathcal{M}^{MN} - \frac{1}{4}\mathcal{M}^{MN}g^{-1}\partial_M g g^{-1}\partial_N g - \frac{1}{4}\mathcal{M}^{MN}\partial_M g_{\mu\nu}\partial_N g^{\mu\nu}).
\end{aligned}
\tag{5.104}
$$

As was seen has all the non-covariant terms canceled and this expression can easily be seen to have the correct weight as in all cases $\lambda(\partial_M) = -1/3$ and the total weight of each term is $\lambda = 1$. Thus this potential transforms into a total derivative

5.2.2 External diffeomorphisms

The action found in previous section has the has the form

$$\mathcal{L} = \mathcal{L}_{EH} + \mathcal{L}_{SC} + \mathcal{L}_{YM} + \mathcal{L}_{top} + \mathcal{L}_{V}, \tag{5.105}$$

where in every term except the generalised Einstein-Hilbert term there exist a coefficient which has not been determined. When considering the external diffeomorphism one will see that these coefficients are fixed since external diffeomorphism mixes the contribution from the different terms together. The parameter of these transformations is allowed to be dependent of the internal coordinates as well as the external. One has therefore $\xi^\mu(x, Y) = \xi^\mu$, where Y^M is the internal coordinates. As is expected these diffeomorphisms takes the form

$$\delta e_\mu{}^a = \xi^\nu \mathcal{D}_\nu e_\mu{}^a + \mathcal{D}_\mu \xi^\nu e_\nu{}^a, \tag{5.106}$$

$$\delta \mathcal{M}^{MN} = \xi^\mu \mathcal{D}_\mu \mathcal{M}^{MN}, \tag{5.107}$$

$$\delta A_\mu^M = \xi^\nu \mathcal{F}_{\nu\mu}^M + \mathcal{M}^{MN} g_{\mu\nu} \partial_N \xi^\nu, \tag{5.108}$$

$$\Delta B_{\mu\nu M} = \frac{1}{16\kappa} \xi^\rho e \varepsilon_{\mu\nu\rho\sigma\tau} \mathcal{F}^{\sigma\tau N} \mathcal{M}_{MN}. \tag{5.109}$$

Inserting the diffeomorphisms and focusing on the variations with respect of δA_μ^M and $\Delta B_{\mu\nu M}$ of the terms in $\mathcal{L}_{YM}$ to find

$$\delta \mathcal{L}_{YM} = 2C_{YM} e \mathcal{M}_{MN} (2\mathcal{D}_{[\mu} \delta A_{\nu]}^M + 10 d^{MKR} \partial_K \Delta B_{\mu\nu R}) \mathcal{F}^{\mu\nu N} + \mathcal{O}(e_\mu{}^a) + \mathcal{O}(\mathcal{M}_{MN}). \tag{5.110}$$

Inserting the expression for the transformations into the above expression, and disregard for the moment the terms from the variation with respect to fünfbein and the scalar metric, the resulting expression becomes

$$\delta \mathcal{L}_{YM} = 4C_{YM} e \mathcal{M}_{MN} \mathcal{F}^{\mu\nu N} \mathcal{D}_\mu (\xi^\rho \mathcal{F}_{\rho\nu}^M + \mathcal{M}^{MK} g_{\nu\rho} \partial_K \xi^\rho) + C_{YM} 20 d^{MKR} e \mathcal{M}_{MN} \mathcal{F}^{\mu\nu N} \frac{1}{16\kappa} \partial_K (\xi^\rho e \varepsilon_{\mu\nu\rho\sigma\tau} \mathcal{F}^{\sigma\tau P} \mathcal{M}_{RP}). \tag{5.111}$$

As was discussed previously the variation of the topological term was

$$\delta \mathcal{L}_{top} = C_{top} \kappa \varepsilon^{\mu\nu\rho\sigma\tau} (\frac{3}{4} d_{MNK} \mathcal{F}_{\mu\nu}^M \mathcal{F}_{\rho\sigma}^N \delta A_\tau^K + 5 d^{MNK} \partial_N \mathcal{H}_{\mu\nu\rho M} \Delta B_{\sigma\tau K}), \tag{5.112}$$

which then becomes

$$\delta \mathcal{L}_{top} = C_{top} \kappa \varepsilon^{\mu\nu\rho\sigma\tau} (\frac{3}{4} d_{MNK} \mathcal{F}_{\mu\nu}^M \mathcal{F}_{\rho\sigma}^N (\xi^\alpha \mathcal{F}_{\alpha\tau}^K + \mathcal{M}^{KR} g_{\tau\alpha} \partial_R \xi^\alpha) + 5 d^{MNK} \partial_N \mathcal{H}_{\mu\nu\rho M} \frac{1}{16\kappa} \xi^\alpha e \varepsilon_{\sigma\tau\alpha\beta\gamma} \mathcal{F}^{\beta\gamma P} \mathcal{M}_{KP}). \tag{5.113}$$

From $\delta\mathcal{L}_{YM} + \delta\mathcal{L}_{top}$ separate the terms with $\mathcal{F}^2$ which appears without a covariant derivative and demanding that they cancel out to

$$\begin{aligned}
C_{top}\frac{3\kappa}{4}\varepsilon^{\mu\nu\rho\sigma\tau}d_{MNK}\mathcal{F}^M_{\mu\nu}\mathcal{F}^N_{\rho\sigma}\mathcal{M}^{KR}g_{\alpha\tau}\partial_R\xi^\alpha \\
+ C_{YM}\frac{5}{4\kappa}d^{MKR}e\mathcal{M}_{MN}\mathcal{F}^{\mu\nu N}\partial_K(\xi^\rho e\varepsilon_{\mu\nu\rho\sigma\tau}\mathcal{F}^{\sigma\tau P}\mathcal{M}_{RP}) \\
= C_{top}\frac{3\kappa}{4}\varepsilon^{\mu\nu\rho\sigma\tau}d_{MNK}\mathcal{F}^M_{\mu\nu}\mathcal{F}^N_{\rho\sigma}\mathcal{M}^{KR}g_{\tau\alpha}\partial_R\xi^\alpha \\
+ C_{YM}\frac{5}{8\kappa}\partial_K(e^2\varepsilon_{\mu\nu\rho\sigma\tau}d^{MKR}\mathcal{M}_{MN}\mathcal{M}_{RP}\mathcal{F}^{\mu\nu N}\mathcal{F}^{\sigma\tau P}\xi^\rho) \\
- C_{YM}\frac{5}{8\kappa}e^2\varepsilon_{\mu\nu\rho\sigma\tau}d^{MKR}\mathcal{M}_{MN}\mathcal{M}_{RP}\mathcal{F}^{\mu\nu N}\mathcal{F}^{\sigma\tau P}\partial_K\xi^\rho,
\end{aligned} \tag{5.114}$$

which becomes after ignoring the total derivative

$$C_{top}\frac{3\kappa}{4}\varepsilon^{\mu\nu\rho\sigma\tau}d_{MNK}\mathcal{F}^M_{\mu\nu}\mathcal{F}^N_{\rho\sigma}\mathcal{M}^{KR}g_{\mu\alpha}\partial_R\xi^\alpha + C_{YM}\frac{5}{8\kappa}\varepsilon^{\lambda\mu\nu\sigma\tau}\mathcal{M}^{KQ}d_{NPQ}\mathcal{F}^N_{\mu\nu}\mathcal{F}^P_{\sigma\tau}g_{\lambda\rho}\partial_K\xi^\rho, \tag{5.115}$$

where one has used that $\mathcal{M}$ is E_6 valued so the invariance of the d-symbol requires that $d^{MNK}\mathcal{M}_{NP}\mathcal{M}_{KQ} = \mathcal{M}^{ML}d_{LPQ}$ and that raising the indices results in inverse fünfbein determinant squared. Thus by requiring that this vanishes one finds

$$\kappa^2 = \frac{5\cdot 4}{3\cdot 8}\left|\frac{C_{YM}}{C_{top}}\right|, \tag{5.116}$$

as well as that the two coefficient carries opposite signs. Next considering the term with $\mathcal{F}^2$ and a covariant derivative. Start by rewriting the term from $\delta\mathcal{L}_{YM}$ as

$$\begin{aligned}
4C_{YM}e\mathcal{M}_{MN}\mathcal{F}^{\mu\nu N}\mathcal{D}_\mu(\xi^\rho\mathcal{F}^M_{\rho\nu}) = 4C_{YM}e\mathcal{M}_{MN}\mathcal{F}^{\mu\nu N}\mathcal{F}^M_{\rho\nu}\mathcal{D}_\mu\xi^\rho \\
- 4C_{YM}e\mathcal{M}_{MN}\mathcal{F}^{\mu\nu N}\xi^\rho\mathcal{D}_\mu\mathcal{F}^M_{\nu\rho}.
\end{aligned} \tag{5.117}$$

By using that

$$3\mathcal{D}_{[\mu}\mathcal{F}^M_{\nu\rho]} = \mathcal{D}_\mu\mathcal{F}^M_{\nu\rho} + \mathcal{D}_\nu\mathcal{F}^M_{\rho\mu} + \mathcal{D}_\rho\mathcal{F}^M_{\mu\nu}, \tag{5.118}$$

and noting that the first and second term on the r.h.s. becomes the same and equal to that term above when they are contracted thus one finds

$$\begin{aligned}
4C_{YM}e\mathcal{M}_{MN}\mathcal{F}^{\mu\nu N}\xi^\rho\mathcal{D}_\mu\mathcal{F}^M_{\nu\rho} = \frac{3}{2}4C_{YM}e\mathcal{M}_{MN}\mathcal{F}^{\mu\nu N}\xi^\rho\mathcal{D}_{[\mu}\mathcal{F}^M_{\nu\rho]} \\
- 2C_{YM}e\mathcal{M}_{MN}\mathcal{F}^{\mu\nu N}\xi^\rho\mathcal{D}_\rho\mathcal{F}^M_{\mu\nu} \\
= 20C_{YM}e\mathcal{M}_{MN}\mathcal{F}^{\mu\nu N}\xi^\rho d^{MPQ}\partial_P\mathcal{H}_{\mu\nu\rho Q} - 2C_{YM}e\mathcal{M}_{MN}\mathcal{F}^{\mu\nu N}\xi^\rho\mathcal{D}_\rho\mathcal{F}^M_{\mu\nu}.
\end{aligned} \tag{5.119}$$

Combining all these steps one finds that

$$\begin{aligned}
4C_{YM}e\mathcal{M}_{MN}\mathcal{F}^{\mu\nu N}\mathcal{D}_\mu(\xi^\rho\mathcal{F}^M_{\rho\nu}) = 4C_{YM}e\mathcal{M}_{MN}\mathcal{F}^{\mu\nu N}\mathcal{F}^M_{\rho\nu}\mathcal{D}_\mu\xi^\rho \\
- 20C_{YM}ed^{MPQ}\mathcal{M}_{MN}\mathcal{F}^{\mu\nu N}\xi^\rho\partial_P\mathcal{H}_{\mu\nu\rho Q} + 2C_{YM}e\mathcal{M}_{MN}\mathcal{F}^{\mu\nu N}\xi^\rho\mathcal{D}_\rho\mathcal{F}^M_{\mu\nu}.
\end{aligned} \tag{5.120}$$

From the topological term there exist a similar term of the form $\mathcal{HF}$ which has to cancel the one from the Yang-Mills like term. Therefore let's separate that term and study it

$$\begin{aligned}\frac{5}{16}C_{top}\varepsilon^{\mu\nu\rho\sigma\tau}d^{MNK}\partial_N\mathcal{H}_{\mu\nu\rho M}\xi^\alpha e\varepsilon_{\sigma\tau\alpha\beta\gamma}\mathcal{F}^{\beta\gamma P}\mathcal{M}_{KP} \\ = \frac{-5\cdot 2}{16}C_{top}e\mathcal{M}_{KP}d^{MNK}\partial_N\mathcal{H}_{\mu\nu\rho}\delta^{\mu\nu\rho}_{\alpha\beta\gamma}\xi^\alpha\mathcal{F}^{\beta\gamma P} \\ = \frac{-15}{4}C_{top}e\mathcal{M}_{KP}d^{MNK}\partial_N\mathcal{H}_{\mu\nu\rho}\xi^\mu\mathcal{F}^{\nu\rho P}.\end{aligned} \quad (5.121)$$

Thus we find that the constants have to be related as

$$C_{top} = -\frac{16}{3}C_{YM} \quad (5.122)$$

in order for the two terms to cancel out, this gives that

$$\kappa^2 = \frac{5\cdot 4}{3\cdot 8}\frac{3}{16} = \frac{5}{32}. \quad (5.123)$$

The remaining terms which are of the form $\mathcal{F}^3$ should cancel as they appear in the action as an exact form in the action, thus should be a total derivative and dissapear. This has not been verified explicitly in this work.

