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A space-time cut finite element method for a time-dependent parabolic model problem

Master's Thesis in Engineering Mathematics and Computational Science

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Department of Mathematical Sciences
Chalmers University of Technology
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Abstract

In this thesis, a space-time finite element method for the heat equation on overlapping meshes is presented. Here, overlapping meshes means that we have a stationary mesh of the solution domain with an additional mesh that is allowed to move around in and through the solution domain. The thesis contains a derivation, an analysis, and results from an implementation of the method. The derivation starts with a strong formulation of the problem and ends with a finite element variational formulation together with adequate function spaces. For the finite element solution, we use continuous Galerkin in space and discontinuous Galerkin in time, with the addition of a discontinuity in the solution on the space-time boundary between the two meshes. In the analysis, we propose an a priori error estimate for the method with discontinuous Galerkin of order zero and one, i.e., dG(0) and dG(1). For dG(1), the error estimate indicates that the movement of the additional mesh decreases the order of convergence of the error, with respect to the time step, from the third to the second order, when the speed of the moving mesh is large enough. The order of convergence with respect to the step size for dG(1), as well as the error convergence for dG(0), are unaffected by the moving mesh and are thus as in the case with only a stationary mesh, presented in [2, 3]. An implementation of the method in one spatial dimension, with piecewise linear elements in space, and dG(0) and dG(1) in time, has also been performed. The numerical results of the implementation show the superiority of using dG(1) instead of dG(0) for overlapping meshes. The numerical results also confirm the behaviour of the error convergence, indicated by the a priori error estimate.

Keywords: partial differential equation, finite element method, space-time cut, time-dependent, parabolic problem, heat equation, overlapping mesh, moving mesh, discontinuous Galerkin, a priori.

Contents

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Carl Lundholm
Orust, June 2015

1 Introduction

Let us start by explaining the title of this thesis, *A space-time cut finite element method for a time-dependent parabolic model problem*. The *time-dependent parabolic model problem* considered here is the heat equation. *A space-time finite element method* means that we use the finite element method (FEM) in both space and time to solve this problem numerically as opposed to, e.g., using FEM in space and some numerical difference scheme in time. The word *cut* represents the concept of using overlapping meshes for this setting. Solving time-dependent partial differential equations (PDEs) with a space-time FEM on overlapping meshes is a relatively new concept and this thesis could be viewed as an introduction to the field. Although not necessary, it is good if the reader is familiar with some of the basics of solving a PDE with the FEM, such as going from the strong formulation of the PDE to the finite element variational formulation, which in turn is used to create a linear system of equations that is solved to yield the finite element solution.

1.1 Overlapping meshes

As always when attempting to solve a PDE in some solution domain with the FEM, the domain is discretized into simplices to form a computational mesh. Let us refer to such a mesh as a *background* mesh. Now, consider a mesh that is allowed to move around over the background mesh. Let us refer to such a mesh as a *moving* mesh. By introducing one or more moving meshes over a background mesh, we arrive at the concept of overlapping meshes.

An application for overlapping meshes is when there is an object in the solution domain and this object has a movement relative to the rest of the domain boundary. For example, imagine that we would like to resolve the fluid flow around a rotating propeller, i.e., solving the Navier-Stokes equations. Taking our moving object to be the rotating propeller, the usual way of creating a computational mesh would be to discretize the space between the domain walls and the propeller. An example of this can be seen in the left illustration of Figure 1. If the propeller starts to rotate, the triangles that have nodes in both the solution domain and on the propeller boundary will start to get deformed. This can be seen in the middle illustration of Figure 1. Eventually, these triangles will turn the part of the mesh around the propeller into a mess that has more resemblance to a magpie's nest than a computational grid. This is shown in the right illustration of Figure 1.

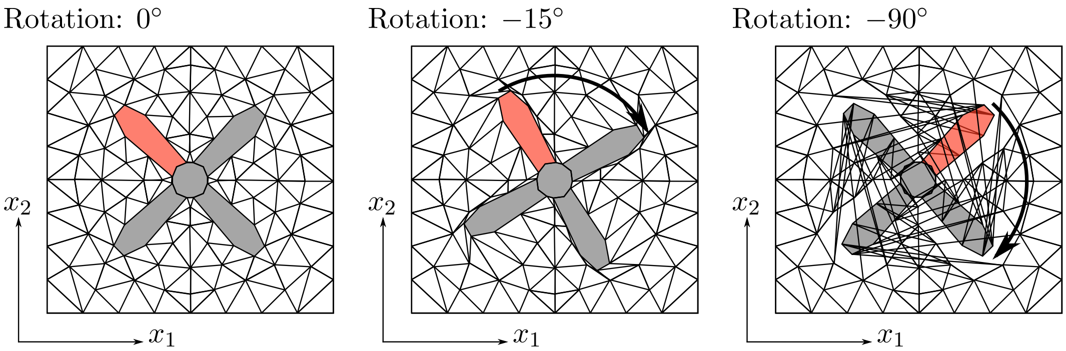

Figure 1: A 2D example of how a moving object can affect the computational mesh of a solution domain when triangles with nodes attached to the object boundary are deformed as a result of the object's movement. The moving object in this example is a rotating propeller with one of its rotor blades coloured differently for reference. *Left*: The initial setting with a propeller in a rectangular domain and a triangle mesh of the solution domain. *Middle*: The propeller is rotated 15 degrees clockwise. Triangles in front of a rotor blade are compressed and triangles behind a rotor blade are streched out. *Right*: The propeller is rotated 90 degrees clockwise and turns the mesh into a mess.

One way of dealing with this inconvenience is to generate a new mesh in the solution domain when the old mesh starts to get messed up. Mesh generation can however be a time consuming procedure, especially in industrial applications, where there are often several dimensions and complicated geometries. A mesh might then have to consist of several million simplices.

By instead using overlapping meshes, a problem such as resolving the fluid flow around a rotating propeller can hopefully be dealt with in a more elegant manner. To apply this concept, one would start by removing the moving object from the solution domain and create the background mesh in the remaining solution domain. The moving object would then be reintroduced into the solution domain, but encapsulated within a moving mesh. An example of this procedure, where the moving object once again is the rotating propeller, can be seen in Figure 2. This transfers the issue of dealing with the object's movement from the object boundary to the joint boundary between the moving mesh and the background mesh.

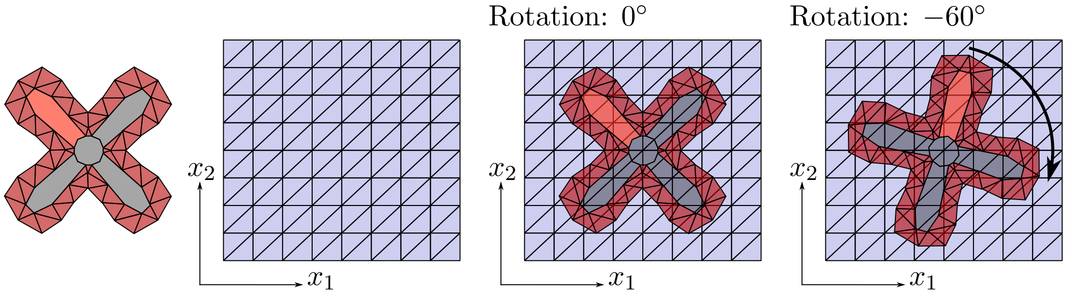

Figure 2: A 2D example of how the concept of overlapping meshes can be applied to handle moving objects in the solution domain. The moving object is again the rotating propeller with one of its rotor blades coloured differently for reference. *Left*: The propeller is removed from the solution domain and encapsulated in a moving mesh with red triangles. The remaining solution domain is triangulated to form a background mesh with light blue triangles. *Middle*: The mesh encapsulated propeller is reintroduced on top of the background mesh. *Right*: The moving mesh with the propeller is rotated 60 degrees clockwise on top of the background mesh.

1.2 Scope and outline of the thesis

One major application for the concept of using FEM for time-dependent PDEs on overlapping meshes is to solve the Navier-Stokes equations in a three-dimensional space domain. However, as previously mentioned, this concept is quite new and is thus in need of research and developed theory. Therefore we will start at the beginning by restricting ourselves to a simpler PDE, the heat equation. This choice of PDE is made since the heat equation is the simplest classical time-dependent PDE. The method developed in this thesis only comprises one moving mesh. We also consider a moving mesh without an encapsulated object, since the focus is shifted from the object boundary to the joint boundary between the meshes when using the concept of overlapping meshes. The notation used in this thesis is for a method in two or three (or more) spatial dimensions, but the translation to one spatial dimension is a trivial matter. For the implementation of the model we have restricted ourselves to a problem in one spatial dimension.

The work of this thesis consists of a derivation, an analysis and an implementation of a space-time cut FEM for the heat equation. The outline of the thesis generally follows the same structure. The derivation of the method and the method itself is presented in Sections 2 – 4. In Section 2, we begin by presenting a strong formulation of the PDE problem. In Section 3, we continue in the usual FEM manner by way of transforming the strong formulation of the problem into a variational formulation. In Section 4, finite element spaces are defined before the finite element variational formulation is finally presented. The analysis of the method spans Sections 5 – 7, and is centred around proving an a priori error estimate. Section 5 contains tools for the proof of the error estimate. In Section 6, stability estimates for the method are presented, which also work as tools for the aforementioned proof. In Section 7, we present the a priori error estimate and its proof. The results from an implementation of the method for one spatial dimension can be found in Section 8. Section 9 consists of conclusions from the results, and in Section 10, some ideas for future work are presented.

2 Problem formulation

Let $\Omega_0 \subset \mathbb{R}^d$, where $d = 1, 2$ or 3, be a bounded domain with boundary $\partial\Omega_0$ and $T > 0$ be a given final time. Let $x \in \mathbb{R}^d$ denote the spatial coordinate and $t \in \mathbb{R}$ denote time. Furthermore, let $G \subset \mathbb{R}^d$ be another bounded domain. The boundary of G is ∂G. The location of G is time-dependent, so G and ∂G are functions of time, i.e., $G = G(t)$ and $\partial G = \partial G(t)$ for $t \in [0, T]$. The movement of G is described by the velocity $\mu : \mathbb{R}^d \times [0, T] \to \mathbb{R}^d$. Let $\Omega_1 = \Omega_0 \setminus G$ and $\Omega_2 = \Omega_0 \cap G$ with boundaries $\partial\Omega_1$ and $\partial\Omega_2$, respectively. For $i = 1, 2$, the set Ω_i and its boundary $\partial\Omega_i$ are functions of time, i.e., $\Omega_i = \Omega_i(t)$ and $\partial\Omega_i = \partial\Omega_i(t)$ for $t \in [0, T]$. Let $\Gamma = \Gamma(t) = \partial\Omega_1(t) \cap \partial\Omega_2(t)$, for $t \in [0, T]$. An illustration of the partition of Ω_0 as a result of G's location is shown in Figure 3.

The time-dependent parabolic model problem considered here is the heat equation on $\Omega_0 \times (0, T]$ with given boundary conditions and initial values. The problem is

$$\begin{cases} \dot{u} - \Delta u = f & \text{in } \Omega_0 \times (0, T], \\ u = 0 & \text{on } \partial\Omega_0 \times [0, T], \\ u = u_0 & \text{in } \Omega_0 \times \{0\}, \end{cases} \tag{2.1}$$

where $\dot{u} = \partial_t u$ and Δ is the Laplace operator. For $t \in [0, T]$, the function $u(\cdot, t) \in H^2(\Omega_0) \cap H_0^1(\Omega_0)$ and the function $f(\cdot, t) \in L_2(\Omega_0)$.

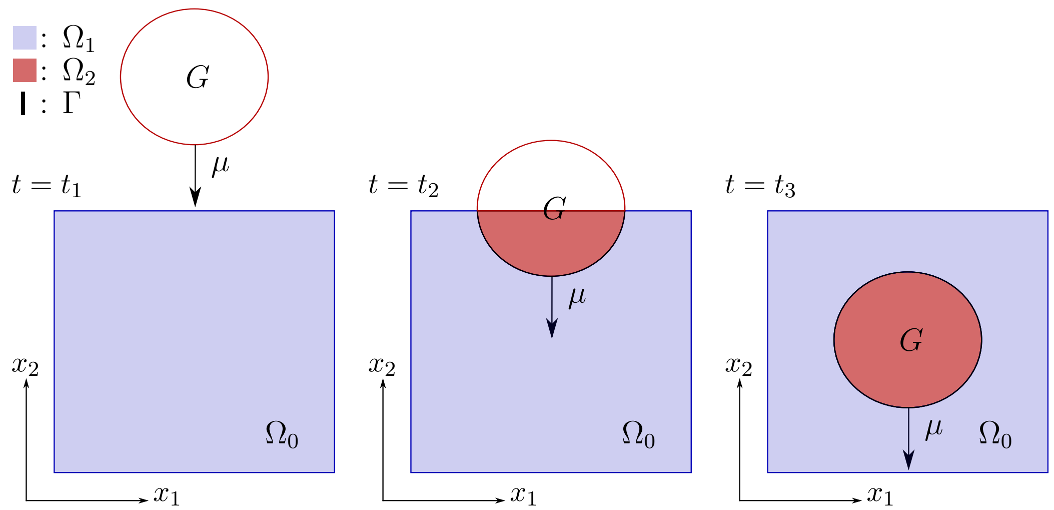

Figure 3: An example of the partition of Ω_0 into Ω_1 (light blue) and Ω_2 (red) for $d = 2$ and three different times $t_1 < t_2 < t_3$, when G is moving with velocity μ.

3 Derivation of the method

3.1 The space-time normal vector $\bar{n}_i$

For $i = 1, 2$, let $\bar{n}_i \in \mathbb{R}^{d+1}$ be the space-time normal vector to the boundary of the space-time volume $\Omega_i \times (t_a, t_b] = \{(x,t) : x \in \Omega_i(t),\ t \in (t_a, t_b]\}$, where t_a and t_b are times such that $0 \leq t_a < t_b \leq T$. We write $\bar{n}_i = (\bar{n}_i^x, \bar{n}_i^t)$, where $\bar{n}_i^x$ denotes the space component and $\bar{n}_i^t$ denotes the time component. The boundary of the space-time volume $\Omega_i \times (t_a, t_b]$ may be partitioned into three parts: the space bottom $\Omega_i(t_a) \times \{t_a\}$, the space top $\Omega_i(t_b) \times \{t_b\}$, and the space-time lateral area $\partial\Omega_i \times (t_a, t_b] = \{(s,t) : s \in \partial\Omega_i(t),\ t \in (t_a, t_b]\}$, where s denotes the spatial coordinate of a boundary element. The space-time normal vector $\bar{n}_i$ to $\Omega_i(t_a) \times \{t_a\}$ and $\Omega_i(t_b) \times \{t_b\}$ is $\bar{n}_i = (\bar{n}_i^x, \bar{n}_i^t) = (\mathbf{0}, -1)$ and $\bar{n}_i = (\bar{n}_i^x, \bar{n}_i^t) = (\mathbf{0}, 1)$, respectively. The space-time lateral area $\partial\Omega_i \times (t_a, t_b]$ may be partitioned further into the two parts $\partial\Omega_i \cap \partial\Omega_0 \times (t_a, t_b]$ and $\Gamma \times (t_a, t_b]$. The space-time normal vector $\bar{n}_i$ to $\partial\Omega_i \cap \partial\Omega_0 \times (t_a, t_b]$ is $\bar{n}_i = (\bar{n}_i^x, \bar{n}_i^t) = (n_i, 0)$, where $n_i \in \mathbb{R}^d$ is the normal vector to $\partial\Omega_i(t)$. The time component is zero since the space coordinates of $\partial\Omega_0$ are fixed in time. The space-time normal vector $\bar{n}_i$ to $\Gamma \times (t_a, t_b]$ can be expressed as a function of n_i and μ, where μ is the velocity of G. To obtain such an expression, consider the two space-time vectors $P = (\mu, 1)$ and $O_i = (n_i, m)$, where m is to be chosen. See Figure 4. Note that P is parallel to $\Gamma \times (t_a, t_b]$ and since $\bar{n}_i$ is orthogonal to $\Gamma \times (t_a, t_b]$, $\bar{n}_i$ is also orthogonal to P. Also note that the space parts of O_i and $\bar{n}_i$ point in the same direction, namely n_i. We want to choose m such that O_i points in the same direction as $\bar{n}_i$, because then it is just to normalize O_i to obtain the desired expression for $\bar{n}_i$. For O_i to point in the same direction as $\bar{n}_i$, O_i and P will have to be orthogonal, i.e.,

$$O_i \cdot P = (n_i, m) \cdot (\mu, 1) = n_i \cdot \mu + m = 0. \tag{3.1}$$

From (3.1), O_i and P are orthogonal if $m = -n_i \cdot \mu$. The desired expression becomes

$$\bar{n}_i = (\bar{n}_i^x, \bar{n}_i^t) = \frac{1}{\sqrt{(n_i \cdot \mu)^2 + 1}} (n_i, -(n_i \cdot \mu)). \tag{3.2}$$

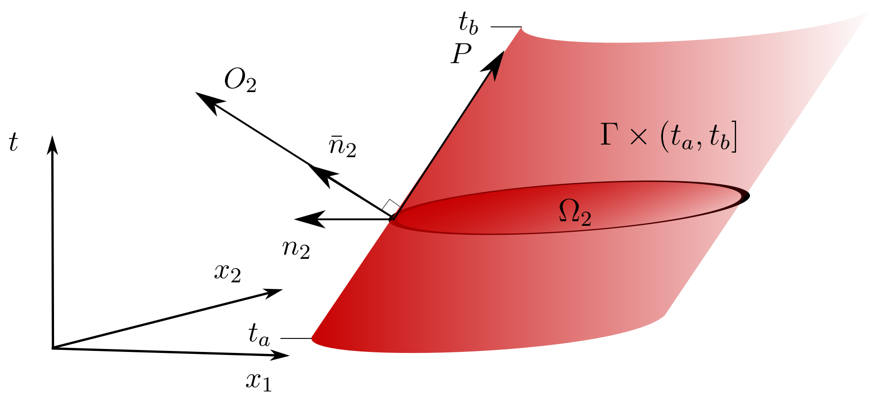

Figure 4: An example, for $d = 2$ and constant μ on $(t_a, t_b]$, of the space-time vectors O_2, P and $\bar{n}_2$ in relation to the space-time surface $\Gamma \times (t_a, t_b]$ shown in red. The space vector n_2 to the boundary of Ω_2, for some time $t \in (t_a, t_b)$, is also present.