Now that all the terms from the topological term has been accounted for one is left with the remaining terms

$$\begin{aligned}\delta\mathcal{L}_{YM} + \delta\mathcal{L}_{top} = 4C_{YM}e\mathcal{M}_{MN}\mathcal{F}^{\mu\nu N}\mathcal{F}^M_{\rho\nu}\mathcal{D}_\mu\xi^\rho + 2C_{YM}e\mathcal{M}_{MN}\mathcal{F}^{\mu\nu N}\xi^\rho\mathcal{D}_\rho\mathcal{F}^M_{\mu\nu} \\ + 4C_{YM}e\mathcal{M}_{MN}\mathcal{F}^{\mu\nu N}\mathcal{D}_\mu(\mathcal{M}^{MK}g_{\nu\rho}\partial_K\xi^\rho).\end{aligned} \quad (5.124)$$

Before considering the other parts of the action lets return to the variation with respect to the scalar metric of the Yang-Mills like term. This gives the contribution

$$C_{YM}e\xi^\rho\mathcal{D}_\rho\mathcal{M}_{MN}\mathcal{F}^{\mu\nu M}\mathcal{F}^N_{\mu\nu}. \quad (5.125)$$

which by partial integration one finds

$$-2C_{YM}e\mathcal{M}_{MN}\xi^\rho\mathcal{F}^{\mu\nu M}\mathcal{D}_\rho\mathcal{F}^N_{\mu\nu} - C_{YM}e\mathcal{M}_{MN}\mathcal{F}^N_{\mu\nu}\mathcal{F}^{\mu\nu M}\mathcal{D}_\rho\xi^\rho + \mathcal{O}(e_\mu{}^a), \quad (5.126)$$

from which one finds that the first term cancel the second one in the result above. Thus the total result from these two terms are given by

$$\begin{aligned}\delta\mathcal{L}_{YM} + \delta\mathcal{L}_{top} = 4C_{YM}e\mathcal{M}_{MN}\mathcal{F}^{\mu\nu N}\mathcal{F}^M_{\rho\nu}\mathcal{D}_\mu\xi^\rho - C_{YM}e\mathcal{M}_{MN}\mathcal{F}^N_{\mu\nu}\mathcal{F}^{\mu\nu M}\mathcal{D}_\rho\xi^\rho \\ + 4C_{YM}e\mathcal{F}^N_{\mu\rho}(\mathcal{D}^\mu\mathcal{M}^{-1}\mathcal{M})^K{}_N\partial_K\xi^\rho + 4C_{YM}e\mathcal{F}^{\mu\nu N}\mathcal{D}_\mu(g_{\nu\rho}\partial_N\xi^\rho) + \mathcal{O}(e_\mu{}^a).\end{aligned} \quad (5.127)$$

Where the next to last term looks like something that will appear in the variation of $\mathcal{L}_{SC}$ therefore it would be enlightening to study that term next. When the addition of the terms found from the variations of the two inverse metrics and fünfbein determinants the first line cancels and the only remaining term is a total derivative which is expected.

Scalar kinetic term contribution

When considering the scalar kinetic term one has to first know how the covariant derivative will change during an external diffeomorphism hence we start with studying the effect of such a diffeomorphism on the term

$$\delta(\mathcal{D}_\mu \mathcal{M}_{MN}). \tag{5.128}$$

Because the covariant derivative contain a Lie derivative one finds two terms

$$\begin{aligned}\delta(\mathcal{D}_\mu \mathcal{M}_{MN}) = -\mathscr{L}_{\delta A_\mu}\mathcal{M}_{MN} + \mathcal{D}_\mu(\xi^\nu \mathcal{D}_\nu \mathcal{M}_{MN}) = -\mathscr{L}_{\delta A_\mu}\mathcal{M}_{MN} \\ + \mathcal{D}_\mu \xi^\nu \mathcal{D}_\nu \mathcal{M}_{MN} + \xi^\nu \mathcal{D}_\mu \mathcal{D}_\nu \mathcal{M}_{MN} = -\mathscr{L}_{\delta A_\mu}\mathcal{M}_{MN} - \xi^\nu \mathscr{L}_{\mathcal{F}_{\mu\nu}}\mathcal{M}_{MN} \\ + \xi^\nu \mathcal{D}_\nu(\mathcal{D}_\mu \mathcal{M}_{MN}) + \mathcal{D}_\mu \xi^\nu \mathcal{D}_\nu \mathcal{M}_{MN}.\end{aligned} \tag{5.129}$$

From this one can see on the last line that this generates a regular exterior Lie derivative $\mathcal{L}_\xi$ as well as some extra terms. By inserting the expression for δA_μ^M one finds that this is

$$\begin{aligned}\delta(\mathcal{D}_\mu \mathcal{M}_{MN}) = \mathcal{L}_\xi(\mathcal{D}_\mu \mathcal{M}_{MN}) - \mathscr{L}_{\xi^\nu \mathcal{F}_{\nu\mu}}\mathcal{M}_{MN} - \xi^\nu \mathscr{L}_{\mathcal{F}_{\mu\nu}}\mathcal{M}_{MN} \\ - \mathscr{L}_{\mathcal{M}^{\cdot K} g_{\mu\nu}\partial_K \xi^\nu}\mathcal{M}_{MN}.\end{aligned} \tag{5.130}$$

From the calculation up until this point the same calculation works for $\delta(\mathcal{D}_\mu \mathcal{M}^{MN})$ and one finds the same results. By a straight-forward computation of the middle two terms one finds that they results in

$$\mathscr{L}_{\xi^\nu \mathcal{F}_{\mu\nu}}\mathcal{M}_{MN} - \xi^\nu \mathscr{L}_{\mathcal{F}_{\mu\nu}}\mathcal{M}_{MN} = 12\mathbb{P}^P{}_Q{}^K{}_{(M}\mathcal{M}_{N)K}\mathcal{F}^Q_{\mu\nu}\partial_P \xi^\nu, \tag{5.131}$$

and

$$\mathscr{L}_{\xi^\nu \mathcal{F}_{\mu\nu}}\mathcal{M}^{MN} - \xi^\nu \mathscr{L}_{\mathcal{F}_{\mu\nu}}\mathcal{M}^{MN} = -12\mathbb{P}^P{}_Q{}^{(M}{}_K\mathcal{M}^{N)K}\mathcal{F}^Q_{\mu\nu}\partial_P \xi^\nu \tag{5.132}$$

for the cause with $\mathcal{D}_\mu \mathcal{M}^{MN}$. Thus when considering the scalar kinetic term one finds

$$\delta\mathcal{L}_{SC} = C_{SC} e g^{\mu\nu}\delta(\mathcal{D}_\mu \mathcal{M}^{MN})\mathcal{D}_\nu \mathcal{M}_{MN} + C_{SC} e g^{\mu\nu}\mathcal{D}_\mu \mathcal{M}^{MN}\delta(\mathcal{D}_\nu \mathcal{M}_{MN}) + \mathcal{O}(e_\mu{}^a), \tag{5.133}$$

which after ignoring the contributions from the fünfbein for the moment becomes

$$\begin{aligned}\delta\mathcal{L}_{SC} = C_{SC} e g^{\mu\nu}(\mathcal{L}_\xi(\mathcal{D}_\mu \mathcal{M}^{MN}) - 12\mathbb{P}^P{}_Q{}^{(M}{}_K\mathcal{M}^{N)K}\mathcal{F}^Q_{\mu\rho}\partial_P \xi^\rho \\ - \mathscr{L}_{\mathcal{M}^{\cdot K} g_{\mu\rho}\partial_K \xi^\rho}\mathcal{M}^{MN})\mathcal{D}_\nu \mathcal{M}_{MN} \\ + C_{SC} e g^{\mu\nu}\mathcal{D}_\mu \mathcal{M}^{MN}(\mathcal{L}_\xi(\mathcal{D}_\nu \mathcal{M}_{MN}) + 12\mathbb{P}^P{}_Q{}^K{}_{(M}\mathcal{M}_{N)K}\mathcal{F}^Q_{\nu\rho}\partial_P \xi^\rho - \mathscr{L}_{\mathcal{M}^{\cdot K} g_{\nu\rho}\partial_K \xi^\rho}\mathcal{M}_{MN}) \\ = C_{SC} e g^{\mu\nu}\mathcal{L}_\xi(\mathcal{D}_\mu \mathcal{M}^{MN}\mathcal{D}_\nu \mathcal{M}_{MN}) - 12 C_{SC} e g^{\mu\nu}\mathbb{P}^P{}_Q{}^M{}_K\mathcal{F}^Q_{\mu\rho}\partial_P \xi^\rho(\mathcal{M}^{-1}\mathcal{D}_\nu \mathcal{M})^K{}_M \\ + 12 C_{SC} e g^{\mu\nu}\mathbb{P}^P{}_Q{}^K{}_M\mathcal{F}^Q_{\nu\rho}\partial_P \xi^\rho(\mathcal{D}_\mu \mathcal{M}^{-1}\mathcal{M})^M{}_K - C_{SC} e g^{\mu\nu}\mathcal{D}_\nu \mathcal{M}_{MN}\mathscr{L}_{\mathcal{M}^{\cdot K} g_{\mu\rho}\partial_K \xi^\rho}\mathcal{M}^{MN} \\ - C_{SC} e g^{\mu\nu}\mathcal{D}_\mu \mathcal{M}^{MN}\mathscr{L}_{\mathcal{M}^{\cdot K} g_{\nu\rho}\partial_K \xi^\rho}\mathcal{M}_{MN}.\end{aligned} \tag{5.134}$$

Now using that $(\mathcal{D}_\mu \mathcal{M}^{-1}\mathcal{M})$ lies in the adjoint representation and is traceless one finds that this becomes

$$\begin{aligned}\delta\mathcal{L}_{SC} = C_{SC} e g^{\mu\nu}\mathcal{L}_\xi(\mathcal{D}_\mu \mathcal{M}^{MN}\mathcal{D}_\nu \mathcal{M}_{MN}) + 24 C_{SC} e g^{\mu\nu}\mathcal{F}^Q_{\mu\rho}\partial_P \xi^\rho(\mathcal{D}_\nu \mathcal{M}^{-1}\mathcal{M})^P{}_Q \\ - C_{SC} e g^{\mu\nu}\mathcal{D}_\nu \mathcal{M}_{MN}\mathscr{L}_{\mathcal{M}^{\cdot K} g_{\mu\rho}\partial_K \xi^\rho}\mathcal{M}^{MN} - C_{SC} e g^{\mu\nu}\mathcal{D}_\mu \mathcal{M}^{MN}\mathscr{L}_{\mathcal{M}^{\cdot K} g_{\nu\rho}\partial_K \xi^\rho}\mathcal{M}_{MN}.\end{aligned} \tag{5.135}$$