3.2 Derivation of the finite element variational formulation

Let V be the function space consisting of all functions v that are zero on $\partial\Omega_0$. The space V is the test space and its elements v are test functions. Multiply the first row in (2.1) with a test function $v \in V$ and integrate over both Ω_0 and $(0, T]$. We have

$$\int_0^T \int_{\Omega_0} (\dot{u} - \Delta u)v \, \mathrm{d}x\mathrm{d}t = \int_0^T \int_{\Omega_0} fv \, \mathrm{d}x\mathrm{d}t, \tag{3.3}$$

for all $v \in V$. Consider the space-time volume $D_0 = \Omega_0 \times (0, T]$. Let $\bar{q} = (-\nabla u, u)$, $\bar{\nabla} = (\nabla, \partial_t)$ and $\bar{x} = (x, t)$. With this new notation, (3.3) becomes

$$\int_{D_0} (\bar{\nabla} \cdot \bar{q})v \, \mathrm{d}\bar{x} = \int_{D_0} fv \, \mathrm{d}\bar{x}. \tag{3.4}$$

Partition the time interval $(0, T]$ into N subintervals $I_n = (t_{n-1}, t_n]$, where $n = 1, \ldots, N$ and $0 = t_0 < t_1 < \ \ldots \ < t_N = T$. For $i = 1, 2$ and $n = 1, \ldots, N$, let the space-time slabs $D_{i,n} = \Omega_i \times I_n = \{(x, t) : x \in \Omega_i(t),\ t \in I_n\}$. See Figure 5. We may then write (3.4) as

$$\sum_{i,n} \int_{D_{i,n}} (\bar{\nabla} \cdot \bar{q})v \, \mathrm{d}\bar{x} = \sum_{i,n} \int_{D_{i,n}} fv \, \mathrm{d}\bar{x}, \tag{3.5}$$

where $\sum_{i,n} = \sum_{i=1}^{2} \sum_{n=1}^{N}$. The right-hand side of (3.5) may be written as

$$\sum_{i,n} \int_{D_{i,n}} fv \, \mathrm{d}\bar{x} = \sum_{i,n} \int_{I_n} (f, v)_{\Omega_i(t)} \, \mathrm{d}t, \tag{3.6}$$

where $(\cdot, \cdot)_{\Omega_i(t)}$ is the $L_2(\Omega_i(t))$-inner product. The boundary of a space-time slab $D_{i,n}$ is $\partial D_{i,n} = \Omega_{i,n-1} \times \{t_{n-1}\} \cup \partial\Omega_i \times I_n \cup \Omega_{i,n} \times \{t_n\}$, where $\Omega_{i,n} = \Omega_i(t_n)$ and $\partial\Omega_i \times I_n = \{(s, t) : s \in \partial\Omega_i(t),\ t \in I_n\}$, where s denotes the spatial coordinate of a boundary element. For $i = 1, 2$, let $\bar{n}_i = (\bar{n}_i^x, \bar{n}_i^t)$ be the space-time normal vector to $\partial D_{i,n}$ with space component $\bar{n}_i^x$ and time component $\bar{n}_i^t$, and let $\bar{s} = (s, t)$. Apply the divergence theorem on the left-hand side of (3.5) to obtain

$$\sum_{i,n} \int_{D_{i,n}} (\bar{\nabla} \cdot \bar{q})v \, \mathrm{d}\bar{x} = \underbrace{\sum_{i,n} \int_{\partial D_{i,n}} \bar{n}_i \cdot \bar{q}v \, \mathrm{d}\bar{s}}_{\text{The boundary integrals}} + \underbrace{\sum_{i,n} \int_{D_{i,n}} -\bar{q} \cdot \bar{\nabla}v \, \mathrm{d}\bar{x}}_{\text{The interior integrals}}. \tag{3.7}$$

Let us consider the boundary integrals and the interior integrals separately.

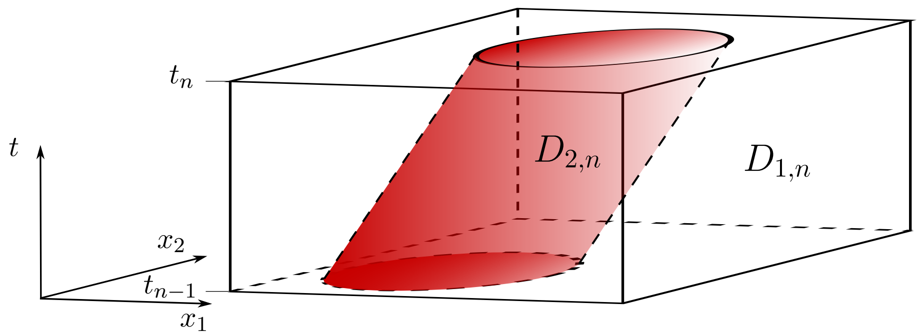

Figure 5: The space-time slabs $D_{1,n}$ and $D_{2,n}$ (red) for $d=2$, for a case when G is immersed in Ω_0 for all $t \in I_n$ and μ is constant on I_n.

3.2.1 The boundary integrals

For $i = 1, 2$ and $n = 1, \ldots, N$, we have the boundary integrals

$$\begin{aligned} \sum_{i,n} \int_{\partial D_{i,n}} \bar{n}_i \cdot \bar{q} v \, \mathrm{d}\bar{s} = & \underbrace{\sum_{i,n} \left(\int_{\Omega_{i,n-1}} \bar{n}_i \cdot \bar{q} v \, \mathrm{d}x + \int_{\Omega_{i,n}} \bar{n}_i \cdot \bar{q} v \, \mathrm{d}x \right)}_{=\mathrm{I}} \\ & + \underbrace{\sum_{i,n} \int_{\partial \Omega_i \times I_n} \bar{n}_i \cdot \bar{q} v \, \mathrm{d}\bar{s}}_{=\mathrm{II}}. \end{aligned} \tag{3.8}$$

Consider I and II in (3.8) separately, starting with I:

$$\begin{aligned} \mathrm{I} &= \sum_{i=1}^{2} \sum_{n=1}^{N} \left(\int_{\Omega_{i,n-1}} (\mathbf{0}, -1) \cdot (-\nabla u, u) v \, \mathrm{d}x + \int_{\Omega_{i,n}} (\mathbf{0}, 1) \cdot (-\nabla u, u) v \, \mathrm{d}x \right) \\ &= \sum_{i=1}^{2} \sum_{n=1}^{N} \left(\int_{\Omega_{i,n-1}} -u_{n-1}^{+} v_{n-1}^{+} \, \mathrm{d}x + \int_{\Omega_{i,n}} u_n^{-} v_n^{-} \, \mathrm{d}x \right) \\ &= \sum_{i=1}^{2} \sum_{n=1}^{N-1} -[(u,v)_{\Omega_{i,n}}]_n - (u_0^{+}, v_0^{+})_{\Omega_{i,0}} + (u_N^{-}, v_N^{-})_{\Omega_{i,N}} \\ &\overset{\text{4th}}{=} \sum_{i=1}^{2} \sum_{n=1}^{N-1} -(u_n^{-}, [v]_n)_{\Omega_{i,n}} - (u_0^{+}, v_0^{+} - v_0^{-})_{\Omega_{i,0}} - (u_N^{-}, v_N^{+} - v_N^{-})_{\Omega_{i,N}} \\ &= \sum_{i=1}^{2} \sum_{n=0}^{N} -(u_n^{-}, [v]_n)_{\Omega_{i,n}}, \end{aligned} \tag{3.9}$$

where $v_n^{\pm} = \lim_{\varepsilon \to +0} v(x, t_n \pm \varepsilon)$ and $[v]_n$ is the jump in v at time t_n, i.e., $[v]_n = v_n^{+} - v_n^{-}$. To obtain the fourth equality in (3.9), consider the jump in $(u,v)_{\Omega_{i,n}}$ at time t_n:

$$
\begin{aligned}
[(u,v)_{\Omega_{i,n}}]_n &= \int_{\Omega_{i,n}} [uv]_n \, \mathrm{d}x = \int_{\Omega_{i,n}} u_n^+ v_n^+ - u_n^- v_n^- \underbrace{- u_n^- v_n^+ + u_n^- v_n^+}_{=0} \, \mathrm{d}x \\
&= \int_{\Omega_{i,n}} \underbrace{[u]_n}_{=0} v_n^+ + u_n^- [v]_n \, \mathrm{d}x = (u_n^-, [v]_n)_{\Omega_{i,n}}.
\end{aligned}
\tag{3.10}
$$

To obtain the last equality we have used $[u]_n = 0$, since we want u to be continuous in time. With the insertion of (3.10), $v_0^- := 0$ and $v_N^+ := 0$ in the third row of (3.9), the fourth equality in (3.9) is obtained. We leave I and consider II:

$$
\begin{aligned}
\mathrm{II} &= \sum_{i,n} \int_{\partial\Omega_i \times I_n} \bar{n}_i \cdot \bar{q} v \, \mathrm{d}\bar{s} = \sum_{i,n} \int_{\partial\Omega_i \cap \partial\Omega_0 \times I_n} \bar{n}_i \cdot \bar{q} v \, \mathrm{d}\bar{s} + \sum_{i,n} \int_{\Gamma_n} \bar{n}_i \cdot \bar{q} v \, \mathrm{d}\bar{s} \\
&= \sum_{i,n} \int_{\Gamma_n} \bar{n}_i \cdot \bar{q} v \, \mathrm{d}\bar{s} = \sum_{n=1}^{N} \int_{\Gamma_n} \bar{n}_1 \cdot \bar{q}_1 v_1 + \bar{n}_2 \cdot \bar{q}_2 v_2 \, \mathrm{d}\bar{s} \overset{\text{5th}}{=} \sum_{n=1}^{N} \int_{\Gamma_n} \bar{n} \cdot [\bar{q} v] \, \mathrm{d}\bar{s} \\
&= \sum_{n=1}^{N} \int_{\Gamma_n} (\bar{n}^x, \bar{n}^t) \cdot ([-\nabla u v], [uv]) \, \mathrm{d}\bar{s} = \sum_{n=1}^{N} \int_{\Gamma_n} -\bar{n}^x \cdot [\nabla u v] + \bar{n}^t [uv] \, \mathrm{d}\bar{s} \\
&= \sum_{n=1}^{N} - \int_{\Gamma_n} [(\partial_{\bar{n}^x} u) v] \, \mathrm{d}\bar{s} + \sum_{n=1}^{N} \int_{\Gamma_n} \bar{n}^t [uv] \, \mathrm{d}\bar{s},
\end{aligned}
\tag{3.11}
$$

where $\Gamma_n = \Gamma \times I_n = \{(s,t) : s \in \Gamma(t),\ t \in I_n\}$, $v_i = \lim_{\varepsilon \to +0} v(\bar{s} - \varepsilon \bar{n}_i)$, and $[v]$ is the jump in v over Γ_n, i.e., $[v] = v_1 - v_2$, and $\partial_{\bar{n}^x} u = \bar{n}^x \cdot \nabla u$. The first sum on the right-hand side in the first row of (3.11) is zero, since the test function v vanishes on $\partial\Omega_0$. To obtain the fifth equality we let $\bar{n} = \bar{n}_1$ on Γ_n. Since $\bar{n}_2 = -\bar{n}_1$ on Γ_n, $\bar{n}_2 = -\bar{n}$ on Γ_n. The space component of $\bar{n}$ is denoted $\bar{n}^x$ and $\bar{n}^x = \bar{n}_1^x$. The time component is denoted $\bar{n}^t$ and $\bar{n}^t = \bar{n}_1^t$. Apply the identity (A.1) from Appendix A to the first term in the last row of (3.11) to obtain

$$
\begin{aligned}
\sum_{n=1}^{N} - \int_{\Gamma_n} [(\partial_{\bar{n}^x} u) v] \, \mathrm{d}\bar{s} &= \sum_{n=1}^{N} - \int_{\Gamma_n} \underbrace{[\partial_{\bar{n}^x} u]}_{=0} \langle v \rangle + \langle \partial_{\bar{n}^x} u \rangle [v] + (\omega_2 - \omega_1) \underbrace{[\partial_{\bar{n}^x} u]}_{=0} [v] \, \mathrm{d}\bar{s} \\
& \sum_{n=1}^{N} - \int_{\Gamma_n} \langle \partial_{\bar{n}^x} u \rangle [v] \, \mathrm{d}\bar{s} \\
&= \sum_{n=1}^{N} \int_{\Gamma_n} - \langle \partial_{\bar{n}^x} u \rangle [v] - \langle \partial_{\bar{n}^x} v \rangle [u] \, \mathrm{d}\bar{s} + S_h(u,v),
\end{aligned}
\tag{3.12}
$$

where $\langle v \rangle = \omega_1 v_1 + \omega_2 v_2$, for $\omega_1,\, \omega_2 \in \mathbb{R}$ and $\omega_1 + \omega_2 = 1$. We want u to be continuous in both space and time and $\partial_{\bar{n}^x} u$ to be continuous in space so to obtain the second equality we take $[u] = 0$ and $[\partial_{\bar{n}^x} u] = 0$. We also add $\langle \partial_{\bar{n}^x} v \rangle [u]$ for symmetry and the symmetric bilinear form S_h, satisfying $S_h(u,v) = 0$, for stability. The explicit expression for the stabilization term is presented and motivated in Section 4.3, since S_h contains some properties that have not yet been discussed. So for now we stick with $S_h(u,v)$. The last term in the last row of (3.11) is

$$
\begin{aligned}
\sum_{n=1}^{N}\int_{\Gamma_n}\bar{n}^t[uv]\,\mathrm{d}\bar{s} &= \sum_{n=1}^{N}\int_{\Gamma_n}\bar{n}^t(u_1v_1-u_2v_2\underbrace{-u_1v_2+u_1v_2}_{=0})\,\mathrm{d}\bar{s}\\
&= \sum_{n=1}^{N}\int_{\Gamma_n}\bar{n}^t(u_1[v]+\underbrace{[u]}_{=0}v_2)\,\mathrm{d}\bar{s}\\
&= \sum_{n=1}^{N}\int_{\Gamma_n}\bar{n}^t u_1[v]\,\mathrm{d}\bar{s} = \sum_{n=1}^{N}\int_{\Gamma_n}\bar{n}^t u_2[v]\,\mathrm{d}\bar{s}.
\end{aligned}
\tag{3.13}
$$

Since we want u to be continuous in both space and time, we use $[u]=0$ to obtain the penultimate equality. To get the last equality we note that instead of adding $u_1v_2-u_1v_2=0$ in the first row, we could add $u_2v_1-u_2v_1=0$. The latter addition changes the index of u in the last row from 1 to 2. With the insertion of (3.12) and (3.13) in (3.11), we obtain

$$
\mathrm{II} = \sum_{n=1}^{N}\left(\int_{\Gamma_n}-\langle\partial_{\bar{n}^x}u\rangle[v]-\langle\partial_{\bar{n}^x}v\rangle[u]\,\mathrm{d}\bar{s}+\int_{\Gamma_n}\bar{n}^t u_1[v]\,\mathrm{d}\bar{s}\right)+S_h(u,v). \tag{3.14}
$$

With the insertion of (3.9) and (3.14) in (3.8), the main result of this subsubsection is obtained. It reads

$$
\begin{aligned}
\sum_{i,n}\int_{\partial D_{i,n}}\bar{n}_i\cdot\bar{q}v\,\mathrm{d}\bar{s} &= \sum_{i=1}^{2}\sum_{n=0}^{N}-(u_n^-,[v]_n)_{\Omega_{i,n}}\\
&+\sum_{n=1}^{N}\left(\int_{\Gamma_n}\bar{n}^t u_1[v]-\langle\partial_{\bar{n}^x}u\rangle[v]-\langle\partial_{\bar{n}^x}v\rangle[u]\,\mathrm{d}\bar{s}\right)+S_h(u,v).
\end{aligned}
\tag{3.15}
$$

3.2.2 The interior integrals

For $i=1,2$ and $n=1,\ldots,N$, we have the interior integrals

$$
\begin{aligned}
\sum_{i,n}\int_{D_{i,n}}-\bar{q}\cdot\bar{\nabla}v\,\mathrm{d}\bar{x} &= \sum_{i,n}\int_{D_{i,n}}-(-\nabla u,u)\cdot(\nabla v,\dot{v})\,\mathrm{d}\bar{x} = \sum_{i,n}\int_{D_{i,n}}\nabla u\cdot\nabla v-u\dot{v}\,\mathrm{d}\bar{x}\\
&= \sum_{i,n}\int_{D_{i,n}}\nabla u\cdot\nabla v\,\mathrm{d}\bar{x}+\sum_{i,n}-\int_{D_{i,n}}u\dot{v}\,\mathrm{d}\bar{x}.
\end{aligned}
\tag{3.16}
$$

The first term in the second row of (3.16) is

$$
\sum_{i,n}\int_{D_{i,n}}\nabla u\cdot\nabla v\,\mathrm{d}\bar{x} = \sum_{i,n}\int_{I_n}(\nabla u,\nabla v)_{\Omega_i(t)}\,\mathrm{d}t. \tag{3.17}
$$