In order to proceed from this stage one has to expand the generalised Lie derivatives remaining therefore we begin with the first of the two

$$\mathscr{L}_{\mathcal{M}^{\cdot K}g_{\mu\rho}\partial_K\xi^\rho}\mathcal{M}^{MN} = \mathcal{M}^{LK}g_{\mu\rho}\partial_K\xi^\rho\partial_L\mathcal{M}^{MN} - 12\mathbb{P}^P{}_Q{}^{(M}{}_R\mathcal{M}^{N)R}\partial_P\mathcal{M}^{QK}g_{\mu\rho}\partial_K\xi^\rho, \tag{5.136}$$

while the second term becomes

$$\mathscr{L}_{\mathcal{M}^{\cdot K}g_{\nu\rho}\partial_K\xi^\rho}\mathcal{M}_{MN} = \mathcal{M}^{LK}g_{\nu\rho}\partial_K\xi^\rho\partial_L\mathcal{M}_{MN} + 12\mathbb{P}^P{}_Q{}^{(M}{}_R\mathcal{M}^{N)R}\partial_P\mathcal{M}^{QK}g_{\nu\rho}\partial_K\xi^\rho. \tag{5.137}$$

From this one finds that

$$\begin{aligned}
&C_{SC}eg^{\mu\nu}\mathcal{D}_\nu\mathcal{M}_{MN}\mathscr{L}_{\mathcal{M}^{\cdot K}g_{\mu\rho}\partial_K\xi^\rho}\mathcal{M}^{MN} + C_{SC}eg^{\mu\nu}\mathcal{D}_\mu\mathcal{M}^{MN}\mathscr{L}_{\mathcal{M}^{\cdot K}g_{\nu\rho}\partial_K\xi^\rho}\mathcal{M}_{MN}\\
&= C_{SC}eg^{\mu\nu}\mathcal{D}_\nu\mathcal{M}_{MN}(\mathcal{M}^{LK}g_{\mu\rho}\partial_K\xi^\rho\partial_L\mathcal{M}^{MN} - 12\mathbb{P}^P{}_Q{}^{(M}{}_R\mathcal{M}^{N)R}\partial_P(\mathcal{M}^{QK}g_{\mu\rho}\partial_K\xi^\rho))\\
&+ C_{SC}eg^{\mu\nu}\mathcal{D}_\mu\mathcal{M}^{MN}(\mathcal{M}^{LK}g_{\nu\rho}\partial_K\xi^\rho\partial_L\mathcal{M}_{MN} + 12\mathbb{P}^P{}_Q{}^R{}_{(M}\mathcal{M}_{N)R}\partial_P(\mathcal{M}^{QK}g_{\nu\rho}\partial_K\xi^\rho))\\
&\quad = C_{SC}e\mathcal{D}_\mu\mathcal{M}_{MN}\mathcal{M}^{LK}\partial_K\xi^\mu\partial_L\mathcal{M}^{MN} + C_{SC}e\mathcal{D}_\mu\mathcal{M}^{MN}\mathcal{M}^{LK}\partial_K\xi^\mu\partial_L\mathcal{M}_{MN}\\
&\qquad - 12C_{SC}e(\mathcal{M}^{-1}\mathcal{D}^\mu\mathcal{M})^R{}_M\mathbb{P}^P{}_Q{}^M{}_R\partial_P(\mathcal{M}^{QK}g_{\mu\rho}\partial_K\xi^\rho)\\
&\qquad + 12C_{SC}e(\mathcal{D}^\mu\mathcal{M}^{-1}\mathcal{M})^M{}_R\mathbb{P}^P{}_Q{}^R{}_M\partial_P(\mathcal{M}^{QK}g_{\mu\rho}\partial_K\xi^\rho)\\
&\quad = C_{SC}e\mathcal{D}_\mu\mathcal{M}_{MN}\mathcal{M}^{LK}\partial_K\xi^\mu\partial_L\mathcal{M}^{MN} + C_{SC}e\mathcal{D}_\mu\mathcal{M}^{MN}\mathcal{M}^{LK}\partial_K\xi^\mu\partial_L\mathcal{M}_{MN}\\
&\qquad + 24C_{SC}e(\mathcal{D}^\mu\mathcal{M}^{-1}\mathcal{M})^P{}_Q\partial_P(\mathcal{M}^{QK}g_{\mu\rho}\partial_K\xi^\rho)\\
&= 2C_{SC}e\mathcal{D}_\mu\mathcal{M}_{MN}\mathcal{M}^{LK}\partial_K\xi^\mu\partial_L\mathcal{M}^{MN} + 24C_{SC}e(\mathcal{D}^\mu\mathcal{M}^{-1}\mathcal{M})^P{}_Q\partial_P(\mathcal{M}^{QK}g_{\mu\rho}\partial_K\xi^\rho).
\end{aligned} \tag{5.138}$$

Combining all the terms one finds

$$\begin{aligned}
&\delta\mathcal{L}_{SC} = C_{SC}eg^{\mu\nu}\mathcal{L}_\xi(\mathcal{D}_\mu\mathcal{M}^{MN}\mathcal{D}_\nu\mathcal{M}_{MN}) + 24C_{SC}e\mathcal{F}^Q_{\mu\rho}\partial_P\xi^\rho(\mathcal{D}^\mu\mathcal{M}^{-1}\mathcal{M})^P{}_Q\\
&- 2C_{SC}e\mathcal{D}_\mu\mathcal{M}_{MN}\mathcal{M}^{LK}\partial_K\xi^\mu\partial_L\mathcal{M}^{MN} - 24C_{SC}e(\mathcal{D}^\mu\mathcal{M}^{-1}\mathcal{M})^P{}_Q\partial_P(\mathcal{M}^{QK}g_{\mu\rho}\partial_K\xi^\rho).
\end{aligned} \tag{5.139}$$

Finally taking into account the contribution from $\delta g^{\mu\nu}$ which can easily be seen from

$$\begin{aligned}
\delta g_{\mu\nu} &= (\xi^\rho\mathcal{D}_\rho e_\mu{}^a + \mathcal{D}_\mu\xi^\rho e_\rho{}^a)e_{\nu a} + e_{\mu a}(\xi^\rho\mathcal{D}_\rho e_\nu{}^a + \mathcal{D}_\nu\xi^\rho e_\rho{}^a)\\
&= \xi^\rho\mathcal{D}_\rho(g_{\mu\nu}) + g_{\rho\nu}\mathcal{D}_\mu\xi^\rho + g_{\mu\rho}\mathcal{D}_\nu\xi^\rho = \mathcal{L}_\xi(g_{\mu\nu}),
\end{aligned} \tag{5.140}$$

together with

$$\delta g^{\mu\nu} = -g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma} \tag{5.141}$$

to give

$$\delta g^{\mu\nu} = \mathcal{L}_\xi(g^{\mu\nu}), \tag{5.142}$$

which was expected. Thus the combined result is

$$\begin{aligned}
&\delta\mathcal{L}_{SC} = C_{SC}e\mathcal{L}_\xi(g^{\mu\nu}\mathcal{D}_\mu\mathcal{M}^{MN}\mathcal{D}_\nu\mathcal{M}_{MN}) + 24C_{SC}e\mathcal{F}^Q_{\mu\rho}\partial_P\xi^\rho(\mathcal{D}^\mu\mathcal{M}^{-1}\mathcal{M})^P{}_Q\\
&- 2C_{SC}e\mathcal{D}_\mu\mathcal{M}_{MN}\mathcal{M}^{LK}\partial_K\xi^\mu\partial_L\mathcal{M}^{MN} - 24C_{SC}e(\mathcal{D}^\mu\mathcal{M}^{-1}\mathcal{M})^P{}_Q\partial_P(\mathcal{M}^{QK}g_{\mu\rho}\partial_K\xi^\rho).
\end{aligned} \tag{5.143}$$

From this one can see that in order to cancel the third term in $\delta\mathcal{L}_{YM} + \delta\mathcal{L}_{top}$ with the second term in the result above the constants has to be related as

$$C_{SC} = -\frac{4}{24}C_{YM}. \tag{5.144}$$

Finally by removing the canceled term and rewriting the last two terms one finds the final result

$$\begin{aligned}\delta\mathcal{L}_{SC} = C_{SC}e\mathcal{L}_{\xi}(g^{\mu\nu}\mathcal{D}_{\mu}\mathcal{M}^{MN}\mathcal{D}_{\nu}\mathcal{M}_{MN}) - 24C_{SC}e\mathcal{D}^{\mu}\mathcal{M}^{PK}\partial_{P}(g_{\mu\rho}\partial_{K}\xi^{\rho})\\ - 24C_{SC}e(\mathcal{M}_{NL}\partial_{M}\mathcal{M}^{LK} + \frac{1}{12}\mathcal{M}^{KL}\partial_{L}\mathcal{M}_{MN})\mathcal{D}_{\mu}\mathcal{M}^{MN}\partial_{K}\xi^{\mu},\end{aligned} \tag{5.145}$$

up to the terms from the variations of the fünfbein determinant. The additional contribution of the terms found by variation of the fünfbein determinant leads to that the first term becomes a total derivative. The last term here contains some terms which one would expect could be cancelled by terms from the potential and therefore we start with examining the first two terms in the potential next.

Potential contributions

Starting with the first term in the potential one finds

$$\begin{aligned}\delta(-\frac{1}{24}\mathcal{M}^{MN}\partial_{M}\mathcal{M}^{KL}\partial_{N}\mathcal{M}_{KL}) &= -\frac{1}{24}\xi^{\mu}\mathcal{D}_{\mu}\mathcal{M}^{MN}\partial_{M}\mathcal{M}^{KL}\partial_{N}\mathcal{M}_{KL}\\ &- \frac{1}{24}\mathcal{M}^{MN}\partial_{M}(\xi^{\mu}\mathcal{D}_{\mu}\mathcal{M}^{KL})\partial_{N}\mathcal{M}_{KL} - \frac{1}{24}\mathcal{M}^{MN}\partial_{M}\mathcal{M}^{KL}\partial_{N}(\xi^{\mu}\mathcal{D}_{\mu}\mathcal{M}_{KL})\\ &= -\frac{1}{24}\xi^{\mu}\mathcal{D}_{\mu}\mathcal{M}^{MN}\partial_{M}\mathcal{M}^{KL}\partial_{N}\mathcal{M}_{KL} - \frac{1}{12}\mathcal{M}^{KL}\partial_{L}\mathcal{M}_{MN}\mathcal{D}_{\mu}\mathcal{M}^{MN}\partial_{K}\xi^{\mu}\\ &- \frac{1}{24}\mathcal{M}^{MN}\left(\partial_{M}\mathcal{D}_{\mu}\mathcal{M}^{KL}\partial_{N}\mathcal{M}_{KL} + \partial_{M}\mathcal{M}^{KL}\partial_{N}\mathcal{D}_{\mu}\mathcal{M}_{KL}\right)\xi^{\mu},\end{aligned} \tag{5.146}$$

where

$$\partial_{M}\mathcal{M}^{KL} = -\mathcal{M}^{KP}\mathcal{M}^{LQ}\partial_{M}\mathcal{M}_{PQ}, \tag{5.147}$$

$$\mathcal{D}_{\mu}\mathcal{M}_{KL} = -\mathcal{M}_{KP}\mathcal{M}_{LQ}\mathcal{D}_{\mu}\mathcal{M}^{PQ}, \tag{5.148}$$

has been used to combine terms. From calculations made earlier one saw that

$$\partial_{M}(\mathscr{L}_{A_{\mu}}\mathcal{M}_{KL}) = \mathscr{L}_{A_{\mu}}(\partial_{M}\mathcal{M}_{KL}) + 12\mathbb{P}^{P}{}_{Q}{}^{R}{}_{(K}\mathcal{M}_{L)R}\partial_{M}\partial_{P}A^{Q}_{\mu}, \tag{5.149}$$

thus one finds that

$$\partial_{M}(\mathcal{D}_{\mu}\mathcal{M}_{KL}) = \mathcal{D}_{\mu}(\partial_{M}\mathcal{M}_{KL}) - 12\mathbb{P}^{P}{}_{Q}{}^{R}{}_{(K}\mathcal{M}_{L)R}\partial_{M}\partial_{P}A^{Q}_{\mu}, \tag{5.150}$$

as well as

$$\partial_{M}(\mathcal{D}_{\mu}\mathcal{M}^{KL}) = \mathcal{D}_{\mu}(\partial_{M}\mathcal{M}^{KL}) + 12\mathbb{P}^{P}{}_{Q}{}^{(K}{}_{R}\mathcal{M}^{L)R}\partial_{M}\partial_{P}A^{Q}_{\mu}. \tag{5.151}$$

Now by applying these results on the last line one finds that the variation becomes

$$
\begin{aligned}
\delta(-\frac{1}{24}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_N\mathcal{M}_{KL}) &= -\frac{1}{24}\xi^\mu\mathcal{D}_\mu(\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_N\mathcal{M}_{KL}) \\
&\quad -\frac{1}{12}\mathcal{M}^{KL}\partial_L\mathcal{M}_{MN}\mathcal{D}_\mu\mathcal{M}^{MN}\partial_K\xi^\mu \\
-\frac{1}{2}\mathcal{M}^{MN}(\mathbb{P}^P{}_Q{}^{(K}{}_R\mathcal{M}^{L)R}\partial_M\partial_P A^Q_\mu\partial_N\mathcal{M}_{KL} &- \partial_M\mathcal{M}^{KL}\mathbb{P}^P{}_Q{}^R{}_{(K}\mathcal{M}_{L)R}\partial_N\partial_P A^Q_\mu)\xi^\mu \\
&= -\frac{1}{24}\xi^\mu\mathcal{D}_\mu(\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_N\mathcal{M}_{KL}) - \frac{1}{12}\mathcal{M}^{KL}\partial_L\mathcal{M}_{MN}\mathcal{D}_\mu\mathcal{M}^{MN}\partial_K\xi^\mu \\
-\frac{1}{2}\mathcal{M}^{MN}\Big(\mathbb{P}^P{}_Q{}^K{}_R\partial_M\partial_P A^Q_\mu(\mathcal{M}^{-1}\partial_N\mathcal{M})^R{}_K &- (\partial_M\mathcal{M}^{-1}\mathcal{M})^K{}_R\mathbb{P}^P{}_Q{}^R{}_K\partial_N\partial_P A^Q_\mu\Big)\xi^\mu \\
&= -\frac{1}{24}\xi^\mu\mathcal{D}_\mu(\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_N\mathcal{M}_{KL}) - \frac{1}{12}\mathcal{M}^{KL}\partial_L\mathcal{M}_{MN}\mathcal{D}_\mu\mathcal{M}^{MN}\partial_K\xi^\mu \\
&\quad + \mathcal{M}^{MN}\xi^\mu(\partial_M\mathcal{M}^{-1}\mathcal{M})^P{}_Q\partial_N\partial_P A^Q_\mu. \qquad (5.152)
\end{aligned}
$$

When considering the second term in the potential the same method applies and one then finds

$$
\begin{aligned}
\delta(\frac{1}{2}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_L\mathcal{M}_{NK}) &= \frac{1}{2}\xi^\mu\mathcal{D}_\mu\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_L\mathcal{M}_{NK} \\
&\quad + \frac{1}{2}\mathcal{M}^{MN}\mathcal{D}_\mu\mathcal{M}^{KL}\partial_M\xi^\mu\partial_L\mathcal{M}_{NK} + \frac{1}{2}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\mathcal{D}_\mu\mathcal{M}_{NK}\partial_L\xi^\mu \\
&\quad + \frac{1}{2}\mathcal{M}^{MN}\Big(\partial_M\mathcal{D}_\mu\mathcal{M}^{KL}\partial_L\mathcal{M}_{NK} + \partial_M\mathcal{M}^{KL}\partial_L\mathcal{D}_\mu\mathcal{M}_{NK}\Big)\xi^\mu, \qquad (5.153)
\end{aligned}
$$

and by using

$$
\mathcal{M}^{-1}\mathcal{D}_\mu\mathcal{M} = -\mathcal{D}_\mu\mathcal{M}^{-1}\mathcal{M}, \qquad (5.154)
$$

and after some relabelling one finds

$$
\begin{aligned}
\delta(\frac{1}{2}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_L\mathcal{M}_{NK}) &= \frac{1}{2}\xi^\mu\mathcal{D}_\mu\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_L\mathcal{M}_{NK} \\
&\quad + \frac{1}{2}\mathcal{M}^{LK}\partial_M\mathcal{M}_{NK}\mathcal{D}_\mu\mathcal{M}^{MN}\partial_L\xi^\mu - \frac{1}{2}\mathcal{M}_{NK}\partial_M\mathcal{M}^{KL}\mathcal{D}_\mu\mathcal{M}^{MN}\partial_L\xi^\mu \\
&\quad + \frac{1}{2}\mathcal{M}^{MN}\Big(\partial_M\mathcal{D}_\mu\mathcal{M}^{KL}\partial_L\mathcal{M}_{NK} + \partial_M\mathcal{M}^{KL}\partial_L\mathcal{D}_\mu\mathcal{M}_{NK}\Big)\xi^\mu \\
&= \frac{1}{2}\xi^\mu\mathcal{D}_\mu\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_L\mathcal{M}_{NK} - \mathcal{M}_{NL}\partial_M\mathcal{M}^{KL}\mathcal{D}_\mu\mathcal{M}^{MN}\partial_K\xi^\mu \\
&\quad + \frac{1}{2}\mathcal{M}^{MN}\Big(\partial_M\mathcal{D}_\mu\mathcal{M}^{KL}\partial_L\mathcal{M}_{NK} + \partial_M\mathcal{M}^{KL}\partial_L\mathcal{D}_\mu\mathcal{M}_{NK}\Big)\xi^\mu. \qquad (5.155)
\end{aligned}
$$

By proceeding as in the previous case the result(s) can be rewritten as

$$
\begin{aligned}
\delta(\frac{1}{2}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_L\mathcal{M}_{NK}) &= \frac{1}{2}\xi^\mu\mathcal{D}_\mu(\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_L\mathcal{M}_{NK}) \\
-\mathcal{M}_{NL}\partial_M\mathcal{M}^{KL}\mathcal{D}_\mu\mathcal{M}^{MN}\partial_K\xi^\mu &+ \frac{1}{2}12\xi^\mu\mathbb{P}^P{}_Q{}^{(K}{}_R\mathcal{M}^{L)R}\partial_M\partial_P A^Q_\mu(\mathcal{M}^{-1}\partial_L\mathcal{M})^M{}_K \\
&\quad - \frac{1}{2}\mathcal{M}^{MN}12\xi^\mu\partial_M\mathcal{M}^{KL}\mathbb{P}^P{}_Q{}^R{}_{(N}\mathcal{M}_{K)R}\partial_L\partial_P A^Q_\mu. \qquad (5.156)
\end{aligned}
$$

The first two terms is what is expected. Now the last two terms have to be worked on a bit more. Taking the third term and expanding it

$$\begin{aligned}
&\frac{1}{2}12\xi^\mu \mathbb{P}^P{}_Q{}^{(K}{}_R\mathcal{M}^{L)R}\partial_M\partial_P A^Q_\mu(\mathcal{M}^{-1}\partial_L\mathcal{M})^M{}_K = \xi^\mu(\frac{1}{3}\mathcal{M}^{KL}\partial_M\partial_P A^P_\mu(\mathcal{M}^{-1}\partial_L\mathcal{M})^M{}_K \\
&+\mathcal{M}^{(L|P|}\partial_M\partial_P A^{K)}_\mu(\mathcal{M}^{-1}\partial_L\mathcal{M})^M{}_K + 10d_{SQR}d^{SP(K}\mathcal{M}^{L)R}\partial_M\partial_P A^Q_\mu(\mathcal{M}^{-1}\partial_L\mathcal{M})^M{}_K) \\
&\qquad = -\frac{1}{3}\xi^\mu\partial_L\mathcal{M}^{ML}\partial_M\partial_P A^P_\mu - \frac{1}{2}\xi^\mu\mathcal{M}^{LP}(\partial_L\mathcal{M}^{-1}\mathcal{M})^M{}_K\partial_M\partial_P A^K_\mu \\
&\quad - \frac{1}{2}\xi^\mu\partial_L\mathcal{M}^{MP}\partial_M\partial_P A^L_\mu - 5d_{SQR}d^{SPK}\xi^\mu\mathcal{M}^{LR}\partial_M\partial_P A^Q_\mu(\partial_L\mathcal{M}^{-1}\mathcal{M})^M{}_K. \qquad (5.157)
\end{aligned}$$

Now taking the fourth term and expanding it one finds

$$\begin{aligned}
&-\frac{1}{2}12\mathcal{M}^{MN}\xi^\mu\partial_M\mathcal{M}^{KL}\mathbb{P}^P{}_Q{}^R{}_{(N}\mathcal{M}_{K)R}\partial_L\partial_P A^Q_\mu = -\frac{1}{3}\xi^\mu\partial_M\mathcal{M}^{ML}\partial_L\partial_P A^P_\mu \\
&-\mathcal{M}^{MN}\xi^\mu\partial_M\mathcal{M}^{KL}\mathcal{M}_{R(K}\partial_{N)}\partial_L A^R_\mu + 10\mathcal{M}^{MN}\xi^\mu\partial_M\mathcal{M}^{KL}d^{PRS}d_{SQ(N}\mathcal{M}_{K)R}\partial_L\partial_P A^Q_\mu \\
&= -\frac{1}{3}\xi^\mu\partial_M\mathcal{M}^{ML}\partial_L\partial_P A^P_\mu - \frac{1}{2}\mathcal{M}^{MN}\xi^\mu(\partial_M\mathcal{M}^{-1}\mathcal{M})^L{}_R\partial_N\partial_L A^R_\mu - \frac{1}{2}\xi^\mu\partial_M\mathcal{M}^{KL}\partial_K\partial_L A^M_\mu \\
&\qquad + 5\mathcal{M}^{MN}\xi^\mu d^{PRS}d_{SQN}(\partial_M\mathcal{M}^{-1}\mathcal{M})^L{}_R\partial_L\partial_P A^Q_\mu. \qquad (5.158)
\end{aligned}$$