Let e_t be the unit vector in time, i.e., $e_t=(\mathbf{0},1)$. The last term in the second row of (3.16) is

$$
\begin{aligned}
\sum_{i,n} -\int_{D_{i,n}} u\dot{v}\,\mathrm{d}\bar{x} &= \sum_{i,n} -\int_{D_{i,n}} ue_t \cdot \bar{\nabla} v\,\mathrm{d}\bar{x} \\
&= \sum_{i,n} \int_{D_{i,n}} (\bar{\nabla}\cdot ue_t) v\,\mathrm{d}\bar{x} + \sum_{i,n} -\int_{\partial D_{i,n}} \bar{n}_i \cdot ue_t v\,\mathrm{d}\bar{s},
\end{aligned}
\tag{3.18}
$$

where the divergence theorem has been applied to obtain the last equality. The first term in the second row of (3.18) is

$$
\begin{aligned}
\sum_{i,n} \int_{D_{i,n}} (\bar{\nabla}\cdot ue_t) v\,\mathrm{d}\bar{x} &= \sum_{i,n} \int_{D_{i,n}} \Big((\nabla, \partial_t)\cdot(\mathbf{0}, u)\Big) v\,\mathrm{d}\bar{x} = \sum_{i,n} \int_{D_{i,n}} \dot{u} v\,\mathrm{d}\bar{x} \\
&= \sum_{i,n} \int_{I_n} (\dot{u}, v)_{\Omega_i(t)}\,\mathrm{d}t.
\end{aligned}
\tag{3.19}
$$

The second term in the second row of (3.18) is

$$
\begin{aligned}
&\sum_{i,n} -\int_{\partial D_{i,n}} \bar{n}_i \cdot ue_t v\,\mathrm{d}\bar{s} = \sum_{i,n} -\int_{\partial D_{i,n}} (\bar{n}_i^x, \bar{n}_i^t)\cdot(\mathbf{0}, uv)\,\mathrm{d}\bar{s} = \sum_{i,n} -\int_{\partial D_{i,n}} \bar{n}_i^t uv\,\mathrm{d}\bar{s} \\
&= \underbrace{\sum_{i,n} \Big(-\int_{\Omega_{i,n-1}} -u_{n-1}^+ v_{n-1}^+\,\mathrm{d}x - \int_{\Omega_{i,n}} u_n^- v_n^-\,\mathrm{d}x \Big)}_{=\,\mathrm{I}} + \underbrace{\sum_{i,n} -\int_{\partial\Omega_i \times I_n} \bar{n}_i^t uv\,\mathrm{d}\bar{s}.}_{=\,\mathrm{II}}
\end{aligned}
\tag{3.20}
$$

Consider I and II in (3.20) separately, starting with I:

$$
\begin{aligned}
\mathrm{I} &= \sum_{i,n} \Big(-\int_{\partial\Omega_{i,n-1}} -u_{n-1}^+ v_{n-1}^+\,\mathrm{d}x - \int_{\partial\Omega_{i,n}} u_n^- v_n^-\,\mathrm{d}x \Big) \\
&= \sum_{i,n} \Big(-(u_n^-, v_n^-)_{\Omega_{i,n}} + (u_{n-1}^+, v_{n-1}^+)_{\Omega_{i,n}} \Big).
\end{aligned}
\tag{3.21}
$$

We leave I like this and consider II:

$$
\begin{aligned}
\mathrm{II} &= \sum_{i,n} -\int_{\partial\Omega_i \times I_n} \bar{n}_i^t uv\,\mathrm{d}\bar{s} = \sum_{i,n} -\int_{\partial\Omega_i \cap \partial\Omega_0 \times I_n} \bar{n}_i^t uv\,\mathrm{d}\bar{s} + \sum_{i,n} -\int_{\Gamma_n} \bar{n}_i^t uv\,\mathrm{d}\bar{s} \\
&= \sum_{i,n} -\int_{\Gamma_n} \bar{n}_i^t uv\,\mathrm{d}\bar{s} = \sum_{n=1}^{N} -\int_{\Gamma_n} \bar{n}_1^t u_1 v_1 + \bar{n}_2^t u_2 v_2\,\mathrm{d}\bar{s} = \sum_{n=1}^{N} -\int_{\Gamma_n} \bar{n}^t [uv]\,\mathrm{d}\bar{s},
\end{aligned}
\tag{3.22}
$$

where the first term on the right-hand side of the first row is zero, since both $\bar{n}_i^t = 0$ and $v = 0$ on $\partial\Omega_0$. To obtain the last equality, we use $\bar{n} = \bar{n}_1$ on Γ_n, which gives $\bar{n}_1^t = \bar{n}^t$ and $\bar{n}_2^t = -\bar{n}^t$ on Γ_n, since $\bar{n}_2 = -\bar{n}_1$ on Γ_n. With the insertion of (3.17) and (3.18) in (3.16), where (3.19) and (3.20) have been inserted in (3.18), and where (3.21) and (3.22) have been inserted in (3.20), the main result of this subsubsection is obtained. It reads

$$
\begin{aligned}
\sum_{i,n}\int_{D_{i,n}} -\bar{q}\cdot\bar{\nabla}v\,\mathrm{d}\bar{x} =& \sum_{i=1}^{2}\sum_{n=1}^{N}\int_{I_n}(\nabla u,\nabla v)_{\Omega_i(t)} + (\dot{u},v)_{\Omega_i(t)}\,\mathrm{d}t \\
&+\sum_{i=1}^{2}\sum_{n=1}^{N}\Big(-(u_n^-,v_n^-)_{\Omega_{i,n}} + (u_{n-1}^+,v_{n-1}^+)_{\Omega_{i,n-1}}\Big) \\
&+\sum_{n=1}^{N} -\int_{\Gamma_n}\bar{n}^t[uv]\,\mathrm{d}\bar{s}.
\end{aligned}
\tag{3.23}
$$

3.2.3 Combining terms from the boundary and the interior integrals

Add the first term on the right-hand side of (3.15) to the second term on the right-hand side of (3.23) to obtain

$$
\begin{aligned}
&\sum_{i=1}^{2}\sum_{n=1}^{N}\Big(-(u_n^-,v_n^-)_{\Omega_{i,n}} + (u_{n-1}^+,v_{n-1}^+)_{\Omega_{i,n-1}}\Big) + \sum_{i=1}^{2}\sum_{n=0}^{N} -(u_n^-,[v]_n)_{\Omega_{i,n}} \\
=&\sum_{i=1}^{2}\sum_{n=1}^{N}\Big(-(u_n^-,v_n^-)_{\Omega_{i,n}} + (u_{n-1}^+,v_{n-1}^+)_{\Omega_{i,n-1}} - (u_n^-,v_n^+)_{\Omega_{i,n}} + (u_n^-,v_n^-)_{\Omega_{i,n}}\Big) \\
&+\sum_{i=1}^{2}\Big(-(u_0^-,v_0^+)_{\Omega_{i,0}} + (u_0^-,v_0^-)_{\Omega_{i,0}}\Big) \\
=&\sum_{i=1}^{2}\sum_{n=1}^{N}\Big((u_{n-1}^+ - u_{n-1}^-,v_{n-1}^+)_{\Omega_{i,n-1}} \underbrace{+(u_n^-,v_n^-)_{\Omega_{i,n}} - (u_n^-,v_n^-)_{\Omega_{i,n}}}_{=0}\Big) \\
&+\sum_{i=1}^{2}\Big((u_0^-,v_0^-)_{\Omega_{i,0}} - (u_N^-,v_N^+)_{\Omega_{i,N}}\Big) = \sum_{i=1}^{2}\sum_{n=1}^{N}([u]_{n-1},v_{n-1}^+)_{\Omega_{i,n-1}},
\end{aligned}
\tag{3.24}
$$

where we have used $v_0^- = v_N^+ = 0$ to obtain the last equality and $u_0^- := u_0$, where u_0 is the right-hand side in the last row of (2.1). Add the first integral term in the last row of (3.15) to the last term on the right-hand side of (3.23) to obtain

$$
\begin{aligned}
&\sum_{n=1}^{N} -\int_{\Gamma_n}\bar{n}^t[uv]\,\mathrm{d}\bar{s} + \sum_{n=1}^{N}\int_{\Gamma_n}\bar{n}^t u_1[v]\,\mathrm{d}\bar{s} = \sum_{n=1}^{N}\int_{\Gamma_n}\bar{n}^t(-[uv]+u_1[v])\,\mathrm{d}\bar{s} \\
=&\sum_{n=1}^{N}\int_{\Gamma_n}\bar{n}^t(-u_1v_1 + u_2v_2 + u_1v_1 - u_1v_2)\,\mathrm{d}\bar{s} \\
=&\sum_{n=1}^{N}\int_{\Gamma_n}\bar{n}^t(-(u_1-u_2)v_2 \underbrace{+u_1v_1 - u_1v_1}_{=0})\,\mathrm{d}\bar{s} = \sum_{n=1}^{N}\int_{\Gamma_n} -\bar{n}^t[u]v_2\,\mathrm{d}\bar{s}.
\end{aligned}
\tag{3.25}
$$

Recall from (3.13) that u_1 is interchangable with u_2 in the second term in the first row of (3.25). With u_2 instead of u_1 in the second term in the first row of (3.25), v_2 changes to v_1 in the last row of (3.25). We thus have

$$\sum_{n=1}^{N} - \int_{\Gamma_n} \bar{n}^t [uv] \, \mathrm{d}\bar{s} + \sum_{n=1}^{N} \int_{\Gamma_n} \bar{n}^t u_\rho [v] \, \mathrm{d}\bar{s} = \sum_{n=1}^{N} \int_{\Gamma_n} -\bar{n}^t [u] v_\sigma \, \mathrm{d}\bar{s}, \tag{3.26}$$

where $\rho, \sigma \in \{1, 2\}$, and $\sigma \neq \rho$. We want the integral on the right-hand side of (3.26) to behave analogously to the time jump term $([u]_{n-1}, v_{n-1}^+)_{\Omega_{i,n-1}}$. To obtain such a behaviour, consider the problem described in Section 2 in one spatial dimension and with G immersed in Ω_0. The moving set G is then an interval that can move to the left and right on the interval Ω_0. For $\mu > 0$, G moves to the right and vice versa. Let $\mu > 0$ on some time interval I_n. Since $\bar{n}^t = \bar{n}_1^t$, $\bar{n}^t < 0$ on the part of Γ_n that is to the left of G and $\bar{n}^t > 0$ on the part of Γ_n that is to the right of G. See Figure 6. This will make $-\bar{n}^t[u] = |\bar{n}^t|(u_1 - u_2)$ to the left of G and $-\bar{n}^t[u] = |\bar{n}^t|(u_2 - u_1)$ to the right of G. This means that the actual jump in u over Γ_n will be $u|_{\Gamma_n}^+ - u|_{\Gamma_n}^-$. Finally by letting $\sigma = \frac{1}{2}(3 + \mathrm{sgn}(\bar{n}^t))$, where sgn is the sign function, we get $v_\sigma = v|_{\Gamma_n}^+$ and the desired analogous behaviour.

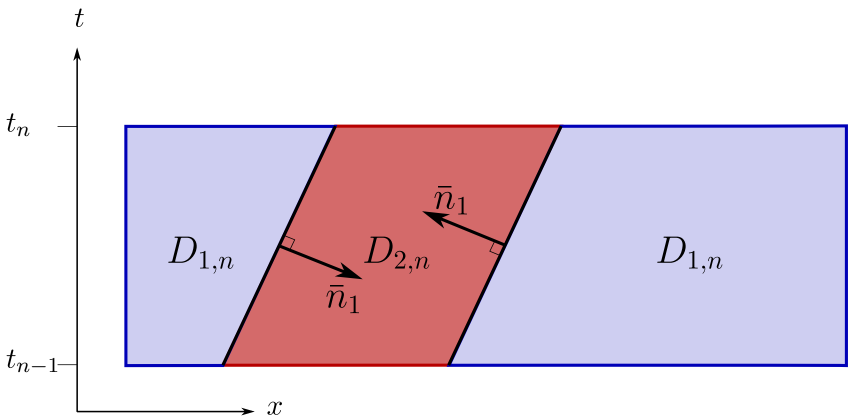

Figure 6: An example, for $d = 1$, of the space-time vector $\bar{n}_1$ on Γ_n (black skewed lines) for a case when G is immersed in Ω_0, for all $t \in I_n$, and $\mu > 0$ is constant on I_n. The space-time slab $D_{1,n}$ is light blue, and $D_{2,n}$ is red.

To obtain the main result of Section 3.2, which will be used in the equation for the finite element variational formulation for the problem described in Section 2, we insert (3.6) and (3.7) in (3.5), where (3.15) and (3.23) have been inserted in (3.7). Finally we apply (3.24) and (3.26). We thus have

$$\begin{aligned} &\sum_{i=1}^{2}\sum_{n=1}^{N}\left(\int_{I_n}(\dot{u},v)_{\Omega_i(t)}\,\mathrm{d}t + ([u]_{n-1},v_{n-1}^{+})_{\Omega_{i,n-1}}\right) \\ &+\sum_{i=1}^{2}\sum_{n=1}^{N}\int_{I_n}(\nabla u,\nabla v)_{\Omega_i(t)}\,\mathrm{d}t \\ &+\sum_{n=1}^{N}\left(\int_{\Gamma_n} -\bar{n}^t[u]v_\sigma - \langle\partial_{\bar{n}^x}u\rangle[v] - \langle\partial_{\bar{n}^x}v\rangle[u]\,\mathrm{d}\bar{s}\right) + S_h(u,v) \\ =&\sum_{i=1}^{2}\sum_{n=1}^{N}\int_{I_n}(f,v)_{\Omega_i(t)}\,\mathrm{d}t. \end{aligned} \tag{3.27}$$

4 Formulation of the method

4.1 The meshes $\mathcal{T}_0$ and $\mathcal{T}_G$

Let the meshes $\mathcal{T}_0$ and $\mathcal{T}_G$ be tessellations of Ω_0 and G, respectively. Both meshes are assumed to consist of simplices. We let M_0 denote the number of interior nodes in $\mathcal{T}_0$, and M_G the total number of interior *and boundary* nodes in $\mathcal{T}_G$. Let $\{\varphi_{0,j}\}_{j=1}^{M_0}$ be the set of polynomial *interior* nodal basis functions of degree $\leq p$ for $\mathcal{T}_0$. Let $\{\varphi_{G,j}\}_{j=1}^{M_G}$ be the set of polynomial nodal basis functions of degree $\leq p$ for $\mathcal{T}_G$. Note that $\mathcal{T}_G$'s basis functions depend on time as well as space, since the set G moves around with velocity μ.

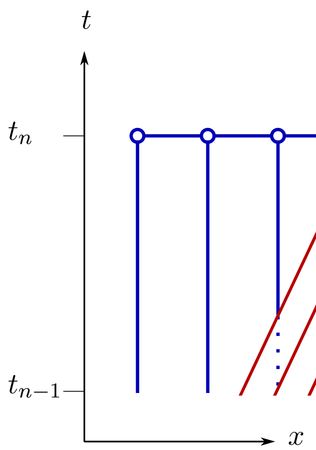

Figure 7: An example of a space-time mesh for I_n for $d = 1$ when μ is positive. At time $t = t_n$, the nodes of the blue background mesh $\mathcal{T}_0$ are marked with circles and the nodes of the red moving mesh $\mathcal{T}_G$ with crosses. The blue vertical lines are thus the nodal trajectories of $\mathcal{T}_0$ and the red skewed vertical lines those of $\mathcal{T}_G$.

4.2 Finite element spaces

The semi-discrete spaces $V_h(t)$ and $V_h(I_n)$

For $t \in (0, T]$, we define the semi-discrete finite element spaces $V_{h,0}$ and $V_{h,G}$ as the spaces of functions that are zero on $\partial\Omega_0$, and continuous piecewise polynomials of degree $\leq p$ on $\mathcal{T}_0$ and $\mathcal{T}_G$, respectively. For $t \in (0, T]$,

$$V_{h,0} := V_{h,p,0} := \left\{ v : v(x,t) = \sum_{j=1}^{M_0} V_j(t)\varphi_{0,j}(x), V_j : (0,T] \to \mathbb{R}, \forall j \right\}, \tag{4.1}$$

$$V_{h,G} := V_{h,p,G} := \left\{ v : v(x,t) = \sum_{j=1}^{M_G} V_j(t)\varphi_{G,j}(x,t), V_j : (0,T] \to \mathbb{R}, \forall j, v|_{\partial\Omega_0} = 0 \right\}. \tag{4.2}$$

We now use these two spaces to create another finite element space. Define the broken finite element space $V_h(t)$ as the space of functions that on $\Omega_1(t)$ is a restriction of functions in $V_{h,0}$ to $\Omega_1(t)$, and on $\Omega_2(t)$, is a restriction of functions in $V_{h,G}$ to $\Omega_2(t)$. For $t \in (0, T]$,

$$V_h(t) := V_{h,p}(t) := \{v : v|_{\Omega_1(t)} = v_0|_{\Omega_1(t)}, \text{ for some } v_0 \in V_{h,0} \text{ and } v|_{\Omega_2(t)} = v_G|_{\Omega_2(t)}, \text{ for some } v_G \in V_{h,G}\}. \tag{4.3}$$

Now we define the space $V_h(I_n)$ as the space of functions that lie in $V_h(t)$, for all $t \in I_n$. For $n = 1, \ldots, N$,

$$V_h(I_n) := V_{h,p}(I_n) := \{v : v(t) \in V_h(t), \forall t \in I_n\}. \tag{4.4}$$

With a general and somewhat relaxed notation, any $v \in V_h(I_n)$ can be represented as

$$v(x,t) = \sum_j V_j(t)\varphi_j(x,t), \tag{4.5}$$

where the φ_j's belong to both $\{\varphi_{0,j}\}_{j=1}^{M_0}$ and $\{\varphi_{G,j}\}_{j=1}^{M_G}$, and the only restriction on the nodal coefficients V_j is that $V_j(t) \in \mathbb{R}$ for all $t \in (0,T]$.

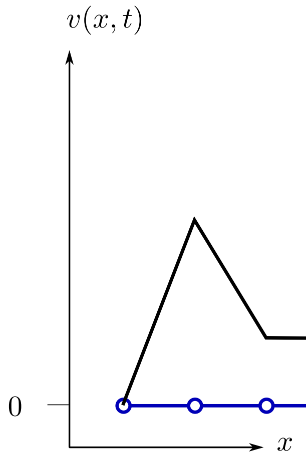

Figure 8: An example of $v(x,t)$ versus x in one spatial dimension, where $v(\cdot,t) \in V_h(t)$, $p = 1$, and time $t \in (0,T]$. The nodes of the blue background mesh $\mathcal{T}_0$ are marked with circles and the nodes of the red moving mesh $\mathcal{T}_G$ with crosses.