In both calculations the section constraint has been used. Now as was the case with the generalised internal diffeomorphisms earlier, the terms with d's that are contracted with $(\partial_M\mathcal{M}^{-1}\mathcal{M})$ above gives zero because of the section constraint in the same way, see discussion at equation (5.85). Therefore one finds that

$$\begin{aligned}
&\delta(\frac{1}{2}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_L\mathcal{M}_{NK}) = \frac{1}{2}\xi^\mu\mathcal{D}_\mu(\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_L\mathcal{M}_{NK}) \\
&\qquad -\mathcal{M}_{NL}\partial_M\mathcal{M}^{KL}\mathcal{D}_\mu\mathcal{M}^{MN}\partial_K\xi^\mu - \frac{2}{3}\xi^\mu\partial_L\mathcal{M}^{ML}\partial_M\partial_P A^P_\mu \\
&\qquad\quad -\xi^\mu\mathcal{M}^{LP}(\partial_L\mathcal{M}^{-1}\mathcal{M})^M{}_K\partial_M\partial_P A^K_\mu - \xi^\mu\partial_L\mathcal{M}^{MP}\partial_M\partial_P A^L_\mu. \qquad (5.159)
\end{aligned}$$

Collecting the results from these two terms one sees that

$$\begin{aligned}
&\delta(-\frac{1}{24}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_N\mathcal{M}_{KL}) + \delta(\frac{1}{2}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_L\mathcal{M}_{NK}) \\
&\quad = -\frac{1}{24}\xi^\mu\mathcal{D}_\mu(\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_N\mathcal{M}_{KL}) + \frac{1}{2}\xi^\mu\mathcal{D}_\mu(\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_L\mathcal{M}_{NK}) \\
&\qquad -(\frac{1}{12}\mathcal{M}^{KL}\partial_L\mathcal{M}_{MN} + \mathcal{M}_{NL}\partial_M\mathcal{M}^{KL})\mathcal{D}_\mu\mathcal{M}^{MN}\partial_K\xi^\mu \\
&\qquad\quad -\frac{2}{3}\xi^\mu\partial_L\mathcal{M}^{ML}\partial_M\partial_P A^P_\mu - \xi^\mu\partial_L\mathcal{M}^{MP}\partial_M\partial_P A^L_\mu \qquad (5.160)
\end{aligned}$$

From this one can see that by choosing the constants to be

$$F = -24C_{SC}, \qquad (5.161)$$

one find that the last line in $\delta\mathcal{L}_{SC}$ cancels. Thus all coefficients have been determined in terms of C_{YM} therefore the remaining things left is to determine the C_{YM} relative to that of $\mathcal{L}_{EH}$ as well as seeing that the remaining terms in the potential doesn't give any non-covariant terms.

To continue one has to look at how the metric determinant transforms under external diffeomorphisms. Using the relation

$$\delta g = g g^{\mu\nu} \delta g_{\mu\nu} \tag{5.162}$$

and that $\delta g_{\mu\nu} = \mathcal{L}_\xi(g_{\mu\nu})$ one finds that this becomes

$$g^{-1}\delta g = g^{\mu\nu}(\xi^\lambda \mathcal{D}_\lambda g_{\mu\nu} + 2g_{\lambda(\mu}\mathcal{D}_{\nu)}\xi^\lambda) = \xi^\lambda g^{-1}\mathcal{D}_\lambda g + 2\mathcal{D}_\lambda \xi^\lambda, \tag{5.163}$$

which means that the metric determinant is a density with weight 2 under external diffeomorphsims. Using that $g = e^2$ one finds that

$$\delta e = \xi^\lambda \mathcal{D}_\lambda e + e\mathcal{D}_\lambda \xi^\lambda, \tag{5.164}$$

that implies that the fünfbein determinant is a density of weight 1 under external diffeomorphisms which is expected and desired in order for all the terms to transform into total derivatives. By using the above formulae one can compute the variation of $\delta(g^{-1}\partial_M g)$, using that

$$\delta g^{-1} = -g^{-2}\delta g, \tag{5.165}$$

one finds

$$\begin{aligned}\delta(g^{-1}\partial_M g) = -g^{-2}(\xi^\lambda \mathcal{D}_\lambda g + 2g\mathcal{D}_\lambda \xi^\lambda)\partial_M g + g^{-1}\mathcal{D}_\lambda g \partial_M \xi^\lambda + \xi^\lambda g^{-1}\partial_M(\mathcal{D}_\lambda g) \\ + 2g^{-1}\partial_M g \mathcal{D}_\lambda \xi^\lambda + 2\partial_M \mathcal{D}_\lambda \xi^\lambda.\end{aligned} \tag{5.166}$$

By straight-forward computation one finds that $\partial_M \mathcal{D}_\mu g = \mathcal{D}_\mu(\partial_M g) - \frac{10}{3}\partial_M \partial_P A^P_\lambda g$, exactly the same calculations as in the gauge transformation case since $\mathcal{D}_\mu = \partial_\mu - \mathscr{L}_{A_\mu}$. By using this relation one finds

$$\delta(g^{-1}\partial_M g) = \xi^\lambda \mathcal{D}_\lambda(g^{-1}\partial_M g) + g^{-1}\mathcal{D}_\lambda g \partial_M \xi^\lambda - \frac{10}{3}\xi^\lambda \partial_M \partial_P A^P_\lambda + 2\partial_M \mathcal{D}_\lambda \xi^\lambda. \tag{5.167}$$

So the variation of the next term in the potential becomes

$$\begin{aligned}\delta(g^{-1}\partial_M g \partial_N \mathcal{M}^{MN}) = \xi^\lambda \mathcal{D}_\lambda(g^{-1}\partial_M g)\partial_N \mathcal{M}^{MN} + g^{-1}\mathcal{D}_\lambda g \partial_M \xi^\lambda \partial_N \mathcal{M}^{MN} \\ - \frac{10}{3}\xi^\lambda \partial_M \partial_P A^P_\lambda \partial_N \mathcal{M}^{MN} + 2\partial_M \mathcal{D}_\lambda \xi^\lambda \partial_N \mathcal{M}^{MN} \\ + g^{-1}\partial_M g \xi^\lambda \mathcal{D}_\lambda(\partial_N \mathcal{M}^{MN}) + g^{-1}\partial_M g \xi^\lambda 12\mathbb{P}^P{}_Q{}^{(M}{}_R \mathcal{M}^{N)R}\partial_N \partial_P A^Q_\lambda,\end{aligned} \tag{5.168}$$

from which one finds

$$\begin{aligned}\delta(g^{-1}\partial_M g \partial_N \mathcal{M}^{MN}) = \xi^\lambda \mathcal{D}_\lambda(g^{-1}\partial_M g \partial_N \mathcal{M}^{MN}) + g^{-1}\mathcal{D}_\lambda g \partial_M \xi^\lambda \partial_N \mathcal{M}^{MN} \\ - \frac{10}{3}\partial_N \mathcal{M}^{MN}\xi^\lambda \partial_M \partial_P A^P_\lambda + 2\partial_M \mathcal{D}_\lambda \xi^\lambda \partial_N \mathcal{M}^{MN} \\ \frac{5}{3}g^{-1}\partial_M g \xi^\lambda \mathcal{M}^{MN}\partial_N \partial_P A^P_\lambda + g^{-1}\partial_M g \xi^\lambda \mathcal{M}^{NP}\partial_N \partial_P A^M_\lambda.\end{aligned} \tag{5.169}$$

Now from the discussion about the fünfbein before, one sees that by including a fünfbein factor in front of this expression leads to the first term becoming a total derivative, and thus can be disregarded. By then converting this into fünfbeins determinants instead one finds that this becomes

$$\begin{aligned}\delta(eg^{-1}\partial_M g \partial_N \mathcal{M}^{MN}) = &-\frac{10}{3}\partial_N \mathcal{M}^{MN}\xi^\lambda \partial_M \partial_P A_\lambda^P \\ &+ 2\mathcal{D}_\lambda e \partial_M \xi^\lambda \partial_N \mathcal{M}^{MN} + 2e\partial_M \mathcal{D}_\lambda \xi^\lambda \partial_N \mathcal{M}^{MN} \\ &+ \frac{10}{3}\partial_M e \xi^\lambda \mathcal{M}^{MN}\partial_N \partial_P A_\lambda^P + 2\partial_M e \xi^\lambda \mathcal{M}^{NP}\partial_N \partial_P A_\lambda^M. \end{aligned} \tag{5.170}$$

Now from rewriting the first term on the second row as

$$2\mathcal{D}_\lambda e \partial_M \xi^\lambda \partial_N \mathcal{M}^{MN} = 2\mathcal{D}_\lambda(e\partial_M \xi^\lambda \partial_N \mathcal{M}^{MN}) - 2e\mathcal{D}_\lambda \partial_M \xi^\lambda \partial_N \mathcal{M}^{MN} - 2e\partial_M \xi^\lambda \mathcal{D}_\lambda \partial_N \mathcal{M}^{MN}, \tag{5.171}$$

and use that ξ^λ carries weight $\lambda = 0$, thus one can commute $\mathcal{D}_\mu$ and ∂_M one finds that the second row, ignoring the total derivative, becomes

$$\begin{aligned}\delta(eg^{-1}\partial_M g \partial_N \mathcal{M}^{MN}) = &-\frac{10}{3}e\partial_N \mathcal{M}^{MN}\xi^\lambda \partial_M \partial_P A_\lambda^P - 2e\partial_M \xi^\lambda \mathcal{D}_\lambda \partial_N \mathcal{M}^{MN} \\ &+ \frac{10}{3}\partial_M e \xi^\lambda \mathcal{M}^{MN}\partial_N \partial_P A_\lambda^P + 2\partial_M e \xi^\lambda \mathcal{M}^{NP}\partial_N \partial_P A_\lambda^M. \end{aligned} \tag{5.172}$$

Moving on to the next term in the potential one finds that it becomes

$$\begin{aligned}\delta(e\mathcal{M}^{MN}g^{-1}\partial_M g g^{-1}\partial_N g) &= \mathcal{D}_\lambda(e\mathcal{M}^{MN}g^{-1}\partial_M g g^{-1}\partial_N g \xi^\lambda) \\ &+ 2\mathcal{M}^{MN}\mathcal{D}_\lambda e \partial_M \xi^\lambda g^{-1}\partial_N g - \frac{20}{3}\xi^\lambda \mathcal{M}^{MN}\partial_M \partial_P A_\lambda^P \partial_N e + 4\mathcal{M}^{MN}\partial_M \mathcal{D}_\lambda \xi^\lambda \partial_N e \\ &+ 2\mathcal{M}^{MN}g^{-1}\partial_M g \mathcal{D}_\lambda e \partial_N \xi^\lambda - \frac{20}{3}\mathcal{M}^{MN}\partial_M e \xi^\lambda \partial_N \partial_P A_\lambda^P + 4\mathcal{M}^{MN}\partial_M e \partial_N \mathcal{D}_\lambda \xi^\lambda \\ &= \mathcal{D}_\lambda(e\mathcal{M}^{MN}g^{-1}\partial_M g g^{-1}\partial_N g \xi^\lambda) + 4\mathcal{M}^{MN}\mathcal{D}_\lambda e \partial_M \xi^\lambda g^{-1}\partial_N g \\ &\quad - \frac{40}{3}\xi^\lambda \mathcal{M}^{MN}\partial_M \partial_P A_\lambda^P \partial_N e + 8\mathcal{M}^{MN}\partial_M \mathcal{D}_\lambda \xi^\lambda \partial_N e. \end{aligned} \tag{5.173}$$

Using these as well as an expression for the last term in the potential one can show that these cancel [31]. The calculation should be analogous to that found in the case of the gauge invariance that was displayed earlier.