The fully discrete spaces V_h^n and V_h

Now we consider a subspace of $V_h(I_n)$, which consists of functions whose nodal coefficients have a polynomial time dependence of degree q or lower. Analogously with the procedure of defining $V_h(t)$, we define two other auxiliary finite element spaces. For $n = 1, \ldots, N$, let $V_{h,0}^n$ and $V_{h,G}^n$ be the spaces of functions that are zero on $\partial\Omega_0$, continuous piecewise polynomials of degree $\leq p$ on $\mathcal{T}_0$ and $\mathcal{T}_G$ for all $t \in I_n$, respectively, and polynomials of degree $\leq q$ in time along the nodal trajectories of both $\mathcal{T}_0$ and $\mathcal{T}_G$ for $t \in I_n$. For $n = 1, \ldots, N$,

$$V_{h,0}^{n} := V_{h,p,0}^{n,q} := \left\{ v : v(x,t) = \sum_{j=1}^{M_0} V_j(t)\varphi_{0,j}(x), V_j \in \mathcal{P}^q(I_n), \forall j \right\}, \tag{4.6}$$

$$V_{h,G}^{n} := V_{h,p,G}^{n,q} := \left\{ v : v(x,t) = \sum_{j=1}^{M_G} V_j(t)\varphi_{G,j}(x,t), V_j \in \mathcal{P}^q(I_n), \forall j, v|_{\partial\Omega_0} = 0 \right\}, \tag{4.7}$$

where $\mathcal{P}^q(I_n)$ is the space of all polynomials of degree $\leq q$ on I_n. We now use these two spaces to create another finite element space. Define the broken finite element space V_h^n as the space of functions that on $D_{1,n}$, is a restriction of functions in $V_{h,0}^n$ to $D_{1,n}$, and on $D_{2,n}$, is a restriction of functions in $V_{h,G}^n$ to $D_{2,n}$. For $n = 1, \ldots, N$,

$$\begin{aligned} V_h^n := V_{h,p}^{n,q} := \{ v : v|_{D_{1,n}} = v_0^n|_{D_{1,n}}, \text{ for some } v_0^n \in V_{h,0}^n \text{ and} \\ v|_{D_{2,n}} = v_G^n|_{D_{2,n}}, \text{ for some } v_G^n \in V_{h,G}^n \}. \end{aligned} \tag{4.8}$$

Finally, we define the finite element space V_h as the space of functions that lie in V_h^n for $n = 1, \ldots, N$.

$$V_h := V_{h,p}^{q} := \{ v : v|_{D_{0,n}} \in V_h^n, \text{ for } n = 1, \ldots, N \}. \tag{4.9}$$

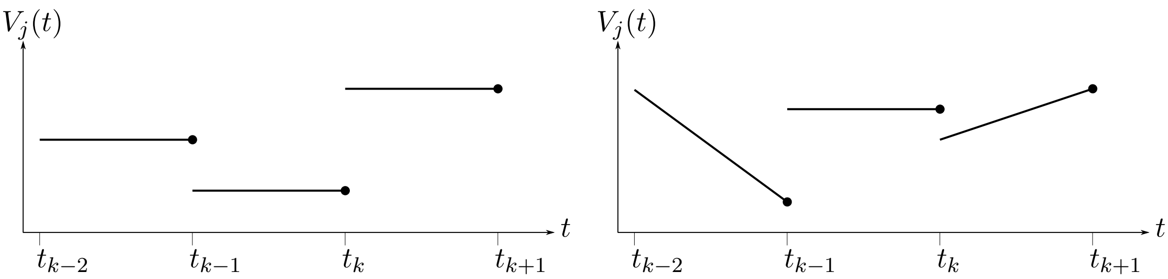

Figure 9: Examples of $V_j(t)$ versus t on three subsequent time subintervals $I_n = (t_{n-1}, t_n]$, for $V_j \in \mathcal{P}^q(I_n)$. *Left*: $q = 0$, i.e., V_j is constant on each I_n. *Right*: $q = 1$, i.e., V_j is at most linear on each I_n.

The spaces $\dot{V}_h^n$ and $\dot{V}_h$

Let $\dot{V}_h^n$ and $\dot{V}_h$ denote the spaces of functions that are time derivatives of functions in V_h^n and V_h, respectively. To define these spaces, we again start with two auxiliary finite element spaces. First consider a function $v_0^n \in V_{h,0}^n$. Such a function can be written as $v_0^n(x,t) = \sum_{j=1}^{M_0} V_j(t)\varphi_{0,j}(x)$. Thus, the time derivative $\dot{v}_0^n$ may be written as $\dot{v}_0^n(x,t) = \sum_{j=1}^{M_0} \dot{V}_j(t)\varphi_{0,j}(x)$, where $\dot{V}_j \in \mathcal{P}^{q-1}(I_n)$, since $V_j \in \mathcal{P}^q(I_n)$. For $n = 1, \ldots, N$, we define $\dot{V}_{h,0}^n$ to be the space of functions that are zero on $\partial\Omega_0$, *continuous* piecewise polynomials of degree $\leq p$ on $\mathcal{T}_0$ for all $t \in I_n$, and polynomials of degree $\leq q-1$ in time for $t \in I_n$. For $n = 1, \ldots, N$,

$$\dot{V}_{h,0}^{n} := \dot{V}_{h,p,0}^{n,q} := \left\{ v : v(x,t) = \sum_{j=1}^{M_0} V_j(t)\varphi_{0,j}(x), V_j \in \mathcal{P}^{q-1}(I_n), \forall j \right\}. \tag{4.10}$$

Note that $\dot{V}^{n,q}_{h,p,0} = V^{n,q-1}_{h,p,0}$. Second, consider a function $v^n_G \in V^n_{h,G}$. Such a function can be written as $v^n_G(x,t) = \sum_{j=1}^{M_G} V_j(t)\varphi_{G,j}(x,t)$. Note that $\varphi_{G,j}$ is also a function of time. The time derivative $\dot{v}^n_G$ is therefore

$$\dot{v}^n_G(x,t) = \sum_{j=1}^{M_G} \Big(\dot{V}_j(t)\varphi_{G,j}(x,t) + V_j(t)\dot{\varphi}_{G,j}(x,t) \Big), \tag{4.11}$$

where $\dot{V}_j \in \mathcal{P}^{q-1}(I_n)$ and $V_j \in \mathcal{P}^q(I_n)$. It is important to note that the extra terms $V_j(t)\dot{\varphi}_{G,j}(x,t)$ make $\dot{v}^n_G$ discontinuous on the edges between simplices in $\mathcal{T}_G$, when $|\mu| > 0$. This is so since the $\dot{\varphi}_{G,j}$'s are discontinuous on these edges when $|\mu| > 0$. Also note that this discontinuity vanishes when $|\mu| = 0$, since then the $\varphi_{G,j}$'s are no longer time-dependent. For $n = 1, \ldots, N$, we therefore define $\dot{V}^n_{h,G}$ to be a subspace of the space of functions that are zero on $\partial\Omega_0$, *discontinuous* piecewise polynomials of degree $\leq p$ on $\mathcal{T}_G$ for all $t \in I_n$ and polynomials of degree $\leq q - 1 + \mathrm{sgn}(|\mu|)$ in time along the nodal trajectories of $\mathcal{T}_G$ for $t \in I_n$, where sgn is the sign function. For $n = 1, \ldots, N$,

$$\begin{aligned} \dot{V}^n_{h,G} := \dot{V}^{n,q}_{h,p,G} := \Big\{ v : v(x,t) = \sum_{j=1}^{M_G} \Big(\dot{V}_j(t)\varphi_{G,j}(x,t) + V_j(t)\dot{\varphi}_{G,j}(x,t) \Big), \\ \dot{V}_j \in \mathcal{P}^{q-1}(I_n), V_j \in \mathcal{P}^q(I_n), \forall j, v|_{\partial\Omega_0} = 0 \Big\}, \end{aligned} \tag{4.12}$$

For $n = 1, \ldots, N$, we now define the broken finite element space $\dot{V}^n_h$ as the space of functions that on $D_{1,n}$, is a restriction of functions in $\dot{V}^n_{h,0}$ to $D_{1,n}$, and on $D_{2,n}$, is a restriction of functions in $\dot{V}^n_{h,G}$ to $D_{2,n}$. For $n = 1, \ldots, N$,

$$\begin{aligned} \dot{V}^n_h := \dot{V}^{n,q}_{h,p} := \{ v : v|_{D_{1,n}} = \dot{v}^n_0|_{D_{1,n}}, \text{ for some } \dot{v}^n_0 \in \dot{V}^n_{h,0} \text{ and} \\ v|_{D_{2,n}} = \dot{v}^n_G|_{D_{2,n}}, \text{ for some } \dot{v}^n_G \in \dot{V}^n_{h,G} \}. \end{aligned} \tag{4.13}$$

Finally, we define the finite element space $\dot{V}_h$ as the space of functions that lie in $\dot{V}^n_h$ for $n = 1, \ldots, N$.

$$\dot{V}_h := \dot{V}^q_{h,p} := \{ v : v|_{D_{0,n}} \in \dot{V}^n_h, \text{ for } n = 1, \ldots, N \}. \tag{4.14}$$

The spaces $\delta_t V_h(I_n)$ and $\delta_t V^n_h$

Recall the relaxed representation of a function $v \in V_h(I_n)$, given by (4.5). For $n = 1, \ldots, N$ and for $v \in \{v : v(x,t) = \sum_j V_j(t)\varphi(x,t), V_j \in C^1(I_n), \forall j\}$, we define the coefficient time differential operator δ_t, by

$$\delta_t v(x,t) = \sum_j \dot{V}_j(t)\varphi(x,t). \tag{4.15}$$

We define the space $\delta_t V_h(I_n)$ by $\delta_t : \{v : v(x,t) = \sum_j V_j(t)\varphi(x,t), V_j \in C^1(I_n), \forall j\} \to \delta_t V_h(I_n)$. The subspace $\delta_t V^n_h$ of $\delta_t V_h(I_n)$ may be defined in an analogous way by $\delta_t : V^n_h \to \delta_t V^n_h$, but here we define it in a more rigorous manner. For $n = 1, \ldots, N$, let $\delta_t V^n_h$ denote the subspace of V_h^n, where the time dependence of the polynomial nodal coefficients is of one degree lower than in V_h^n. Recall the definitions of the spaces $V_{h,p,0}^{n,q}$ and $V_{h,p,G}^{n,q}$, given by (4.6) and (4.7), respectively. The spaces $V_{h,p,0}^{n,q-1}$ and $V_{h,p,G}^{n,q-1}$ are defined in the same way, but with $q-1$ instead of q. For $n = 1, \dots, N$, we now define the broken finite element space $\delta_t V_h^n$ as the space of functions that on $D_{1,n}$, is a restriction of functions in $V_{h,p,0}^{n,q-1}$ to $D_{1,n}$, and on $D_{2,n}$, is a restriction of functions in $V_{h,p,G}^{n,q-1}$ to $D_{2,n}$. For $n = 1, \dots, N$,

$$\begin{aligned}\delta_t V_h^n := \delta_t V_{h,p}^{n,q} := \{v : v|_{D_{1,n}} = v_0^n|_{D_{1,n}},\ \text{for some } v_0^n \in V_{h,p,0}^{n,q-1} \text{ and} \\ v|_{D_{2,n}} = v_G^n|_{D_{2,n}},\ \text{for some } v_G^n \in V_{h,p,G}^{n,q-1}\}.\end{aligned} \tag{4.16}$$

A partition of functions in $\dot{V}_h^n$

Recall that the functions $\dot{v} \in \dot{V}_h^n$ are the time derivatives of functions $v \in V_h^n$, and the relaxed representation of a function in $V_h(I_n)$, given by (4.5). With this in mind and since $V_h^n \subset V_h(I_n)$, we may partition $\dot{v} \in \dot{V}_h^n$ as follows:

$$\begin{aligned}\dot{v} =& \frac{\partial}{\partial t}\Big(\sum_j V_j(t)\varphi_j(x,t)\Big) = \sum_j \Big(\dot{V}_j(t)\varphi_j(x,t) + V_j(t)\dot{\varphi}_j(x,t)\Big) \\ =& \sum_j \dot{V}_j(t)\varphi_j(x,t) + \sum_j V_j(t)\dot{\varphi}_j(x,t) \\ =& \sum_j \dot{V}_j(t)\varphi_j(x,t) + \sum_j V_j(t)\big(-\hat{\mu}\cdot\nabla\varphi_j(x,t)\big) \\ =& \ \delta_t v - \hat{\mu}\cdot\nabla v,\end{aligned} \tag{4.17}$$

where we have applied (A.3) from Lemma A.2 to obtain the penultimate equality, and used the definition of the coefficient differential operator δ_t, given by (4.15), in the last equality.

4.3 The stabilization term S_h

Here we will give an explicit expression for the symmetric stabilization bilinear form S_h, introduced in Section 3.2. Following the notation for overlapping meshes proposed by Massing, Larson, Logg and Rognes [10], we define the subset $\mathcal{T}_{0,\Gamma}(t)$ of the tessellation $\mathcal{T}_0$, for $t \in [0,T]$, by

$$\mathcal{T}_{0,\Gamma}(t) := \{K \in \mathcal{T}_0 : |K \cap \Omega_i(t)| > 0,\ i = 1,2\}, \tag{4.18}$$

where K denotes a simplex and $|\cdot|$ the Lebesgue measure. The set $\mathcal{T}_{0,\Gamma}(t)$ consists of all the simplices in $\mathcal{T}_0$ that are cut by $\Gamma(t)$. We also define the set $\Omega_O(t)$, for $t \in [0,T]$, by

$$\Omega_O(t) := \mathcal{T}_{0,\Gamma}(t) \cap \Omega_2(t). \tag{4.19}$$

The index O is short for overlap. For a simplex $K \in \mathcal{T}_{0,\Gamma}(t)$, the part $|K \cap \Omega_1(t)|$ can become arbitrarily small. This can cause the gap in the model's finite element solution on $\Gamma(t)$ to become very big, resulting in the model producing unstable and inaccurate solutions. The purpose of S_h is thus to handle the discontinuity on $\Gamma(t)$ in such a way that the model is hopefully prevented from producing unreliable solutions. We define the symmetric stabilization bilinear form S_h, for $n = 1, \ldots, N$, by

$$S_h(w,v) := \int_{\Gamma_n} |\bar{n}^x| \gamma h_K^{-1} [w][v] \, \mathrm{d}\bar{s} + \int_{I_n} |\bar{n}^x| (\lambda [\nabla w], [\nabla v])_{\Omega_O(t)} \, \mathrm{d}t, \tag{4.20}$$

where $|\bar{n}^x|$ is the absolute value of the space component of the space-time vector $\bar{n}$, the stabilization parameters $\gamma, \lambda \geq 0$, and $h_K := h_K(x) := h_{K_0}$ for $x \in K_0$, where h_{K_0} is the diameter of simplex $K_0 \in \mathcal{T}_0$. The purpose of the first stabilization term is to reduce the gap in a function on $\Gamma(t)$. The second stabilization term tries to smooth out the transition of a function between the two meshes $\mathcal{T}_0$ and $\mathcal{T}_G$, by penalizing the presence of a kink in a function on $\Gamma(t)$.

4.4 Finite element variational formulation

Consider the problem described in Section 2 where the time interval $(0, T]$ has been partitioned into N subintervals $I_n = (t_{n-1}, t_n]$, where $0 = t_0 < t_1 < \ \ldots \ < t_N = T$ and $n = 1, \ldots, N$. With (3.27), the finite element variational formulation for this problem is: Find $u_h \in V_h$ such that

$$\begin{aligned}
&\sum_{i=1}^{2}\sum_{n=1}^{N}\Big(\int_{I_n} (\dot{u}_h, v)_{\Omega_i(t)} \, \mathrm{d}t + ([u_h]_{n-1}, v_{n-1}^+)_{\Omega_{i,n-1}}\Big) \\
&+ \sum_{i=1}^{2}\sum_{n=1}^{N}\int_{I_n} (\nabla u_h, \nabla v)_{\Omega_i(t)} \, \mathrm{d}t \\
&+ \sum_{n=1}^{N}\Big(\int_{\Gamma_n} -\bar{n}^t [u_h] v_\sigma - \langle \partial_{\bar{n}^x} u_h \rangle [v] - \langle \partial_{\bar{n}^x} v \rangle [u_h] + |\bar{n}^x| \gamma h_K^{-1} [u_h][v] \, \mathrm{d}\bar{s}\Big) \\
&+ \sum_{n=1}^{N}\int_{I_n} |\bar{n}^x| (\lambda [\nabla u_h], [\nabla v])_{\Omega_O(t)} \, \mathrm{d}t \\
=& \sum_{i=1}^{2}\sum_{n=1}^{N}\int_{I_n} (f, v)_{\Omega_i(t)} \, \mathrm{d}t,
\end{aligned} \tag{4.21}$$

for all $v \in V_h$, where V_h is defined by (4.9), $(\cdot,\cdot)_{\Omega_i(t)}$ is the $L_2(\Omega_i(t))$-inner product, $[v]_n$ is the jump in v at time t_n, i.e., $[v]_n = v_n^+ - v_n^-$, $v_n^\pm = \lim_{\varepsilon \to +0} v(x, t_n \pm \varepsilon)$, $\Omega_{i,n} = \Omega_i(t_n)$, $\Gamma_n = \Gamma \times I_n = \{(s,t) : s \in \Gamma(t), t \in I_n\}$, where s denotes the spatial coordinate of a boundary element, $\bar{n}$ is the space-time normal vector to Γ_n with space and time components $\bar{n}^x$ and $\bar{n}^t$, respectively, $[v]$ is the jump in v over Γ_n, i.e., $[v] = v_1 - v_2$, $v_i = \lim_{\varepsilon \to +0} v(\bar{s} - \varepsilon \bar{n}_i)$, $\bar{s} = (s,t)$, if $\bar{n} = \bar{n}_1$, then $\sigma = \frac{1}{2}(3 + \mathrm{sgn}(\bar{n}^t))$ and if $\bar{n} = \bar{n}_2$, then $\sigma = \frac{1}{2}(3 - \mathrm{sgn}(\bar{n}^t))$, where sgn is the sign function, $\langle v \rangle = \omega_1 v_1 + \omega_2 v_2$, where $\omega_1, \omega_2 \in \mathbb{R}$ and $\omega_1 + \omega_2 = 1$, $\partial_n v = n \cdot \nabla v$, $\gamma \geq 0$ is a stabilization parameter, $h_K = h_K(x) = h_{K_0}$ for $x \in K_0$, where h_{K_0} is the diameter of simplex $K_0 \in \mathcal{T}_0$, $\lambda \geq 0$ is a stabilization parameter, and the domain $\Omega_O(t)$ is defined by (4.19).

5 Analytical preliminaries

In this section, we present necessary tools for the stability analysis and the a priori error analysis.

5.1 The bilinear form $A_{h,t}$

Define the symmetric bilinear form $A_{h,t}$ by

$$\begin{aligned} A_{h,t}(w,v) := &\sum_{i=1}^{2}(\nabla w, \nabla v)_{\Omega_i(t)} - (\langle \partial_{\bar{n}^x} w\rangle, [v])_{\Gamma(t)} - (\langle \partial_{\bar{n}^x} v\rangle, [w])_{\Gamma(t)} \\ &+ |\bar{n}^x|(\gamma h_K^{-1}[w],[v])_{\Gamma(t)} + |\bar{n}^x|(\lambda[\nabla w],[\nabla v])_{\Omega_O(t)}, \end{aligned} \tag{5.1}$$

where $(w,v)_{\Gamma(t)}$ is the $L_2(\Gamma(t))$-inner product. For $n = 1,\dots,N$ and two functions w and v, we write

$$\int_{\Gamma_n} wv \,\mathrm{d}\bar{s} = \int_{I_n} (w,v)_{\Gamma(t)} \,\mathrm{d}t. \tag{5.2}$$

Using (5.2) and the bilinear form $A_{h,t}$, we may write the finite element variational formulation (4.21) as: Find $u_h \in V_h$ such that

$$\begin{aligned} &\sum_{i=1}^{2}\sum_{n=1}^{N}\int_{I_n}(\dot{u}_h, v)_{\Omega_i(t)}\,\mathrm{d}t + \sum_{n=1}^{N}\int_{I_n} A_{h,t}(u_h,v)\,\mathrm{d}t \\ &+ \sum_{i=1}^{2}\sum_{n=1}^{N}([u_h]_{n-1}, v_{n-1}^{+})_{\Omega_{i,n-1}} + \sum_{n=1}^{N}\int_{\Gamma_n} -\bar{n}^t[u_h]v_\sigma\,\mathrm{d}\bar{s} \\ =&\sum_{i=1}^{2}\sum_{n=1}^{N}\int_{I_n}(f,v)_{\Omega_i(t)}\,\mathrm{d}t, \end{aligned} \tag{5.3}$$

for all $v \in V_h$. With $\mathcal{T}_{0,\Gamma}(t)$, defined by (4.18), and $\Gamma_K(t) := \Gamma_{K_0}(t) = K_0 \cap \Gamma(t)$, we define the following two mesh dependent norms:

$$\begin{aligned} \|w\|^2_{1/2,h,\Gamma(t)} &:= \sum_{K\in\mathcal{T}_{0,\Gamma}(t)} h_K^{-1}\|w\|^2_{\Gamma_K(t)}, \\ \|w\|^2_{-1/2,h,\Gamma(t)} &:= \sum_{K\in\mathcal{T}_{0,\Gamma}(t)} h_K\|w\|^2_{\Gamma_K(t)}. \end{aligned} \tag{5.4}$$

Note that

$$\begin{aligned} (w,v)_{\Gamma(t)} &= \int_{\Gamma(t)} (h_K^{1/2}w)(h_K^{-1/2}v)\,\mathrm{d}s \le \left(\int_{\Gamma(t)} h_K w^2\,\mathrm{d}s\right)^{1/2}\left(\int_{\Gamma(t)} h_K^{-1} v^2\,\mathrm{d}s\right)^{1/2} \\ &= \left(\sum_{K\in\mathcal{T}_{0,\Gamma}(t)} h_K\int_{\Gamma_K(t)} w^2\,\mathrm{d}s\right)^{1/2}\left(\sum_{K\in\mathcal{T}_{0,\Gamma}(t)} h_K^{-1}\int_{\Gamma_K(t)} v^2\,\mathrm{d}s\right)^{1/2} \\ &= \|w\|_{-1/2,h,\Gamma(t)}\|v\|_{1/2,h,\Gamma(t)}, \end{aligned} \tag{5.5}$$

and

$$\|w\|^2_{1/2,h,\Gamma(t)} = \sum_{K\in\mathcal{T}_{0,\Gamma}(t)} h_K^{-1}\|w\|^2_{\Gamma_K(t)} \geq h^{-1} \sum_{K\in\mathcal{T}_{0,\Gamma}(t)} \|w\|^2_{\Gamma_K(t)} = h^{-1}\|w\|^2_{\Gamma(t)}, \tag{5.6}$$

where $h = \max_{K_l\in\mathcal{T}_0\cup\mathcal{T}_G}(h_{K_l})$. With H^1 denoting the Sobolev space $W^{1,2}$, and the two mesh dependent norms, we define the norm $|||\cdot|||_t$ of a function $w \in H^1(\Omega_1(t),\Omega_2(t)) = \{w : w|_{\Omega_1(t)} \in H^1(\Omega_1(t)),\ w|_{\Omega_2(t)} \in H^1(\Omega_2(t))\}$ by

$$|||w|||_t^2 := \sum_{i=1}^{2} \|\nabla w\|^2_{\Omega_i(t)} + \|\langle\partial_{\bar{n}^x} w\rangle\|^2_{-1/2,h,\Gamma(t)} + \|[w]\|^2_{1/2,h,\Gamma(t)} + \|[\nabla w]\|^2_{\Omega_O(t)}. \tag{5.7}$$

We are now ready to state the following lemma on the coercivity of $A_{h,t}$ on $V_h(t) \subset H^1(\Omega_1(t),\Omega_2(t))$ with respect to $|||\cdot|||_t$.

Lemma 5.1 (Coercivity of $A_{h,t}$)**.** *Let the bilinear form $A_{h,t}$ and the norm $|||\cdot|||_t$ be defined by* (5.1) *and* (5.7)*, respectively. For $t \in (0,T]$, there is a constant $\alpha > 0$ such that*

$$A_{h,t}(v,v) \geq \alpha|||v|||_t^2, \quad \forall v \in V_h(t). \tag{5.8}$$

Proof. Following the proof of Hansbo and Hansbo [5], we start by inserting $v \in V_h(t)$ into $A_{h,t}$:

$$\begin{aligned} A_{h,t}(v,v) =& \sum_{i=1}^{2}(\nabla v,\nabla v)_{\Omega_i(t)} - (\langle\partial_{\bar{n}^x} v\rangle,[v])_{\Gamma(t)} - (\langle\partial_{\bar{n}^x} v\rangle,[v])_{\Gamma(t)} \\ &+ |\bar{n}^x|(\gamma h_K^{-1}[v],[v])_{\Gamma(t)} + |\bar{n}^x|(\lambda[\nabla v],[\nabla v])_{\Omega_O(t)} \\ =& \sum_{i=1}^{2} \|\nabla v\|^2_{\Omega_i(t)} - 2(\langle\partial_{\bar{n}^x} v\rangle,[v])_{\Gamma(t)} + |\bar{n}^x|\gamma\|[v]\|^2_{1/2,h,\Gamma(t)} + |\bar{n}^x|\lambda\|[\nabla v]\|^2_{\Omega_O(t)}. \end{aligned} \tag{5.9}$$

The second term in the last row of (5.9) with opposite sign is

$$\begin{aligned} 2(\langle\partial_{\bar{n}^x} v\rangle,[v])_{\Gamma(t)} &\leq 2\|\langle\partial_{\bar{n}^x} v\rangle\|_{-1/2,h,\Gamma(t)}\|[v]\|_{1/2,h,\Gamma(t)} \\ &\leq \frac{1}{\varepsilon}\|\langle\partial_{\bar{n}^x} v\rangle\|^2_{-1/2,h,\Gamma(t)} + \varepsilon\|[v]\|^2_{1/2,h,\Gamma(t)} \\ &= \frac{2}{\varepsilon}\|\langle\partial_{\bar{n}^x} v\rangle\|^2_{-1/2,h,\Gamma(t)} - \frac{1}{\varepsilon}\|\langle\partial_{\bar{n}^x} v\rangle\|^2_{-1/2,h,\Gamma(t)} + \varepsilon\|[v]\|^2_{1/2,h,\Gamma(t)} \\ &\leq \frac{2C_I}{\varepsilon}\sum_{i=1}^{2}\|\nabla v\|^2_{\Omega_i(t)} - \frac{1}{\varepsilon}\|\langle\partial_{\bar{n}^x} v\rangle\|^2_{-1/2,h,\Gamma(t)} + \varepsilon\|[v]\|^2_{1/2,h,\Gamma(t)}, \end{aligned} \tag{5.10}$$

where $\varepsilon > 0$ is to be chosen and $C_I > 0$. We have used (5.5) to obtain the first inequality. To obtain the last inequality, we have used the inverse inequality from Lemma A.3. Inserting (5.10) in (5.9), gives

$$
\begin{aligned}
A_{h,t}(v,v) \geq& \sum_{i=1}^{2} \|\nabla v\|^2_{\Omega_i(t)} - \frac{2C_I}{\varepsilon}\sum_{i=1}^{2} \|\nabla v\|^2_{\Omega_i(t)} + \frac{1}{\varepsilon}\|\langle\partial_{\bar{n}^x} v\rangle\|^2_{-1/2,h,\Gamma(t)} \\
&- \varepsilon\|[v]\|^2_{1/2,h,\Gamma(t)} + |\bar{n}^x|\gamma\|[v]\|^2_{1/2,h,\Gamma(t)} + |\bar{n}^x|\lambda\|[\nabla v]\|^2_{\Omega_O(t)} \\
=&\left(1 - \frac{2C_I}{\varepsilon}\right)\sum_{i=1}^{2} \|\nabla v\|^2_{\Omega_i(t)} + \frac{1}{\varepsilon}\|\langle\partial_{\bar{n}^x} v\rangle\|^2_{-1/2,h,\Gamma(t)} \\
&+ (|\bar{n}^x|\gamma - \varepsilon)\|[v]\|^2_{1/2,h,\Gamma(t)} + |\bar{n}^x|\lambda\|[\nabla v]\|^2_{\Omega_O(t)}.
\end{aligned}
\tag{5.11}
$$

By taking $\varepsilon > 2C_I$, e.g. $\varepsilon = 4C_I$, and $\gamma > \varepsilon/|\bar{n}^x|$ we may obtain (5.8) from (5.11).

□

Note that by using (5.6) in (5.11), we have

$$
\begin{aligned}
A_{h,t}(v,v) \geq& \left(1 - \frac{2C_I}{\varepsilon}\right)\sum_{i=1}^{2} \|\nabla v\|^2_{\Omega_i(t)} + \frac{1}{\varepsilon}\|\langle\partial_{\bar{n}^x} v\rangle\|^2_{-1/2,h,\Gamma(t)} \\
&+ \left(\frac{|\bar{n}^x|\gamma - \epsilon}{h}\right)\|[v]\|^2_{\Gamma(t)} + |\bar{n}^x|\lambda\|[\nabla v]\|^2_{\Omega_O(t)}.
\end{aligned}
\tag{5.12}
$$

Remark. *The identity (5.12) will be used later in Corollary 6.2.*

5.2 The bilinear form B_h

Define the bilinear form B_h by

$$
\begin{aligned}
B_h(w,v) :=& \sum_{i=1}^{2}\sum_{n=1}^{N}\int_{I_n} (\dot{w}, v)_{\Omega_i(t)}\,\mathrm{d}t + \sum_{n=1}^{N}\int_{I_n} A_{h,t}(w,v)\,\mathrm{d}t \\
&+ \sum_{i=1}^{2}\sum_{n=1}^{N-1}([w]_n, v_n^+)_{\Omega_{i,n}} + \sum_{i=1}^{2}(w_0^+, v_0^+)_{\Omega_{i,0}} + \sum_{n=1}^{N}\int_{\Gamma_n} -\bar{n}^t[w]v_\sigma\,\mathrm{d}\bar{s}.
\end{aligned}
\tag{5.13}
$$

We may then write (4.21) in compact form as: Find $u_h \in V_h$ such that

$$
B_h(u_h, v) = \sum_{i=1}^{2}(u_0, v_0^+)_{\Omega_i(0)} + \sum_{i=1}^{2}\int_0^T (f,v)_{\Omega_i(t)}\,\mathrm{d}t, \tag{5.14}
$$

for all $v \in V_h$. By partially integrating the first term in (5.13), the bilinear form B_h can be expressed differently, as noted in the following Lemma:

Lemma 5.2 (Partial integration w.r.t. time in B_h)**.** *The bilinear form B_h, defined in* (5.13)*, can be written as*

$$
\begin{aligned}
B_h(w,v) = &\sum_{i=1}^{2}\sum_{n=1}^{N}\int_{I_n}(w,-\dot{v})_{\Omega_i(t)}\,\mathrm{d}t + \sum_{n=1}^{N}\int_{I_n} A_{h,t}(w,v)\,\mathrm{d}t \\
&+\sum_{i=1}^{2}\sum_{n=1}^{N-1}(w_n^-,-[v]_n)_{\Omega_{i,n}} + \sum_{i=1}^{2}(w_N^-,v_N^-)_{\Omega_{i,N}} + \sum_{n=1}^{N}\int_{\Gamma_n}\bar{n}^t w_\rho [v]\,\mathrm{d}\bar{s},
\end{aligned} \tag{5.15}
$$

where $\rho = \frac{1}{2}(3 - \mathrm{sgn}(\bar{n}^t))$.

Proof. With $\sum_{i,n} = \sum_{i=1}^{2}\sum_{n=1}^{N}$, the first term in (5.13) is

$$
\begin{aligned}
\sum_{i,n}\int_{I_n}(\dot{w},v)_{\Omega_i(t)}\,\mathrm{d}t &= \sum_{i,n}\int_{D_{i,n}}\dot{w}v\,\mathrm{d}\bar{x} \\
&= \sum_{i,n}\int_{D_{i,n}} -w\dot{v}\,\mathrm{d}\bar{x} + \sum_{i,n}\int_{\partial D_{i,n}}\bar{n}^t wv\,\mathrm{d}\bar{s},
\end{aligned} \tag{5.16}
$$

where $D_{i,n} = \Omega_i \times I_n$, and $\partial D_{i,n}$ is the boundary of $D_{i,n}$. To obtain the second equality in (5.16), we have used the divergence theorem on $\int_{D_{i,n}}\bar{\nabla}\cdot(wve_t)\,\mathrm{d}\bar{x}$, where $\bar{\nabla} = (\nabla, \partial t)$ and $e_t = (\mathbf{0}, 1)$, i.e., the unit vector in time. The first term in the second row of (5.16) is

$$
\sum_{i,n}\int_{D_{i,n}} -w\dot{v}\,\mathrm{d}\bar{x} = \sum_{i,n}\int_{I_n}(w,-\dot{v})_{\Omega_i(t)}\,\mathrm{d}t. \tag{5.17}
$$

The second term in the second row of (5.16) is

$$
\sum_{i,n}\int_{\partial D_{i,n}}\bar{n}^t wv\,\mathrm{d}\bar{s} = \underbrace{\sum_{i,n}\left(\int_{\Omega_{i,n-1}}\bar{n}^t wv\,\mathrm{d}\bar{s} + \int_{\Omega_{i,n}}\bar{n}^t wv\,\mathrm{d}\bar{s}\right)}_{=\,\mathrm{I}} + \underbrace{\sum_{i,n}\int_{\partial\Omega_i\times I_n}\bar{n}^t wv\,\mathrm{d}\bar{s}}_{=\,\mathrm{II}}. \tag{5.18}
$$

Consider I and II in (5.18) separately, starting with I:

$$
\begin{aligned}
\mathrm{I} &= \sum_{i,n}\left(\int_{\Omega_{i,n-1}}\bar{n}^t wv\,\mathrm{d}\bar{s} + \int_{\Omega_{i,n}}\bar{n}^t wv\,\mathrm{d}\bar{s}\right) = \sum_{i,n}\left(\int_{\Omega_{i,n-1}} -wv\,\mathrm{d}\bar{s} + \int_{\Omega_{i,n}} wv\,\mathrm{d}\bar{s}\right) \\
&= \sum_{i,n}\left((w_n^-,v_n^-)_{\Omega_{i,n}} - (w_{n-1}^+,v_{n-1}^+)_{\Omega_{i,n-1}}\right).
\end{aligned} \tag{5.19}
$$

We leave I like this and consider II:

$$
\begin{aligned}
\mathrm{II} &= \sum_{i,n}\int_{\partial\Omega_i\times I_n}\bar{n}^t wv\,\mathrm{d}\bar{s} = \sum_{i,n}\int_{\partial\Omega_i\cap\partial\Omega_0\times I_n}\bar{n}^t wv\,\mathrm{d}\bar{s} + \sum_{i,n}\int_{\Gamma_n}\bar{n}^t wv\,\mathrm{d}\bar{s} \\
&= \sum_{i,n}\int_{\Gamma_n}\bar{n}^t wv\,\mathrm{d}\bar{s} = \sum_{n=1}^{N}\int_{\Gamma_n}\bar{n}_1^t w_1 v_1 + \bar{n}_2^t w_2 v_2\,\mathrm{d}\bar{s} = \sum_{n=1}^{N}\int_{\Gamma_n}\bar{n}^t[wv]\,\mathrm{d}\bar{s},
\end{aligned} \tag{5.20}
$$

where we have used $\bar{n}^t = 0$ on $\partial\Omega_0$ to get the third equality, and $\bar{n} = \bar{n}_1$ and $[v] = v_1 - v_2$ to obtain the last equality. Inserting (5.17) and (5.18) in (5.16), where (5.19) and (5.20) have been inserted in (5.18), we have

$$\begin{aligned}\sum_{i,n} \int_{I_n} (\dot{w}, v)_{\Omega_i(t)}\,\mathrm{d}t = &\sum_{i,n} \int_{I_n} (w, -\dot{v})_{\Omega_i(t)}\,\mathrm{d}t \\ &+ \sum_{i,n} \Big((w_n^-, v_n^-)_{\Omega_{i,n}} - (w_{n-1}^+, v_{n-1}^+)_{\Omega_{i,n-1}} \Big) \\ &+ \sum_{n=1}^{N} \int_{\Gamma_n} \bar{n}^t [wv]\,\mathrm{d}\bar{s}.\end{aligned} \tag{5.21}$$