Finally the last constant to be determined was the C_{YM} in relation to the $\mathcal{L}_{EH}$. This was not done in this work however in [17] the result was found to be $C_{YM} = -1/4$. Using this in the relations derived earlier one finds that

$$C_{YM} = -\frac{1}{4}, \tag{5.174}$$

$$C_{top} = \frac{4}{3}, \tag{5.175}$$

$$C_{SC} = \frac{1}{24}, \tag{5.176}$$

$$F = -1. \tag{5.177}$$

Using these coefficients one has that the (psuedo)Lagrangian is

$$\mathcal{L} = e\Big(\hat{\mathcal{R}} - \frac{1}{4}\mathcal{M}_{MN}\mathcal{F}^{M}_{\mu\nu}\mathcal{F}^{\mu\nu N} + \frac{1}{24}g^{\mu\nu}\mathcal{D}_\mu\mathcal{M}_{MN}\mathcal{D}_\nu\mathcal{M}^{MN} - V(\mathcal{M}, g)\Big)$$
$$+ \frac{4}{3}\kappa \int_{\mathcal{M}_6} d_{MNK}\mathcal{F}^M \wedge \mathcal{F}^N \wedge \mathcal{F}^K - 40 d^{MNK}\mathcal{H}_M \wedge \partial_N \mathcal{H}_K, \quad (5.178)$$

with $\kappa^2 = 5/32$.

5.3 E$_6$-like exceptional field theories

In previous section we showed the action for the case with E$_6$ was invariant under internal diffeomorphisms and was given uniquely from the requirement that it should be invariant under external diffeomorphisms. From this one could see that much of the computations relied on two properties of the d symbols. Firstly that it was a fully symmetric tensor under the E$_6$ transformations, and secondly that there existed a cubic relation. However as was shown earlier there exists a fully symmetric d-tensor in all the cases of the magical supergravities, which is confirmed by computations with the computer program LiE [32]. Similarly one can compute with LiE that under

$$\vee^4 R(\lambda) = R(\lambda) + ... \quad (5.179)$$

where $\vee^k$ represents the symmetric tensor product k times and $R(\lambda)$ is the coordinate representation, which are for $D = 5$ the $R(\lambda)$ is the following representations where λ is given in the parentheses $\mathbf{6}([0,2]), \mathbf{9}([0,1,1,0]), \mathbf{15}([0,1,0,0,0])$ and $\mathbf{27}([1,0,0,0,0,0])$ for the division algebra in increasing dimension. The ellipses represent higher representation for all the cases from the magical square. This shows that there must exist a relation like

$$d^{P(MN}d^{KL)Q}d_{PRQ} = y\delta_R^{(M}d^{NKL)}, \quad (5.180)$$

where y is a constant specified for the different cases. Therefore one expects that the Y-tensor has the same expression however possibly with different scaling factors, i.e.

$$Y^{MN}{}_{PQ} = x d^{MNR}d_{RPQ}, \quad (5.181)$$

where x is the constant. The constant can be determined from evaluating

$$Y^{MN}{}_{NQ} = x d^{MNR}d_{RNQ} = x\delta^M_Q, \quad (5.182)$$

and by the general expression

$$Y^{MN}{}_{NQ} = -\tilde{k}t^{\alpha N}{}_N t_\alpha{}^M{}_Q + \beta\delta^N_N\delta^M_Q + \delta^M_Q, \quad (5.183)$$

with $\tilde{k} = \frac{2}{(\alpha_i,\alpha_i)}$. However since the generators are anti-symmetric and that in all cases $\beta = k(\lambda,\lambda) - 1 = 1/3$, one finds that this becomes

$$x\delta^M_Q = (\frac{1}{3}\dim(R_1(\lambda)) + 1)\delta^M_Q, \quad (5.184)$$

and since $\dim(R(\lambda)) = 3(\nu + 1)$ where ν is the dimension of the division algebra under consideration, which gives the final result

$$x = \nu + 2. \tag{5.185}$$

Thus from this one can see that the projection operator is given by

$$\mathbb{P}^M{}_N{}^K{}_L = kt_\alpha{}^M{}_N t^{\alpha K}{}_L = \frac{k}{3}\delta^M_N\delta^K_L + k\delta^K_N\delta^M_L - k(\nu+2)d^{MKR}d_{NLR}, \tag{5.186}$$

where k is determined from

$$\mathbb{P}^M{}_N{}^N{}_M = k(t_\alpha t^\alpha)^M{}_M = 2kC_2(\lambda)\dim(R(\lambda)) = \dim(\mathfrak{g}), \tag{5.187}$$

where $C_2(\lambda)$ is the Casimir operator defined as

$$C_2(\lambda) = \frac{1}{2}(\lambda, \lambda + 2\rho) = \frac{1}{2}(\lambda, \lambda) + (\lambda, \rho), \tag{5.188}$$

with ρ being the Weyl vector. Through computation of the for E_6-like theories the following result was found $C_2(\lambda) = (\dim(R(\lambda)) - 1)/3$. Now one can then compute the constants k for the cases we discussed earlier and find the results of table 5.1.

Table 5.1: The table show the coefficient k for the different groups that corresponds to $D = 5$ magical supergravities.

Group	k
$SL(3,\mathbb{R})$	1/2
$SL(3,\mathbb{C})$	1/3
$SU(6)$	1/4
$E_{6(-26)}$	1/6

Next one could determine the constant y for the different cases by using the cubic relation discussed earlier

$$d_{R(MN}d^{RQS}d_{KL)S} = y\delta^Q_{(M}d_{NKL)} \tag{5.189}$$

and multiplying with δ^M_Q on both sides. This gives

$$d_{NKL} = y\frac{1}{4}(\dim(R(\lambda)) + 3)d_{NKL}, \tag{5.190}$$

which gives

$$y = \frac{4}{3(\nu+2)}. \tag{5.191}$$

From this it is easy to convince oneself that if $\nu = 8$, i.e. the octonionic case, this reduces to the known case of E_6. In order to find a unified theory one has to modify some relations where $10 \mapsto \nu+2$, like the definition of $\Delta B_{\mu\nu M}$, as well as the coefficient for the term involving $\mathcal{H}$ in the topological term. Through the derivation of the coefficients by the internal and external diffeomorphisms one find that the relative coefficients have to be modified for some terms. The first of which is in the scalar kinetic term which is found to be

$$\mathcal{L}_{SC} = \frac{k}{4} e g^{\mu\nu} \mathcal{D}_\mu \mathcal{M}^{MN} \mathcal{D}_\nu \mathcal{M}_{MN}, \tag{5.192}$$

by simply following the derivation and noting that there is a factor of 4 from the two terms and the symmetry of $\mathcal{M}$ one finds another two, as well as a factor of $\frac{1}{k}$ because of the projector. Next by similar reasoning one has that the potential becomes

$$\begin{aligned} V(\mathcal{M}, g) = &-\frac{k}{4}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_N\mathcal{M}_{KL} + \frac{1}{2}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_L\mathcal{M}_{NK} \\ &- \frac{1}{2}g^{-1}\partial_M g\partial_N\mathcal{M}^{MN} - \frac{1}{4}\mathcal{M}^{MN}g^{-1}\partial_M g g^{-1}\partial_N g - \frac{1}{4}\mathcal{M}^{MN}\partial_M g^{\mu\nu}\partial_N g_{\mu\nu}. \end{aligned} \tag{5.193}$$

5.4 E$_7$- and E$_8$-like ExFT

Like with the case of previous section one would like generalise [18] and [19] to apply to all the cases of $D = 4$ and $D = 3$ magical supergravities. From computations one could see that in all cases within the same dimension one has $\beta = 1/2$ and $\beta = 1$ respectively. Since there are some differences between the two actions this section has been split into two subsections dealing with the differences. However the scalar kinetic term is common within both theories and have the normalisation

$$\mathcal{L}_{SC} = \frac{k}{4} e g^{\mu\nu} \mathcal{D}_\mu \mathcal{M}^{MN} \mathcal{D}_\nu \mathcal{M}_{MN}, \tag{5.194}$$

in all the dimensions considered here.

5.4.1 E$_7$-like

Like the case of the E$_6$ one would like to rescale the generators of the project in the same fashion, i.e. equation (5.187). In the case presented in [18] one has that the projection operator had the form

$$\mathbb{P}^M{}_N{}^K{}_L = k t^\alpha{}_N{}^M t_{\alpha L}{}^K = \frac{k}{2}\delta_N^M\delta_L^K + k\delta_N^K\delta_L^M + k t_{\alpha NL} t^{\alpha MK} - \frac{k}{2}\Omega^{MK}\Omega_N, \tag{5.195}$$

which utilise the antisymmetric tensor Ω^{MN}. This sets a requirement that such a tensor has to exist in all of the different cases that are considered, i.e., one has to have

$$\wedge^2 R(\lambda) = \mathbf{1} + ... \tag{5.196}$$

in order for the theory to work. From simple computations using LiE one finds that this holds for all the groups in the $D = 4$-case [32]. Apart from that one also has that

$$\vee^4 R(\lambda) = \mathbf{1} + ... \tag{5.197}$$

thus there exist a fully symmetric C_{MNKL}. This is required for the formalism to agree with [30]. The potential has the same form as in the case of E_6-like theories, however the coefficient k has to be altered using the relation (5.187). The value of k for the different cases are given in the table 5.2.

Table 5.2: The table shows the coefficient k for the different groups that corresponds to $D = 4$ magical supergravities.

Group	k
$Sp(6)$	1/5
$SU(3,3)$	1/6
$SO^*(12)$	1/8
$\mathrm{E}_{7(-25)}$	1/12

This concludes the modifications needed for the different cases of E_6-like exceptional field theories found from the magical supergravities in $D = 4$.

5.4.2 E_8-like

For the case of $D = 3$ one knows that in all cases the coordinate representation is the adjoint representation as was needed for the formalism in [19]. Proceeding as with the case of E_7 and using that the representation is the adjoint one finds that

$$\mathbb{P}^{M}{}_{N}{}^{K}{}_{L} = kt^{PM}{}_{N}t_{P}{}^{K}{}_{L} = kf^{M}{}_{NP}f^{PK}{}_{L}, \tag{5.198}$$

where $f^{MN}{}_{K}$ is the structure constant. Next modification is the potential. The potential in the case of E_8 has the same terms as in the cases E_6 and E_7 although with an extra term and is thus given by

$$\begin{aligned} V(\mathcal{M}, g) = &-\frac{k}{4}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_N\mathcal{M}_{KL} + \frac{1}{2}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_L\mathcal{M}_{NK} \\ &+ \frac{k^2}{2}f^{NQ}{}_{P}f^{MS}{}_{R}\mathcal{M}^{PK}\partial_M\mathcal{M}_{QK}\mathcal{M}^{RL}\partial_N\mathcal{M}_{SL} \\ &- \frac{1}{2}g^{-1}\partial_M g\partial_N\mathcal{M}^{MN} - \frac{1}{4}\mathcal{M}^{MN}g^{-1}\partial_M g g^{-1}\partial_N g - \frac{1}{4}\mathcal{M}^{MN}\partial_M g^{\mu\nu}\partial_N g_{\mu\nu}. \end{aligned} \tag{5.199}$$

As with the previous two cases of generalisations one has to compute the coefficient k for all the dimensions however the relations in eq. 5.187 with the case of the adjoint representation becomes

$$k = \frac{1}{2C_2(\lambda)} = \frac{1}{2g^{\vee}}, \tag{5.200}$$

where $g^{\vee}$ is the dual Coxeter number. The value for k is then found in the table 5.3.

Table 5.3: The table show the coefficient k for the different groups that corresponds to $D = 3$ magical supergravities.

Group	k
$\mathrm{F}_{4(4)}$	1/18
$\mathrm{E}_{6(2)}$	1/24
$\mathrm{E}_{7(-5)}$	1/36
$\mathrm{E}_{8(-24)}$	1/60

This concludes the modifications to account for all the $D = 3$ groups.