Inserting (5.21) in (5.13), we have

$$\begin{aligned}B_h(w, v) = &\sum_{i=1}^{2} \sum_{n=1}^{N} \int_{I_n} (w, -\dot{v})_{\Omega_i(t)}\,\mathrm{d}t + \sum_{n=1}^{N} \int_{I_n} A_{h,t}(w, v)\,\mathrm{d}t \\ &+ \sum_{i=1}^{2} \sum_{n=1}^{N-1} ([w]_n, v_n^+)_{\Omega_{i,n}} + \sum_{i=1}^{2} (w_0^+, v_0^+)_{\Omega_{i,0}} \\ &+ \sum_{i=1}^{2} \sum_{n=1}^{N} \Big((w_n^-, v_n^-)_{\Omega_{i,n}} - (w_{n-1}^+, v_{n-1}^+)_{\Omega_{i,n-1}} \Big) \\ &+ \sum_{n=1}^{N} \int_{\Gamma_n} -\bar{n}^t [w] v_\sigma\,\mathrm{d}\bar{s} + \sum_{n=1}^{N} \int_{\Gamma_n} \bar{n}^t [wv]\,\mathrm{d}\bar{s}.\end{aligned} \tag{5.22}$$

The terms in the second and third rows of (5.22) are combined and rewritten as

$$
\begin{aligned}
&\sum_{i=1}^{2}\sum_{n=1}^{N-1}([w]_n, v_n^+)_{\Omega_{i,n}} + \sum_{i=1}^{2}(w_0^+, v_0^+)_{\Omega_{i,0}} \\
&+\sum_{i=1}^{2}\sum_{n=1}^{N}\Big((w_n^-, v_n^-)_{\Omega_{i,n}} - (w_{n-1}^+, v_{n-1}^+)_{\Omega_{i,n-1}}\Big) \\
=&\sum_{i=1}^{2}\sum_{n=1}^{N-1}\Big((w_n^+, v_n^+)_{\Omega_{i,n}} - (w_n^-, v_n^+)_{\Omega_{i,n}} + (w_n^-, v_n^-)_{\Omega_{i,n}} - (w_{n-1}^+, v_{n-1}^+)_{\Omega_{i,n-1}}\Big) \\
&+\sum_{i=1}^{2}\Big((w_0^+, v_0^+)_{\Omega_{i,0}} + (w_N^-, v_N^-)_{\Omega_{i,N}} - (w_{N-1}^+, v_{N-1}^+)_{\Omega_{i,N-1}}\Big) \\
=&\sum_{i=1}^{2}\sum_{n=1}^{N-1}\Big((w_n^-, v_n^-)_{\Omega_{i,n}} - (w_n^-, v_n^+)_{\Omega_{i,n}}\Big) \\
&+\sum_{i=1}^{2}\Big(\underbrace{(w_0^+, v_0^+)_{\Omega_{i,0}} - (w_0^+, v_0^+)_{\Omega_{i,0}}}_{=0} + (w_N^-, v_N^-)_{\Omega_{i,N}} \\
&\underbrace{+(w_{N-1}^+, v_{N-1}^+)_{\Omega_{i,N-1}} - (w_{N-1}^+, v_{N-1}^+)_{\Omega_{i,N-1}}}_{=0}\Big) \\
=&\sum_{i=1}^{2}\sum_{n=1}^{N-1}(w_n^-, -[v]_n)_{\Omega_{i,n}} + \sum_{i=1}^{2}(w_N^-, v_N^-)_{\Omega_{i,N}}.
\end{aligned}
\tag{5.23}
$$

The terms in the last row of (5.22) are combined and rewritten as

$$
\begin{aligned}
&\sum_{n=1}^{N}\int_{\Gamma_n} -\bar{n}^t[w]v_\sigma \,\mathrm{d}\bar{s} + \sum_{n=1}^{N}\int_{\Gamma_n} \bar{n}^t[wv]\,\mathrm{d}\bar{s} = \sum_{n=1}^{N}\int_{\Gamma_n} \bar{n}^t([wv] - [w]v_\sigma)\,\mathrm{d}\bar{s} \\
=&\sum_{n=1}^{N}\int_{\Gamma_n} \bar{n}^t(w_1v_1 - w_2v_2 - w_1v_\sigma + w_2v_\sigma)\,\mathrm{d}\bar{s} = \sum_{n=1}^{N}\int_{\Gamma_n} \bar{n}^t w_\rho[v]\,\mathrm{d}\bar{s},
\end{aligned}
\tag{5.24}
$$

where $\rho = \frac{1}{2}(3 - \mathrm{sgn}(\bar{n}^t))$, when $\sigma = \frac{1}{2}(3 + \mathrm{sgn}(\bar{n}^t))$ and $\bar{n} = \bar{n}_1$. These expressions make $\rho, \sigma \in \{1, 2\}$ and $\rho \neq \sigma$. Inserting (5.23) and (5.24) in (5.22), we obtain (5.15).

□

5.3 Consistency and Galerkin orthogonality

To show Galerkin orthogonality for the bilinear form B_h, we need the following lemma on consistency.

Lemma 5.3 (Consistency)**.** *The solution u to* (2.1) *also solves* (4.21).

Proof. Insert u in place of u_h in the expression on the left-hand side of (4.21). From the regularity of u, we have, for $n = 1, \ldots, N$, $[u]_{n-1} = 0$, $[u] = 0$ and $[\nabla u] = 0$. Writing $\sum_{i,n} = \sum_{i=1}^{2}\sum_{n=1}^{N}$, the left-hand side of (4.21) with u becomes

$$\sum_{i,n}\int_{I_n}(\dot{u},v)_{\Omega_i(t)}\,\mathrm{d}t+\sum_{i,n}\int_{I_n}(\nabla u,\nabla v)_{\Omega_i(t)}\,\mathrm{d}t+\sum_{n=1}^{N}\int_{\Gamma_n}-\langle\partial_{\bar{n}^x}u\rangle[v]\,\mathrm{d}\bar{s}. \tag{5.25}$$

The second term in (5.25) is

$$\begin{aligned}\sum_{i,n}\int_{I_n}(\nabla u,\nabla v)_{\Omega_i(t)}\,\mathrm{d}t &=\sum_{i,n}\int_{D_{i,n}}\nabla u\cdot\nabla v\,\mathrm{d}\bar{x}\\ &=\sum_{i,n}\int_{D_{i,n}}-\Delta uv\,\mathrm{d}\bar{x}+\sum_{i,n}\int_{\partial D_{i,n}}\bar{n}^x\cdot\nabla uv\,\mathrm{d}\bar{s},\end{aligned} \tag{5.26}$$

where $D_{i,n}=\Omega_i\times I_n$, and $\partial D_{i,n}$ is the boundary of $D_{i,n}$. To obtain the second equality in (5.26), we have used the divergence theorem on $\int_{D_{i,n}}\bar{\nabla}\cdot(\nabla uv,0)\,\mathrm{d}\bar{x}$, where $\bar{\nabla}=(\nabla,\partial t)$. The first term in the second row of (5.26) is

$$\sum_{i,n}\int_{D_{i,n}}-\Delta uv\,\mathrm{d}\bar{x}=\sum_{i,n}\int_{I_n}(-\Delta u,v)_{\Omega_i(t)}\,\mathrm{d}t. \tag{5.27}$$

The second term in the second row of (5.26) is

$$\begin{aligned}\sum_{i,n}\int_{\partial D_{i,n}}\bar{n}^x\cdot\nabla uv\,\mathrm{d}\bar{s}&=\sum_{i,n}\left(\int_{\Omega_{i,n-1}}\bar{n}^x\cdot\nabla uv\,\mathrm{d}x+\int_{\Omega_{i,n}}\bar{n}^x\cdot\nabla uv\,\mathrm{d}x\right)\\ &\quad+\sum_{i,n}\int_{\partial\Omega_i\times I_n}\bar{n}^x\cdot\nabla uv\,\mathrm{d}\bar{s}\overset{\text{2nd}}{=}\sum_{i,n}\int_{\partial\Omega_i\times I_n}\bar{n}^x\cdot\nabla uv\,\mathrm{d}\bar{s}\\ &=\sum_{i,n}\int_{\partial\Omega_i\cap\partial\Omega_0\times I_n}\bar{n}^x\cdot\nabla uv\,\mathrm{d}\bar{s}+\sum_{i,n}\int_{\Gamma_n}\bar{n}^x\cdot\nabla uv\,\mathrm{d}\bar{s}\\ &\overset{\text{4th}}{=}\sum_{i,n}\int_{\Gamma_n}\bar{n}^x\cdot\nabla uv\,\mathrm{d}\bar{s}=\sum_{n=1}^{N}\int_{\Gamma_n}\bar{n}_1^x\cdot\nabla u_1v_1+\bar{n}_2^x\cdot\nabla u_2v_2\,\mathrm{d}\bar{s}\\ &\overset{\text{6th}}{=}\sum_{n=1}^{N}\int_{\Gamma_n}\bar{n}^x\cdot[\nabla uv]\,\mathrm{d}\bar{s}=\sum_{n=1}^{N}\int_{\Gamma_n}[(\partial_{\bar{n}^x}u)v]\,\mathrm{d}\bar{s}\\ &\overset{\text{8th}}{=}\sum_{n=1}^{N}\int_{\Gamma_n}[\partial_{\bar{n}^x}u]\langle v\rangle+\langle\partial_{\bar{n}^x}u\rangle[v]+(\omega_2-\omega_1)[\partial_{\bar{n}^x}u][v]\,\mathrm{d}\bar{s}\\ &=\sum_{n=1}^{N}\int_{\Gamma_n}\langle\partial_{\bar{n}^x}u\rangle[v]\,\mathrm{d}\bar{s},\end{aligned} \tag{5.28}$$

where we have used $\bar{n}^x=0$ on $\Omega_{i,n}$, for $i=1,2$ and $n=0,\dots,N$, to obtain the the second equality, $v=0$ on $\partial\Omega_0$ to get the fourth equality, $\bar{n}=\bar{n}_1$ and $[v]=v_1-v_2$ to obtain the sixth equality, applied (A.1) to get the eight equality and finally, to obtain the last equality, we have used $[\partial_{\bar{n}^x}u]=0$, which follows from the regularity of u. Inserting (5.27) and (5.28) in (5.26), we obtain

$$\sum_{i,n}\int_{I_n}(\nabla u,\nabla v)_{\Omega_i(t)}\,\mathrm{d}t=\sum_{i,n}\int_{I_n}(-\Delta u,v)_{\Omega_i(t)}\,\mathrm{d}t+\sum_{n=1}^{N}\int_{\Gamma_n}\langle\partial_{\bar{n}^x}u\rangle[v]\,\mathrm{d}\bar{s}. \tag{5.29}$$

With the insertion of (5.29) in (5.25), we obtain

$$\begin{aligned}&\sum_{i,n}\int_{I_n}(\dot{u},v)_{\Omega_i(t)}\,\mathrm{d}t+\sum_{i,n}\int_{I_n}(\nabla u,\nabla v)_{\Omega_i(t)}\,\mathrm{d}t+\sum_{n=1}^{N}\int_{\Gamma_n}-\langle\partial_{\bar{n}^x}u\rangle[v]\,\mathrm{d}\bar{s}\\ =&\sum_{i,n}\int_{I_n}(\dot{u}-\Delta u,v)_{\Omega_i(t)}\,\mathrm{d}t.\end{aligned} \tag{5.30}$$

From (2.1), $\dot{u}-\Delta u=f$. This completes the proof. □

From Lemma 5.3, we have that u solves (4.21). Since (5.14) is just another way of writing (4.21), u solves (5.14) as well. From this, and with the error $e\equiv u-u_h$, we have Galerkin orthogonality for B_h.

Corollary 5.1 (Galerkin orthogonality)**.**

$$B_h(e,v)=0,\quad \textit{for all } v\in V_h. \tag{5.31}$$

5.4 The discrete dual problem

We now consider the function $z_h\in V_h$ defined by

$$B_h(v,z_h)=\sum_{i=1}^{2}(v_N^-,z_{h,N}^+)_{\Omega_{i,N}}, \tag{5.32}$$

for all $v\in V_h$. From (5.32), the function z_h is the solution to a discrete dual problem to (2.1). With the alternative way of expressing B_h from Lemma 5.2, we may write (5.32) as the following discrete dual problem that goes backwards in time: Find $z_h\in V_h$ such that

$$\begin{aligned}&\sum_{i=1}^{2}\sum_{n=1}^{N}\int_{I_n}(v,-\dot{z}_h)_{\Omega_i(t)}\,\mathrm{d}t+\sum_{n=1}^{N}\int_{I_n}A_{h,t}(v,z_h)\,\mathrm{d}t\\ &+\sum_{i=1}^{2}\sum_{n=1}^{N-1}(v_n^-,-[z_h]_n)_{\Omega_{i,n}}+\sum_{i=1}^{2}(v_N^-,z_{h,N}^-)_{\Omega_{i,N}}+\sum_{n=1}^{N}\int_{\Gamma_n}\bar{n}^t v_\rho[z_h]\,\mathrm{d}\bar{s}\\ =&\sum_{i=1}^{2}(v_N^-,z_{h,N}^+)_{\Omega_{i,N}},\end{aligned} \tag{5.33}$$

for all $v\in V_h$. Thus, we may consider z_h to be the finite element solution to the following continuous dual problem:

$$\begin{cases}-\dot{z}-\Delta z=0 & \text{in } \Omega_0\times(T,0],\\ z=0 & \text{on } \partial\Omega_0\times[T,0],\\ z=z_{h,N}^+ & \text{in } \Omega_0\times\{T\}.\end{cases} \tag{5.34}$$

5.5 Ritz projection and discrete Laplacian

The Ritz projection operator $R_t : H^1(\Omega_1(t), \Omega_2(t)) \to V_h(t)$ is defined by

$$A_{h,t}(R_t w, v) = A_{h,t}(w, v), \quad \text{for all } v \in V_h(t), \tag{5.35}$$

where H^1 denotes the Sobolev space $W^{1,2}$. Based on an estimate for $w - R_t w$, for the case with only a background mesh, presented in [3], we propose a corresponding estimate for our model.

Conjecture 5.1 (An estimate for $w - R_t w$). *For $t \in (0, T]$, a function $w \in H^1(\Omega_1(t), \Omega_2(t))$, and the Ritz projection operator R_t defined by* (5.35)*, we have for $i = 1, 2$:*

$$\|w - R_t w\|_{\Omega_i(t)} \leq C \min_{1 \leq j \leq 2} \|h^j D^j w\|_{\Omega_i(t)}, \tag{5.36}$$

where $\|w\|_{\Omega_i(t)} = \|w\|_{L_2(\Omega_i(t))}$, $C > 0$ is a constant, h is the largest diameter of a simplex in $\mathcal{T}_0 \cup \mathcal{T}_G$, and $D^j w = \max\{|D^\alpha w| : |\alpha| = j\}$, where $D^\alpha = \partial^{|\alpha|}/\partial x^{\alpha_1} \ldots \partial x^{\alpha_d}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_d$.

The discrete Laplacian $\Delta_{h,t} : H^1(\Omega_1(t), \Omega_2(t)) \to V_h(t)$ is defined by

$$-\sum_{i=1}^{2} (\Delta_{h,t} w, v)_{\Omega_i(t)} = A_{h,t}(w, v), \quad \text{for all } v \in V_h(t), \tag{5.37}$$

where H^1 denotes the Sobolev space $W^{1,2}$.

5.6 Interpolants

The auxiliary interpolation operators $\tilde{I}_{0,n}$ and $\tilde{I}_{G,n}$

We will first define two auxiliary interpolation operators that will be used to define our main interpolation operator. The idea is similar to how the broken finite element spaces were defined in Section 4. For $n = 1, \ldots, N$, we start by considering the two space-time curves $S_{0,n}(x)$ and $S_{G,n}(x)$, defined by

$$S_{0,n}(x) := \{(x, t) : t_{n-1} < t \leq t_n\} = \{x\} \times I_n, \text{ for } x \in \Omega_0, \tag{5.38a}$$

$$S_{G,n}(x) := \{(y(x, t), t) : y(x, t) = x - \int_t^{t_n} \mu(\tau) \, \mathrm{d}\tau, \ t_{n-1} < t \leq t_n\}, \text{ for } x \in G(t_n). \tag{5.38b}$$

Note that $S_{0,n}(x)$ and $S_{G,n}(x)$ are parallel to the nodal trajectories of $\mathcal{T}_0$ and $\mathcal{T}_G$, respectively. For $n = 1, \ldots, N$, the segments $L_{0,n}(x)$ and $L_{G,n}(x)$ of $S_{0,n}(x)$ and $S_{G,n}(x)$, respectively, are defined by

$$L_{0,n}(x) := S_{0,n}(x) \cap D_{1,n}, \text{ for } x \in \Omega_0, \tag{5.39a}$$

$$L_{G,n}(x) := S_{G,n}(x) \cap D_{2,n}, \text{ for } x \in G(t_n). \tag{5.39b}$$

See Figure 10 for an illustration of the space-time curves $S_{0,n}(x)$, $S_{G,n}(x)$, and the segments $L_{0,n}(x)$, $L_{G,n}(x)$ for some x's.

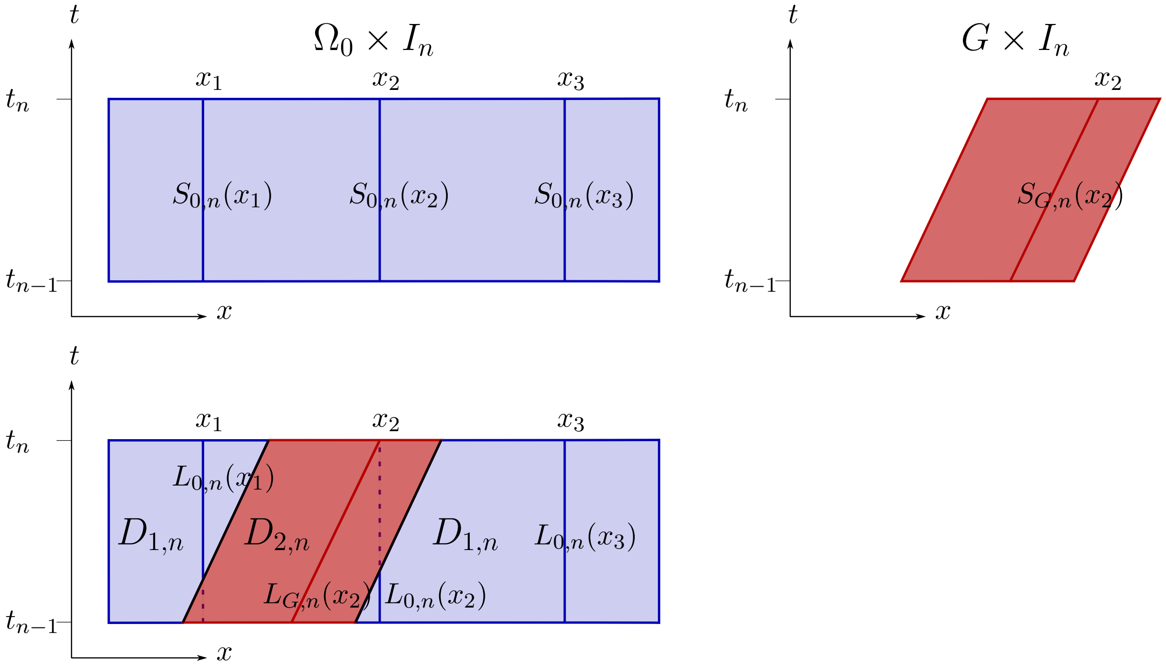

Figure 10: An example, for $d = 1$, of the space-time curves $S_{0,n}(x)$, $S_{G,n}(x)$, $L_{0,n}(x)$, and $L_{G,n}(x)$ for the three x's, $x_1 < x_2 < x_3$, when μ is constant on I_n. *Top left*: The space-time curves $S_{0,n}(x_1)$, $S_{0,n}(x_2)$, and $S_{0,n}(x_3)$ in $\Omega_0 \times I_n$ (light blue) for $x_1, x_2, x_3 \in \Omega_0$. *Top right*: The space-time curve $S_{G,n}(x_2)$ in $G \times I_n$ (red) for $x_2 \in G(t_n)$. *Bottom*: The resulting space-time segments $L_{0,n}(x_1)$, $L_{0,n}(x_2)$, and $L_{0,n}(x_3)$ in $D_{1,n}$ (light blue), and the resulting space-time segment $L_{G,n}(x_2)$ in $D_{2,n}$ (red).

For $n = 1, \ldots, N$, and functions $w_0 : S_{0,n}(x) \to \mathbb{R}$ and $w_G : S_{G,n}(x) \to \mathbb{R}$, the auxiliary interpolation operators $\tilde{I}_{0,n}$ and $\tilde{I}_{G,n}$ are uniquely defined by the interpolants $\tilde{I}_{0,n}w_0 \in \mathcal{P}^q(S_{0,n}(x))$ and $\tilde{I}_{G,n}w_G \in \mathcal{P}^q(S_{G,n}(x))$, respectively, that fulfil the following properties:

$$(\tilde{I}_{0,n}w_0)_n^- = w_{0,n}^-, \tag{5.40a}$$
$$(\tilde{I}_{G,n}w_G)_n^- = w_{G,n}^-, \tag{5.40b}$$

and for $q \geq 1$,

$$\int_{L_{0,n}(x)} (\tilde{I}_{0,n}w_0)v \, \mathrm{d}s = \int_{L_{0,n}(x)} w_0 v \, \mathrm{d}s, \quad \text{for all } v \in \mathcal{P}^{q-1}(L_{0,n}(x)), \tag{5.41a}$$
$$\int_{L_{G,n}(x)} (\tilde{I}_{G,n}w_G)v \, \mathrm{d}s = \int_{L_{G,n}(x)} w_G v \, \mathrm{d}s, \quad \text{for all } v \in \mathcal{P}^{q-1}(L_{G,n}(x)), \tag{5.41b}$$

where ds denotes an infinitesimal arc length of either $S_{0,n}(x)$ or $S_{G,n}(x)$. We may now extend this definition to interpolate functions $w : D_{1,n} \to \mathbb{R}$ (and $w : D_{2,n} \to \mathbb{R}$) by noting that any $\bar{x} \in D_{1,n}$ (or $D_{2,n}$) belongs to precisely one segment $L_{0,n}(x)$ (or $L_{G,n}(x)$).

The main interpolation operator $\tilde{I}_n$

We are now ready to define our main interpolation operator $\tilde{I}_n$ in terms of $\tilde{I}_{0,n}$ and $\tilde{I}_{G,n}$. For $n = 1, \dots, N$, and a function $w : D_{0,n} \to \mathbb{R}$, we define the interpolation operator $\tilde{I}_n$ by

$$(\tilde{I}_n w)|_{D_{1,n}} := \tilde{I}_{0,n}(w|_{D_{1,n}}) \text{ and } (\tilde{I}_n w)|_{D_{2,n}} := \tilde{I}_{G,n}(w|_{D_{2,n}}). \tag{5.42}$$

Immediate consequences of (5.42) are that for $n = 1, \dots, N$, and $i = 1, 2$,

$$(\tilde{I}_n w)_n^- = w_n^-, \tag{5.43a}$$

and for $q \geq 1$,

$$\int_{I_n} (\tilde{I}_n w, v)_{\Omega_i(t)} \, \mathrm{d}t = \int_{I_n} (w, v)_{\Omega_i(t)} \, \mathrm{d}t, \quad \text{for all } v \in \delta_t V_h^n. \tag{5.43b}$$

Note that (5.43b) holds for $q = 0$ as well, since then the function space $\delta_t V_h^n$ only consists of the zero function.

Lemma 5.4 (Interpolation estimate)**.** *Let μ and its time derivative be bounded on I_n and let $\tilde{I}_n$ be defined as in* (5.42)*. Then, for $q = 0, 1$, $\tilde{I}_n$ is bounded and there is a constant $C = C(\mu) > 0$ such that, for $i = 1, 2$, and for any function $w : D_{0,n} \to \mathbb{R}$ with sufficient regularity,*

$$\|w - \tilde{I}_n w\|_{\Omega_i, I_n} \leq C k_n^{q+1} \|\dot{w}^{(q+1)}\|_{\Omega_i, I_n} + E_{q,(\Omega_i, I_n)}(w), \tag{5.44}$$

where $\|w\|_{\Omega_i, I_n} = \max_{t \in I_n} \|w\|_{\Omega_i(t)}$, $k_n = t_n - t_{n-1}$, $\dot{w}^{(q+1)} = \partial^{q+1} w / \partial t^{q+1}$, and $E_{q,(\Omega_i, I_n)}(w)$ is

$$E_{q,(\Omega_i, I_n)}(w) = \max_{r \in \{0, \dots, q\}} \{|\dot{\mu}^{(r)}|_{I_n}\} C k_n^{q+1} \Big(\|\nabla w\|_{\Omega_i, I_n} + q \big(\|H(w)\mu\|_{\Omega_i, I_n} + \|\nabla \dot{w}\|_{\Omega_i, I_n} \big) \Big), \tag{5.45}$$

where $\dot{\mu}^{(0)} = \mu$, $C = C(\mu) > 0$, and $H(w)$ is the Hessian matrix of w. The vector $H(w)\mu$ is the result of a matrix multiplication between matrix $H(w)$ and vector μ.

Proof. We start by deriving explicit expressions for $\tilde{I}_n w$, involving w, for $q = 0, 1$. From these explicit expressions, boundedness of $\tilde{I}_n$ will follow. We then use these expressions to derive estimates for $w - \tilde{I}_n w$, from which (5.44) will be derived. But first we introduce some useful notation and estimates. Let j denote an element in the index set $\{0, G\}$, and let $S_j = S_{j,n}(x)$, defined by (5.38), and $L_j = L_{j,n}(x)$, defined by (5.39). Furthermore, let r be a parametrization of S_j with respect to time, i.e., $r(t) = (y(t), t)$, where $y(t) = x - \int_t^{t_n} \hat{\mu}(\tau) \, \mathrm{d}\tau$ for some $x \in \Omega_0$ and $\hat{\mu}$ is defined by (A.2). Note that $dy/dt = \hat{\mu}$ and that the relation between the parametrization r and the infinitesimal arc length ds is $|dr| = ds$. We thus have

$$\frac{dr(t)}{dt} = \left(\frac{dy(t)}{dt}, \frac{dt}{dt} \right) = (\hat{\mu}, 1) \quad \Rightarrow \quad \left| \frac{dr}{dt} \right| = \frac{ds}{dt} = (|\hat{\mu}|^2 + 1)^{1/2}. \tag{5.46}$$

We use (5.46) to obtain the following estimate:

$$\frac{dt}{ds} = (|\hat{\mu}|^2 + 1)^{-1/2} \leq 1. \tag{5.47}$$

With $r(t) = (y(t), t)$, the first and second time derivatives of a function w along S_j are

$$\left.\frac{dw}{dt}\right|_{S_j} = \frac{dw(y(t),t)}{dt} = \nabla w \cdot \frac{dy}{dt} + \frac{\partial w}{\partial t} = \nabla w \cdot \hat{\mu} + \dot{w}. \tag{5.48a}$$

$$\begin{aligned} \left.\frac{d^2 w}{dt^2}\right|_{S_j} &= \frac{d}{dt}\Big(\nabla w \cdot \hat{\mu} + \dot{w}\Big) = \frac{d}{dt}\Big(\nabla w \cdot \hat{\mu}\Big) + \nabla \dot{w} \cdot \hat{\mu} + \dot{w}^{(2)} \\ &= (H(w)\hat{\mu}) \cdot \hat{\mu} + \nabla w \cdot \frac{d\hat{\mu}}{dt} + \nabla \dot{w} \cdot \hat{\mu} + \dot{w}^{(2)}, \end{aligned} \tag{5.48b}$$

where $H(w)$ is the Hessian matrix of w. The vector $H(w)\hat{\mu}$ is thus the resulting vector from the matrix multiplication between matrix $H(w)$ and vector $\hat{\mu}$. Now, let $r_{n-1} = r(t_{n-1}^+)$ and $r_n = r(t_n)$ denote the endpoints of the space-time curve S_j, and let r_a and r_b denote the endpoints of the segment L_j of S_j. A curve L_j could of course consist of more than one segment, e.g., $L_j = L_{j,1} \cup L_{j,2}$ and $L_{j,1} \cap L_{j,2} = \emptyset$. But without loss of generality, the curve L_j only consists of one segment in this proof. With the new notation, we derive an estimate for the length of S_j, i.e., $|S_j|$.

$$|S_j| = \int_{S_j} \mathrm{d}s = \int_{r_{n-1}}^{r_n} \mathrm{d}s = \int_{t_{n-1}}^{t_n} (|\hat{\mu}|^2 + 1)^{1/2}\,\mathrm{d}t \leq (|\mu|_{I_n}^2 + 1)^{1/2} k_n = Ck_n, \tag{5.49}$$

where $|\mu|_{I_n} = \max_{t\in I_n} |\mu(t)|$ and $C = C(\mu) > 0$. To obtain the third equality, we have used the general formula for the arc length of a curve with the infinitesimal arc length being $\mathrm{d}s = (|\hat{\mu}|^2 + 1)^{1/2}\,\mathrm{d}t$, from (5.46). Since ds denotes an infinitesimal arc length of S_j, the variable s could be thought of as a length associated with a point $r(t)$ on S_j. We define s by

$$s(r(t)) := s_{n-1} + \int_{r_{n-1}}^{r(t)} \mathrm{d}s, \tag{5.50}$$

where $s_{n-1} = s(r_{n-1})$ is some predefined reference length at the point r_{n-1}. In the same way, we let s_n, a, and b denote the lengths associated with the points r_n, r_a, and r_b, respectively. Furthermore, we let l denote the length associated with a point $r_l \in L_j$. With the new notation, the lengths of S_j and L_j, may be written $|S_j| = s_n - s_{n-1}$ and $|L_j| = b - a$, respectively. Line integrals with the integrand being a function of s, may thus be treated in the same manner as regular one variable integrals. We are now well equipped to start deriving expressions for $\tilde{I}_n w$ and $w - \tilde{I}_n w$.

$\tilde{I}_n w$ for $q = 0$

For $q = 0$, and $r_l \in L_j$,

$$(\tilde{I}_n w)(r_l) = w_n^-, \tag{5.51}$$

from (5.43a). The identity (5.51) indicates that $\tilde{I}_n$ is bounded for $q = 0$. Using (5.51), we get

$$
\begin{aligned}
(w - \tilde{I}_n w)|_{L_j} &= w(r_l) - w_n^- = -\int_{r_l}^{r_n} \frac{dw}{dr}\,\mathrm{d}s \le \int_{r_l}^{r_n} \left|\frac{dt}{ds}\right|\left|\frac{dw}{dt}\right| \mathrm{d}s \le \int_{r_{n-1}}^{r_n} \left|\frac{dw}{dt}\right| \mathrm{d}s \\
&\le \int_{r_{n-1}}^{r_n} |\dot{w}| + |\mu||\nabla w|\,\mathrm{d}s
\end{aligned}
\tag{5.52}
$$

where we have used the absolute value and the fact that $|dr| = ds$ together with the chain rule to obtain the first inequality, and $dt/ds \le 1$, from (5.47), to obtain the second inequality. To obtain the last inequality, we have used (5.48a). By taking the $\|\cdot\|_{\Omega_i,I_n}$-norm of (5.52), we obtain

$$
\begin{aligned}
\|w - \tilde{I}_n w\|_{\Omega_i,I_n} &\le \|\int_{r_{n-1}}^{r_n} |\dot{w}| + |\mu||\nabla w|\,\mathrm{d}s\|_{\Omega_i,I_n} \le |S_j|\Big(\|\dot{w}\|_{\Omega_i,I_n} + |\mu|_{I_n}\|\nabla w\|_{\Omega_i,I_n}\Big) \\
&\le Ck_n\|\dot{w}\|_{\Omega_i,I_n} + |\mu|_{I_n} Ck_n\|\nabla w\|_{\Omega_i,I_n},
\end{aligned}
\tag{5.53}
$$

where we have used (5.49) in the last step. This proves (5.44) for $q = 0$.

$\tilde{I}_n w$ for $q = 1$

For $q = 1$, the procedure is trickier. We start by considering the following integral:

$$
\begin{aligned}
\int_{L_j} (s - s_n)\frac{d(\tilde{I}_n w)(r)}{dr}\,\mathrm{d}s &= \frac{d(\tilde{I}_n w)(r_b)}{dr}\int_{r_a}^{r_b} (s - s_n)\,\mathrm{d}s \\
&= \frac{d(\tilde{I}_n w)(r_b)}{dr}\frac{1}{2}\big((b - s_n)^2 - (a - s_n)^2\big),
\end{aligned}
\tag{5.54}
$$

where we have used the fact that $d(\tilde{I}_n w)/dr$ is constant on L_j for $q = 1$. We may also use this fact to treat the integral as:

$$
\begin{aligned}
\int_{L_j} (s - s_n)\frac{d(\tilde{I}_n w)(r)}{dr}\,\mathrm{d}s &= \int_{L_j} (s - s_n)\frac{(\tilde{I}_n w)(r) - (\tilde{I}_n w)(r_n)}{s - s_n}\,\mathrm{d}s \\
&= \int_{L_j} (\tilde{I}_n w)(r)\,\mathrm{d}s - \int_{L_j} (\tilde{I}_n w)(r_n)\,\mathrm{d}s \\
&= \int_{L_j} w(r)\,\mathrm{d}s - \int_{L_j} w_n^-\,\mathrm{d}s = \int_{L_j} w - w_n^-\,\mathrm{d}s,
\end{aligned}
\tag{5.55}
$$

where we have used (5.42) together with (5.41), and (5.43a) to obtain the third equality. By Taylor expansion of $(\tilde{I}_n w)(r_l)$ at r_b for $r_l \in L_j$, we get

$$
\begin{aligned}
(\tilde{I}_n w)(r_l) &= (\tilde{I}_n w)(r_n) + (l - s_n)\frac{d(\tilde{I}_n w)(r_b)}{ds} \\
&= w_n^- + \frac{2(l - s_n)}{(b - s_n)^2 - (a - s_n)^2}\int_{L_j} w - w_n^-\,\mathrm{d}s \\
&= w_n^- + \alpha\int_{L_j} w - w_n^-\,\mathrm{d}s,
\end{aligned}
\tag{5.56}
$$

where $\alpha = 2(l - s_n)/\big((b - s_n)^2 - (a - s_n)^2\big)$. To obtain the second equality, we have used (5.43a), and combined (5.54) with (5.55). The identity (5.56) indicates that $\tilde{I}_n$ is bounded for $q = 1$. We note that the denominator in α may be written as

$$(b - s_n)^2 - (a - s_n)^2 = -(b - a)\big(2s_n - (a + b)\big) = -(b - a)(s_n - a)\bigg(1 + \frac{s_n - b}{s_n - a}\bigg). \quad (5.57)$$