5.5 Solving the section constraint

In previous work the general method for solving the section constraint for the special case of real forms has been explored for the finite dimensional Lie algebras as well as Kac-Moody algebras. This will be reviewed in the first subsection by closely following the work by Cederwall and Palmkvist [16]. In order for the exceptional field theory formalism to work for the cases with magical supergravities one has to know how to solve the section constrain for general real forms of the Lie algebras. Therefore is the second subsection devoted to expanding this method to include the general real forms of the algebras.

5.5.1 Section constraint for split real form

The section constraint could be divided into two different constraints. The first one is the weak sections condition which states that the momentum has to be in a minimal orbit of the structure group. This corresponds to requiring that $\partial_M \partial_N$ lies in the dual to the maximal weight in

$$R(\lambda) \otimes R(\lambda) \tag{5.201}$$

i.e. the dual to $R(2\lambda)$. This is a non-linear space. The second constraint consists of the requirement for the product of any two momenta to lie within the lowest symmetric and anti-symmetric modules of the above mentioned tensor product. By introduction of the following notation,

$$R_2^S = \vee^2 R(\lambda) \ominus R(2\lambda), \tag{5.202}$$

$$R_2^A = \wedge^2 R(\lambda) \ominus \bigoplus_{i \in \mathscr{F}_k} R(2\lambda - \alpha_i), \tag{5.203}$$

where $\mathscr{F}_k$ is defined as all the states $|2\lambda - \alpha_i\rangle = |\lambda\rangle \otimes f_i |\lambda\rangle - f_i |\lambda\rangle \otimes |\lambda\rangle$ with $\lambda_i = (\lambda, \alpha_i^\vee)$ such that

$$\mathscr{F}_k \subset \{i | \lambda_i = 1, k = \frac{2}{(\alpha_i, \alpha_i)}\}, \tag{5.204}$$

one would formulate the section constraint as

$$(\partial \otimes \partial)\Big|_{\overline{R_2^S \oplus R_2^A}} = 0. \tag{5.205}$$

The remaining question is how could this be constructed algebraically. Starting by defining the Casimir operator for a general Kac-Moody algebra by

$$C_2 = \frac{1}{2}\eta_{\alpha\beta} : T^\alpha T^\beta : + (h, \varrho), \tag{5.206}$$

where $\eta_{\alpha\beta}$ is the killing form, ϱ is the Weyl vector defined to be $(\rho, \alpha_i^\vee) = 1$ and : ... : indicates normal ordering. Through explicit calculation using that $[e_{-\alpha}, e_\alpha] = h_\alpha$

one can rewrite this expression as

$$C_2 = \sum_{\alpha\in\Delta_+} e_{-\alpha}e_\alpha + \frac{1}{2}(h,h) + (h,\varrho). \tag{5.207}$$

Here Δ_+ represent the set of positive fundamental roots. One can then evaluate this operator on any element in an irreducible representation and find the same value. By taking the highest weight state $|\lambda\rangle$ one finds that the sum will disappear because the highest weight state could otherwise be increased through $e_\alpha |\lambda\rangle$ hence it would contradict the assumption that it was the highest weight state. The reining terms becomes

$$\begin{aligned} C_2(R(\lambda))|\lambda\rangle &= \left[\frac{1}{2}(h,h)+(h,\varrho)\right]|\lambda\rangle = \left[\frac{1}{2}(\alpha_i^\vee,\alpha_j^\vee)h_ih_j + (\alpha_i^\vee,\varrho)h_i\right]|\lambda\rangle \\ &= \left[\frac{1}{2}(\alpha_i^\vee,\alpha_j^\vee)\lambda_i\lambda_j + (\alpha_i^\vee,\varrho)\lambda_i\right]|\lambda\rangle = \left[\frac{1}{2}(\lambda,\lambda)+(\lambda,\varrho)\right]|\lambda\rangle\,, \end{aligned} \tag{5.208}$$

thus one has

$$C_2(R(\lambda)) = \frac{1}{2}(\lambda,\lambda+2\varrho). \tag{5.209}$$

In which $R(\lambda)$ has been taken to be an irreducible module. Through simply employing the above relation for $R(2\lambda)$ one finds

$$\begin{aligned} C_2(R(2\lambda)) = \frac{1}{2}(2\lambda,2\lambda+2\rho) = \frac{1}{2}2(\lambda,2\lambda+2\rho) = \frac{1}{2}2(\lambda,\lambda+2\rho) + (\lambda,\lambda) \\ = 2C_2(R(\lambda)) + (\lambda,\lambda). \end{aligned} \tag{5.210}$$

A similar calculation can be done for the modules found from anti-symmetric tensor products as well to find the result

$$C_2(R(2\lambda-\alpha_i)) = 2C_2(R(\lambda)) + (\lambda,\lambda) - \lambda_i(\alpha_i,\alpha_i), \tag{5.211}$$

with the root α_i fulfilling that $(\lambda,\alpha_i^\vee) = \lambda_i > 0$.

With the relations for the Casimir operator one could then find an algebraic constraint for the elements which are found in the minimal orbit

$$0 = [C_2(R(2\lambda)) - 2C_2(R(\lambda)) - (\lambda,\lambda)]\,|\lambda\rangle\otimes|\lambda\rangle\,. \tag{5.212}$$

This requirement can then be simplified through insertion of the definition of the Casimir operator. One then finds

$$\begin{aligned} 0 = &\left[\frac{1}{2}\eta_{\alpha\beta} : T^\alpha T^\beta : +(h,\varrho)\right]|\lambda\rangle\otimes|\lambda\rangle \\ &- \left(\left(\left[\frac{1}{2}\eta_{\alpha\beta} : T^\alpha T^\beta : +(h,\varrho)\right]|\lambda\rangle\right)\otimes|\lambda\rangle + |\lambda\rangle\otimes\left(\left[\frac{1}{2}\eta_{\alpha\beta} : T^\alpha T^\beta : +(h,\varrho)\right]|\lambda\rangle\right)\right) \\ &- (\lambda,\lambda)\,|\lambda\rangle\otimes|\lambda\rangle = \left[\eta_{\alpha\beta}T^\alpha\otimes T^\beta - (\lambda,\lambda)\right]|\lambda\rangle\otimes|\lambda\rangle\,, \end{aligned} \tag{5.213}$$

since the first term above contain terms where the generators both acts on the same factor in the tensor product and terms where one of the generator act on the first and the other on the second while the second line removes the double generators actions. For the anti-symmetric part one has

$$\begin{aligned}C_2(R(2\lambda-\alpha_i)) &= \frac{1}{2}(2\lambda-\alpha_i, 2\lambda-\alpha_i+2\rho) = \frac{1}{2}(2\lambda, 2\lambda-\alpha_i+2\rho)-\frac{1}{2}(\alpha_i, 2\lambda-\alpha_i+2\rho)\\ &= \frac{1}{2}2(\lambda,\lambda+2\rho)+(\lambda,\lambda)-2(\lambda,\alpha_i)+\frac{1}{2}(\alpha_i,\alpha_i)-(\alpha_i,\rho)\\ &= 2C_2(R(\lambda))+(\lambda,\lambda)-(\lambda,\alpha_i^\vee)(\alpha_i,\alpha_i)+\frac{1}{2}(\alpha_i,\alpha_i)-(\alpha_i,\frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha)\\ &= 2C_2(R(\lambda))+(\lambda,\lambda)-\lambda_i(\alpha_i,\alpha_i). \end{aligned} \tag{5.214}$$

Through application of this on the anti-symmetric state $|2\lambda-\alpha_i\rangle$ through similar reasoning as the case with the symmetric part of the tensor product one finds that this becomes

$$0 = \Big[\eta_{\alpha\beta}T^\alpha\otimes T^\beta-(\lambda,\lambda)-\lambda_i(\alpha_i,\alpha_i)\Big]\,|2\lambda-\alpha_i\rangle\,. \tag{5.215}$$

Assuming the existence of another vector $|q\rangle$ which also lie within the section would then fulfil the two requirements which states that the symmetric and anti-symmetric part of the tensor product are annihilated, i.e.

$$0 = \Big[\eta_{\alpha\beta}T^\alpha\otimes T^\beta-(\lambda,\lambda)\Big]\,(|\lambda\rangle\otimes|q\rangle+|q\rangle\otimes|\lambda\rangle) \tag{5.216}$$

and

$$0 = \Big[\eta_{\alpha\beta}T^\alpha\otimes T^\beta-(\lambda,\lambda)-\lambda_i(\alpha_i,\alpha_i)\Big]\,(|\lambda\rangle\otimes|q\rangle-|q\rangle\otimes|\lambda\rangle). \tag{5.217}$$

This are solved trivially by $|q\rangle = |\lambda\rangle$ hence one can take $|q\rangle = e_{-\alpha_i}|\lambda\rangle$ and after insertion into the symmetric constraint leads to

$$\begin{aligned}[\eta_{\alpha\beta}T^\alpha\otimes T^\beta-(\lambda,\lambda)]\,|\lambda\rangle\otimes|q\rangle &= [\sum_{\alpha\in\Delta_+} e_{-\alpha}\otimes e_\alpha+e_\alpha\otimes e_{-\alpha}+(h,h)-(\lambda,\lambda)]\,|\lambda\rangle\otimes|q\rangle\\ &= \sum_{\alpha\in\Delta_+} e_{-\alpha}|\lambda\rangle\otimes e_\alpha e_{-\alpha_i}|\lambda\rangle-(\lambda,\alpha_i)\,|\lambda\rangle\otimes e_{-\alpha_i}|\lambda\rangle\\ &= e_{-\alpha_i}|\lambda\rangle\otimes(\lambda,\alpha_i)\,|\lambda\rangle-(\lambda,\alpha_i)\,|\lambda\rangle\otimes e_{\alpha_i}|\lambda\rangle\,, \end{aligned} \tag{5.218}$$

which is then shows that

$$\Big[\eta_{\alpha\beta}T^\alpha\otimes T^\beta-(\lambda,\lambda)\Big]\,(|\lambda\rangle\otimes|q\rangle+|q\rangle\otimes|\lambda\rangle)=0 \tag{5.219}$$

is satisfied. Through demanding that the vector $|q\rangle$ itself lies within the minimal orbit, i.e. satisfies

$$[\eta_{\alpha\beta}T^\alpha\otimes T^\beta-(\lambda,\lambda)]\,|q\rangle\otimes|q\rangle=0, \tag{5.220}$$

one finds that

$$\begin{aligned}0 = \Big[\eta_{\alpha\beta}T^\alpha\otimes T^\beta-(\lambda,\lambda)\Big]\,e_{-\alpha_i}|\lambda\rangle\otimes e_{-\alpha_i}|\lambda\rangle &= (1+\sigma)\sum_{\alpha\in\Delta_+} e_\alpha e_{-\alpha_i}|\lambda\rangle\otimes e_{-\alpha}e_{-\alpha_i}|\lambda\rangle\\ &+[(\lambda-\alpha_i,\lambda-\alpha_i)-(\lambda,\lambda)]e_{-\alpha_i}|\lambda\rangle\otimes e_{-\alpha_i}|\lambda\rangle\,, \end{aligned} \tag{5.221}$$

where σ symbolise the permutation of the order in the tensor product. This gives two requirements to be fulfilled. The first is that the latter term vanishes, by expanding this one finds

$$(\lambda - \alpha_i, \lambda - \alpha_i) - (\lambda, \lambda) = -2(\lambda, \alpha_i) + (\alpha_i, \alpha_i) = (1 - \lambda_i)(\alpha_i, \alpha_i), \tag{5.222}$$

which then shows that $\lambda_i = 1$. The second constraint is that the first term should vanish, now by noting that the terms $\alpha \neq \alpha_i$ annihilates since the $e_\alpha |\lambda\rangle = 0$. Thus one has

$$\lambda(1 + \sigma)[|\lambda\rangle \otimes e_{-\alpha_i} e_{-\alpha_i} |\lambda\rangle] = 0 \tag{5.223}$$

which gives the second constraint $e^2_{-\alpha_i} |\lambda\rangle = 0$. Thus one has that $|\lambda\rangle$ and $e_{-\alpha_i} |\lambda\rangle$ is part of the section if $\lambda_i = 1$ together with $e^2_{-\alpha_i} |\lambda\rangle = 0$.

Next, following [2], assume that there exist another vector $|p\rangle = e_{-\alpha_i - \beta} |\lambda\rangle$ that is also in the section. By imposing the symmetric constraint for $(1+\sigma) |\lambda\rangle \otimes |p\rangle$ one finds that

$$0 = \sum_{\alpha \in \Delta_+} e_{-\alpha} |\lambda\rangle \otimes e_\alpha e_{-\alpha_i - \beta} |\lambda\rangle + [(\lambda, \lambda - \alpha_i - \beta) - (\lambda, \lambda)] |\lambda\rangle \otimes e_{-\alpha_i - \beta} |\lambda\rangle\,, \tag{5.224}$$

and as with the case $|q\rangle$ one finds that the sum for $\alpha = \alpha_i + \beta$ cancels the remaining term from the last term in the equation above. Now inserting this for $|p\rangle \otimes |p\rangle$ and going through the same calculation as with the case $|q\rangle$ one has

$$0 = \lambda(1+\sigma)[|\lambda\rangle \otimes e^2_{-\alpha_i - \beta} |\lambda\rangle] - [2(\lambda, \alpha_i + \beta) - (\alpha_i + \beta, \alpha_i + \beta)] e_{-\alpha_i - \beta} |\lambda\rangle \otimes e_{-\alpha_i - \beta} |\lambda\rangle\,. \tag{5.225}$$

The first term is once again that no twice lowering could be included, the second term, once expanded, gives that

$$(\lambda_i - 1)(\alpha_i, \alpha_i) + 2(\lambda, \beta) - 2(\beta, \alpha_i) - (\beta, \beta) = 0, \tag{5.226}$$

the first term is then zero as $\lambda_i = 1$ and the remaining terms are zero if $(\beta, \lambda) = 0$ and in the case of β being the neighbouring node α_{i+1} has of the same length as α_i. Thus through this process one can expand the section until the diagram ends, a node with different length is encountered or a node α_j with $\lambda_j \neq 0$ is encountered.

This method can then visually be seen as walking along a ”gravity line” in a Dynkin diagram from the node which corresponds to the representation $R(\lambda)$ until one of the previously presented cases happens. This has then chosen a section which consists of $|\lambda\rangle$ acted on with f_i with the generators e_i, f_i and h_i which then forms the sub-algebra $\mathfrak{sl}(n)$, where n is the number of nodes within the section. However another h_j can be included to then form $\mathfrak{gl}(n)$ which is the full structure algebra of the section as was desired.

One can from the symmetric and anti-symmetric constraint derive a single constraint by

$$\begin{aligned}[\eta_{\alpha\beta} T^\alpha \otimes T^\beta - (\lambda, \lambda) - (\alpha_i, \alpha_i)](1-\sigma) |p\rangle \otimes |q\rangle - [\eta_{\alpha\beta} T^\alpha \otimes T^\beta - (\lambda, \lambda)](1+\sigma) |p\rangle \otimes |q\rangle \\ = \Big(2[-\eta_{\alpha\beta} T^\alpha \otimes T^\beta - (\lambda, \lambda)] + (\alpha_i, \alpha_i)(\sigma - 1)\Big) |p\rangle \otimes |q\rangle = 0,\end{aligned} \tag{5.227}$$

thus one could define

$$\sigma Y = k[-\eta_{\alpha\beta} T^{\alpha} \otimes T^{\beta} - (\lambda, \lambda)] + \sigma - 1, \tag{5.228}$$

where $k = 2/(\alpha_i, \alpha_i)$, which satisfies

$$Y \left|p\right\rangle \otimes \left|q\right\rangle = 0. \tag{5.229}$$

This form agrees with the Y tensor which was used previously.

5.5.2 Section constraint for general real form

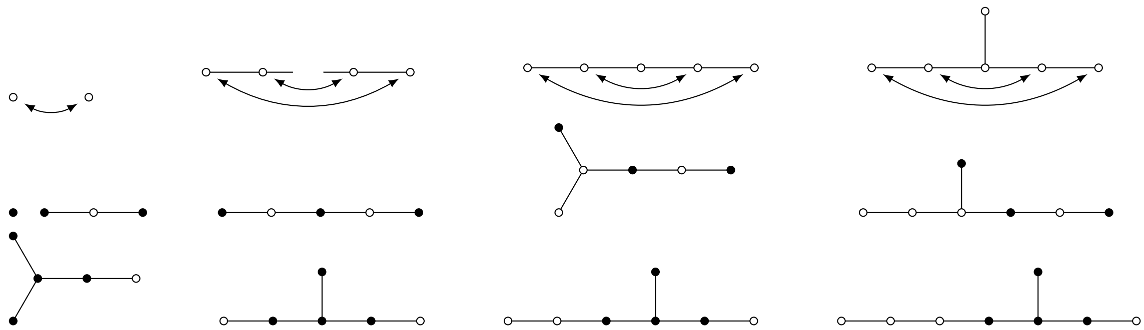
By studying the involutions corresponding to the different nodes in the diagrams as was done in 2.4 one can start to solve the section constraint through the same method used in [16] and explained in section 5.5.1 for the chain of white nodes until one encounters a white node which is connected to a black node or a node which have an arrow pointing on it. These cannot be included in the section because in the case of a white node connected to a black node the involution maps the generators non-trivially which shows that these generators are not real and hence cannot be included. The linear subspace which should be found from the solution to the section constraint should be real. When one tries to extend to a black node one finds that the lowering operator becomes a creation operator under the involution and hence annihilates the states therefore also showing that these are not real generators. Thus one can not include a white node that is connected to a black node or a black node. The case of encountering nodes with arrows pointed on them leads to similar results as the mapping of such a node leads to inclusions of non-real generators. Thus the final rules could be stated as

- The section which is found from starting on a black node or a node with an arrow would be trivial, zero dimensional.
- Starting on a white node one can continue the section through inclusion of neighbouring connected white nodes with the method for solving the section constraint found in [16].
- The section stops if a white node that is connected to a black node is encountered. Note that this white node is not included in the gravity line.
- The section stops if a node with arrows is encountered.
- The section stops if a short node is encountered.
- The section stops if the diagram ends.
- If a multiconnected node is encountered, a choice of branch to follow has to be made and then proceed until one of the above stopping conditions is met.

This concludes the general results and the next subsections is devoted to the examples found from the magical supergravities.

5.5.2.1 Magical supergravities

From table 4.1 one can see which group is the symmetry group and thus can represent them using Satake diagrams, which were discussed in section 2.4, and one finds that the table translates into:



The structure from the table displays that the diagrams are built up in every case by adding a white node on the node which corresponded to the coordinate representation. One should note the R-symmetry node in the quaternion case (left most black node) corresponding to a $SU(2)$ that enters the symmetry group when compactification to $D = 5$.

The coordinate representations for $D = 5$ that was discussed earlier was the representations with the Dynkin label $\lambda = [0, 2],\ [0, 1, 1, 0],\ [0, 1, 0, 0, 0]$ and $[1, 0, 0, 0, 0, 0]$. Now the node which the coordinate representation are found on for the lower D diagrams is always the newly added node. By applying the method for solution to the section constraint one finds the gravity line for the different dimensions and division algebras all give the same $GL(n)$ group where $n = 6 - D$ in which D is the dimension compactified to. This reflects well that the same number of physical coordinates are selected from the section constraint and the number of coordinates agrees with that found from compactification of the magical supergravities.

6

Discussion

Throughout this thesis the subjects of magical supergravities as well as extended geometry has been gradually introduced through Chapter 2 which introduced the needed mathematics which was used to create and tie the structure groups together as well as the necessary tools for decoding the Satake diagrams used to describe said structure groups. In the chapter which followed, chapter 3, the supergravities which are the most commonly encountered were introduced and the different kinds of dualities together with how they appear were discussed as a motivation for their study through a geometrical perspective. That chapter also served as an introduction to the mechanisms of supergravity which was then explored in the subsequent chapter. Chapter 4 was then devoted to explaining the intricacies found in magical supergravities, specifically how the fields which take values in modules of the different internal structure algebras combine upon compactification. In chapter 5 the subject of extended geometry was introduced and later an example of such a theory was discussed in detail. This chapter also presented the generalisations of the exceptional field theories which was presented by Hohm and Samtleben in [17–19] to the appropriate cases which was found from the magical supergravities. The final part of the chapter was devoted to understanding how the section constraint could be solved in the four different cases which could be encountered in a Satake diagram.

When the magical supergravities were studied one saw that if the division algebra was taken to be complex or quaternions one found that in the adjoint representation of the target symmetry group there existed an extra representation module, i.e. **1** and **3** respectively. These two was said to be from a right multiplication by a unit element in the division algebra. However in the case that the division algebra is the reals one would then not find a Lie group but a $\mathbb{Z}_2$. Since this is not a Lie group it would not appear in the table but would have to be included if it exists. The existence of this extra group is only speculated and a more thorough analysis of this has to be conducted in order to determine its existence for sure. The same decomposition also contain the exception with the octonions as well. Since the unit octonions do not form a group by themselves [33] one would not expect that they appear in that form. They are however combined into the adjoint module of $SO(1,9)$ as it can be expressed as 2×2-matrices with octonionic entries.

During the construction of the exceptional field theories for the magical square one saw that the same structure for the action could be applied to each of the different division algebras. This might not be that surprising as the coordinate representations for all of them behaves the same way because of their construction as hermitian matrices, however over different division algebras. Upon solving the section constraint for the different division algebras one found that they all yielded the same dimensional section would be expected as that reflect how many physical coordinates are to be selected and since the same amount of dimensions were compactified they should agree.

When the section constraint was solved for the groups involved in the magical supergravities the quaternionic case resulted in an extra $SU(2)$ group as well as the node which was (partially) included in the solution of the section constraint was the "bridge" between the $SU(2)$ group and the $SO(1,5)$ which was added upon compactification. This broke this so one found that extra R-symmetry. However there is a speculation that in the case of $\mathbb{C}$ one find a $U(1)$ as the section selected by the section constraint partially includes a node with an arrow on it.

This work has in some sense filled in some gap in the understanding of how the solution to section can be constructed even in cases where the real form is not the split one. This lends itself as a tool for further studies of exceptional field theories in cases where the structure group has real form which is not the split real form. This thesis has only worked with the case of global symmetries and constructed the exceptional field theories for them. Therefore it would be interesting to study how this could be done in the case of gauged magical supergravities instead and present an excellent problem that remains to be dealt with.

The form of the magical supergravities that are formulated in $D = 6$ are very similar to what is expected in the case of the E_5 which is the found from the $D = 11$ supergavity compactified to $D = 6$ [34, 35]. One should note that E_5 is not an exceptional groups but is $SO(5,5)$, as it is found by removing a the left most node in $E_{6(6)}$. Much like the cases previously found in this thesis, one has that the octionic case of the magical supergravities in the same dimension is another real form compared to that of the already studied exceptional field theories. However in the case of $SO(1,9)$ one has that the solution to the section constraint is zero-dimensional. It would be interesting to compare the magical supergravities form in $D = 6$ to the $SO(5,5)$-ExFT in more detail.

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