Using (5.56), we get

$$\begin{aligned}
(w - \tilde{I}_n w)|_{L_j} &= w(r_l) - w_n^- - \alpha \int_{L_j} w - w_n^- \,\mathrm{d}s = -\int_{r_l}^{r_n} \frac{dw}{dr}\,\mathrm{d}s - \alpha \int_{r_a}^{r_b} w - w_n^- \,\mathrm{d}s \\
&= -\int_{r_l}^{r_n} \frac{dw}{dr}\,\mathrm{d}s + \alpha \int_{r_a}^{r_b} \frac{dw}{dr}(s - a)\,\mathrm{d}s - \alpha(w(r_b) - w_n^-)(b - a) \\
&= -\int_{r_l}^{r_n} \frac{dw}{dr}\,\mathrm{d}s + \alpha \int_{r_a}^{r_b} \frac{dw}{dr}(s - a)\,\mathrm{d}s + \alpha(b - a)\int_{r_b}^{r_n} \frac{dw}{dr}\,\mathrm{d}s \\
&= \int_{r_l}^{r_n} \frac{d^2w}{dr^2}(s - l)\,\mathrm{d}s \underbrace{- \frac{dw(r_n)}{dr}(s_n - l)}_{\text{2nd}} \\
&\quad - \alpha \int_{r_a}^{r_b} \frac{d^2w}{dr^2}\frac{(s - a)^2}{2}\,\mathrm{d}s + \underbrace{\alpha \frac{dw(r_b)}{dr}\frac{(b - a)^2}{2}}_{\text{4th}} \\
&\quad - \alpha(b - a)\int_{r_b}^{r_n} \frac{d^2w}{dr^2}\bigg(s - \frac{(a + b)}{2}\bigg)\,\mathrm{d}s \\
&\quad + \underbrace{\alpha(b - a)\bigg(\frac{dw(r_n)}{dr}\bigg(s_n - \frac{(a + b)}{2}\bigg) - \frac{dw(r_b)}{dr}\bigg(b - \frac{(a + b)}{2}\bigg)\bigg)}_{\text{6th}},
\end{aligned} \quad (5.58)$$

where we have used partial integration to obtain the equalities. We continue by considering the terms involving dw/dr separately, i.e., the second, fourth and sixth terms in the last right-hand side of (5.58):

$$
\begin{aligned}
&-\frac{dw(r_n)}{dr}(s_n-l)+\alpha\frac{dw(r_b)}{dr}\frac{(b-a)^2}{2}\\
&+\alpha(b-a)\left(\frac{dw(r_n)}{dr}\left(s_n-\frac{(a+b)}{2}\right)-\frac{dw(r_b)}{dr}\left(b-\frac{(a+b)}{2}\right)\right)\\
=&\frac{dw(r_n)}{dr}\left(\alpha(b-a)\left(s_n-\frac{(a+b)}{2}\right)-(s_n-l)\right)\\
&+\frac{dw(r_b)}{dr}\alpha(b-a)\underbrace{\left(\frac{(b-a)}{2}-\left(b-\frac{(a+b)}{2}\right)\right)}_{=0}\\
=&\frac{dw(r_n)}{dr}\left(\frac{2(l-s_n)}{(b-s_n)^2-(a-s_n)^2}(b-a)\left(s_n-\frac{(a+b)}{2}\right)-(s_n-l)\right)\\
=&\frac{dw(r_n)}{dr}(l-s_n)\left(\frac{(b-a)(2s_n-(a+b))}{(b-s_n)^2-(a-s_n)^2}+1\right)\\
=&\frac{dw(r_n)}{dr}(l-s_n)\left(\underbrace{\frac{(b-a)(2s_n-(a+b))}{-(b-a)(2s_n-(a+b))}}_{=-1}+1\right)\\
=&\,0,
\end{aligned}
\tag{5.59}
$$

where we have used (5.57) in the penultimate equality. By inserting (5.59) in (5.58), we get

$$
\begin{aligned}
(w-\tilde{I}_n w)|_{L_j}=&\int_{r_l}^{r_n}\frac{d^2w}{dr^2}(s-l)\,\mathrm{d}s-\alpha\int_{r_a}^{r_b}\frac{d^2w}{dr^2}\frac{(s-a)^2}{2}\,\mathrm{d}s\\
&-\alpha(b-a)\int_{r_b}^{r_n}\frac{d^2w}{dr^2}\left(s-\frac{(a+b)}{2}\right)\mathrm{d}s\\
\le&\,|s_n-l|\int_{r_l}^{r_n}\left|\frac{d^2w}{dr^2}\right|\mathrm{d}s+|b-a|\int_{r_a}^{r_b}\left|\frac{d^2w}{dr^2}\right|\mathrm{d}s\\
&+2|s_n-s_{n-1}|\int_{r_b}^{r_n}\left|\frac{d^2w}{dr^2}\right|\mathrm{d}s\\
\le&\,4|s_n-s_{n-1}|\int_{r_{n-1}}^{r_n}\left|\frac{dt}{ds}\right|^2\left|\frac{d^2w}{dt^2}\right|\mathrm{d}s\le 4|S_j|\int_{r_{n-1}}^{r_n}\left|\frac{d^2w}{dt^2}\right|\mathrm{d}s\\
\le&\,4|S_j|\int_{r_{n-1}}^{r_n}|\dot{w}^{(2)}|+|\mu||H(w)\mu|+|\dot{\mu}||\nabla w|+|\mu||\nabla\dot{w}|\,\mathrm{d}s
\end{aligned}
\tag{5.60}
$$

where we have used the explicit expression for α together with (5.57), and the absolute value to obtain the first inequality. In the second inequality, we have used $|dr|=ds$ and the chain rule, and in the third inequality, we have used $dt/ds\le 1$, from (5.47). Finally, we have used (5.48b), to obtain the last inequality. By taking the $\|\cdot\|_{\Omega_i,I_n}$-norm of (5.60), we obtain

$$
\begin{aligned}
\|w - \tilde{I}_n w\|_{\Omega_i, I_n} &\le 4|S_j| \| \int_{r_{n-1}}^{r_n} |\dot{w}^{(2)}| + |\mu||H(w)\mu| + |\dot{\mu}||\nabla w| + |\mu||\nabla \dot{w}| \,\mathrm{d}s \|_{\Omega_i, I_n} \\
&\le 4|S_j|^2 \|\dot{w}^{(2)}\|_{\Omega_i, I_n} \\
&\quad + 4|S_j|^2 \Big(|\mu|_{I_n} \|H(w)\mu\|_{\Omega_i, I_n} + |\dot{\mu}|_{I_n} \|\nabla w\|_{\Omega_i, I_n} + |\mu|_{I_n} \|\nabla \dot{w}\|_{\Omega_i, I_n} \Big) \\
&\le C k_n^2 \|\dot{w}^{(2)}\|_{\Omega_i, I_n} \\
&\quad + \max\{|\mu|_{I_n}, |\dot{\mu}|_{I_n}\} C k_n^2 \Big(\|H(w)\mu\|_{\Omega_i, I_n} + \|\nabla w\|_{\Omega_i, I_n} + \|\nabla \dot{w}\|_{\Omega_i, I_n} \Big)
\end{aligned} \tag{5.61}
$$

where we have used (5.49) in the last inequality. This proves (5.44) for $q = 1$. The proof of Lemma 5.4 is thus completed.

□

The extension operator $\mathcal{E}_s^t$

Recall the relaxed representation of a function $v \in V_h(I_n)$, given by (4.5). We define the extension operator $\mathcal{E}_s^t : V_h(s) \to V_h(t)$ by

$$\mathcal{E}_s^t v = \sum_j V_j(s)\varphi_j(x,t). \tag{5.62}$$

For $v \in \{v : v(x,t) = \sum_j V_j(t)\varphi(x,t), V_j \in C^1(I_n), \forall j\}$ and $t_{n-1} < t < t_n$, where $n = 1, \ldots, N$, we may write

$$V_j(t) = V_j(t_n^-) + \int_{t_n}^t \dot{V}_j(s)\,\mathrm{d}s, \tag{5.63}$$

since $\int_{t_n}^t \dot{V}_j(s)\,\mathrm{d}s = \int_{t_n^-}^t \dot{V}_j(s)\,\mathrm{d}s = V_j(t) - V_j(t_n^-)$. With (5.62) and (5.63), we may partition v in the following way:

$$\begin{aligned} v &= \sum_j V_j(t)\varphi_j(x,t) = \sum_j \left(V_j(t_n^-) + \int_{t_n}^t \dot{V}_j(s)\,\mathrm{d}s \right)\varphi_j(x,t) \\ &= \sum_j V_j(t_n^-)\varphi_j(x,t) + \sum_j \left(\int_{t_n}^t \dot{V}_j(s)\,\mathrm{d}s \right)\varphi_j(x,t) \\ &= \sum_j V_j(t_n^-)\varphi_j(x,t) + \int_{t_n}^t \sum_j \dot{V}_j(s)\varphi_j(x,t)\,\mathrm{d}s \\ &= \mathcal{E}_n^t v + \int_{t_n}^t \mathcal{E}_s^t \delta_t v\,\mathrm{d}s, \end{aligned} \tag{5.64}$$

where $\mathcal{E}_n^t v = \mathcal{E}_{t_n^-}^t v$. Note that, for $q \geq 1$, the function $\mathcal{E}_n^t v \in \delta_t V_h^n$. Now consider a function $v \in V_h^n$ for $q = 1$. The nodal coefficient functions are then piecewise linear in time and hence their derivatives are constant on every I_n, i.e., $\dot{V}_j(s) = \dot{V}_j(t)$ for all $s \in (t, t_n]$ and $t \in I_n$. We may thus rewrite $\int_{t_n}^t \mathcal{E}_s^t \delta_t v\,\mathrm{d}s$ as

$$\int_{t_n}^t \mathcal{E}_s^t \delta_t v\,\mathrm{d}s = \int_{t_n}^t \sum_j \dot{V}_j(s)\varphi_j(x,t)\,\mathrm{d}s = \int_{t_n}^t \sum_j \dot{V}_j(t)\varphi_j(x,t)\,\mathrm{d}s = (t - t_n)\delta_t v. \tag{5.65}$$

6 Stability analysis

The stability analysis in this section is based on a stability analysis for the case with only a background mesh, presented in [2, 3]. The main result of this section is the following stability estimate and its counterpart for the discrete dual problem:

Theorem 6.1 (Stability estimate)**.** *Let u_h be the solution to* (4.21) *with $f \equiv 0$ and let u_0 be the initial value of the analytic solution to the problem presented in Section 2. We then have*

$$\begin{aligned}
&\sum_{i=1}^{2}\sum_{n=1}^{N}\int_{I_n} \|\dot{u}_h\|_{\Omega_i(t)} + \|\nabla u_h\|_{\Omega_i(t)}\,\mathrm{d}t \\
&+\sum_{i=1}^{2}\sum_{n=1}^{N}\int_{I_n} \|\Delta_{h,t}u_h\|_{\Omega_i(t)} + |\mu|_{I_n}^{-1}\|\Delta_{h,t}\mathcal{E}_n^t u_h - \mathcal{E}_n^t\Delta_{h,n}u_h\|_{\Omega_i(t)}\,\mathrm{d}t \\
&+\sum_{i=1}^{2}\sum_{n=1}^{N}\|[u_h]_{n-1}\|_{\Omega_{i,n-1}} + \sum_{i=1}^{2}\|u_{h,N}^-\|_{\Omega_{i,N}} + \sum_{n=1}^{N}\|[u_h]\|_{\Gamma_n} \\
&\leq C_1\bigg(\sum_{i=1}^{2}\|u_0\|_{\Omega_i(0)}^2\bigg)^{1/2},
\end{aligned} \tag{6.1}$$

where $C_1 = C(\log(t_N/k_1)+1)^{1/2}$ and $C>0$, and $|\mu|_{I_n} = \max_{t\in I_n}|\mu(t)|$.

The counterpart of (6.1) for z_h is a crucial tool in the proof of the a priori error estimate presented in Theorem 7.1 in Section 7.

Corollary 6.1 (Stability estimate for z_h)**.** *The corresponding stability estimate to* (6.1) *for the finite element solution z_h to the discrete dual problem* (5.33) *is*

$$\begin{aligned}
&\sum_{i=1}^{2}\sum_{n=1}^{N}\int_{I_n} \|\dot{z}_h\|_{\Omega_i(t)} + \|\nabla z_h\|_{\Omega_i(t)}\,\mathrm{d}t \\
&+\sum_{i=1}^{2}\sum_{n=1}^{N}\int_{I_n} \|\Delta_{h,t}z_h\|_{\Omega_i(t)} + |\mu|_{I_n}^{-1}\|\Delta_{h,t}\mathcal{E}_n^t z_h - \mathcal{E}_n^t\Delta_{h,n}z_h\|_{\Omega_i(t)}\,\mathrm{d}t \\
&+\sum_{i=1}^{2}\sum_{n=1}^{N}\|[z_h]_{n}\|_{\Omega_{i,n}} + \sum_{i=1}^{2}\|z_{h,N}^-\|_{\Omega_{i,N}} + \sum_{n=1}^{N}\|[z_h]\|_{\Gamma_n} \\
&\leq C_N\bigg(\sum_{i=1}^{2}\|z_{h,N}^+\|_{\Omega_{i,N}}^2\bigg)^{1/2},
\end{aligned} \tag{6.2}$$

where $C_N = C(\log(t_N/k_N)+1)^{1/2}$ and $C>0$, and $|\mu|_{I_n} = \max_{t\in I_n}|\mu(t)|$.

To prove Theorem 6.1 and thus also Corollary 6.1, we need two other stability estimates for the finite element problem (4.21). We start by letting $f \equiv 0$ in (5.3). We have: Find $u_h \in V_h$ such that

$$
\begin{aligned}
&\sum_{i=1}^{2}\sum_{n=1}^{N}\int_{I_n}(\dot{u}_h, v)_{\Omega_i(t)}\,\mathrm{d}t + \sum_{n=1}^{N}\int_{I_n} A_{h,t}(u_h, v)\,\mathrm{d}t \\
&+ \sum_{i=1}^{2}\sum_{n=1}^{N}([u_h]_{n-1}, v_{n-1}^{+})_{\Omega_{i,n-1}} + \sum_{n=1}^{N}\int_{\Gamma_n} -\bar{n}^t [u_h] v_\sigma \,\mathrm{d}\bar{s} = 0,
\end{aligned}
\tag{6.3}
$$

for all $v \in V_h$.

6.1 The first auxiliary stability estimate

The first of the two auxiliary stability estimates is presented as the following lemma.

Lemma 6.1 (The first auxiliary stability estimate)**.** *Let u_h be the solution to* (4.21) *with $f \equiv 0$ and let u_0 be the initial value of the analytic solution to the problem presented in Section 2. We then have*

$$
\begin{aligned}
&\sum_{i=1}^{2}\|u_{h,N}^{-}\|_{\Omega_{i,N}}^2 + 2\sum_{n=1}^{N}\int_{I_n} A_{h,t}(u_h, u_h)\,\mathrm{d}t \\
&+ \sum_{i=1}^{2}\sum_{n=1}^{N}\|[u_h]_{n-1}\|_{\Omega_{i,n-1}}^2 + \sum_{n=1}^{N}\||\bar{n}^t|[u_h]\|_{\Gamma_n}^2 = \sum_{i=1}^{2}\|u_0\|_{\Omega_i(0)}^2.
\end{aligned}
\tag{6.4}
$$

Proof. Start by taking $v = 2u_h \in V_h$ in (6.3). With $\sum_{i,n} = \sum_{i=1}^{2}\sum_{n=1}^{N}$, we have

$$
\begin{aligned}
&\underbrace{\sum_{i,n}\int_{I_n} 2(\dot{u}_h, u_h)_{\Omega_i(t)}\,\mathrm{d}t}_{=\,\mathrm{I}} + \underbrace{2\sum_{n=1}^{N}\int_{I_n} A_{h,t}(u_h, u_h)\,\mathrm{d}t}_{=\,\mathrm{II}} \\
&+ \underbrace{\sum_{i,n} 2([u_h]_{n-1}, u_{h,n-1}^{+})_{\Omega_{i,n-1}}}_{=\,\mathrm{III}} + \underbrace{\sum_{n=1}^{N}\int_{\Gamma_n} -2\bar{n}^t [u_h] u_{h,\sigma}\,\mathrm{d}\bar{s}}_{=\,\mathrm{IV}} = 0.
\end{aligned}
\tag{6.5}
$$

We consider the terms in (6.5) separately, starting with the first:

$$
\begin{aligned}
\mathrm{I} &= \sum_{i,n}\int_{I_n} 2(\dot{u}_h, u_h)_{\Omega_i(t)}\,\mathrm{d}t = \sum_{i,n}\int_{D_{i,n}} \partial_t(u_h^2)\,\mathrm{d}\bar{x} = \sum_{i,n}\int_{D_{i,n}} \bar{\nabla}\cdot(u_h^2 e_t)\,\mathrm{d}\bar{x} \\
&= \sum_{i,n}\int_{\partial D_{i,n}} \bar{n}_i \cdot (u_h^2 e_t)\,\mathrm{d}\bar{s} = \sum_{i,n}\int_{\partial D_{i,n}} \bar{n}_i^t u_h^2\,\mathrm{d}\bar{s} \\
&= \sum_{i,n}\left(\int_{\Omega_{i,n}} (u_{h,n}^{-})^2\,\mathrm{d}x - \int_{\Omega_{i,n-1}} (u_{h,n-1}^{+})^2\,\mathrm{d}x\right) + \sum_{i,n}\int_{\Gamma_n} \bar{n}_i^t u_{h,i}^2\,\mathrm{d}\bar{s} \\
&= \sum_{i=1}^{2}\sum_{n=1}^{N}\left(\|u_{h,n}^{-}\|_{\Omega_{i,n}}^2 - \|u_{h,n-1}^{+}\|_{\Omega_{i,n-1}}^2\right) + \sum_{n=1}^{N}\int_{\Gamma_n} \bar{n}^t [u_h^2]\,\mathrm{d}\bar{s},
\end{aligned}
\tag{6.6}
$$

where $D_{i,n} = \Omega_i \times I_n$, $\bar{\nabla} = (\nabla, \partial_t)$, e_t is the unit vector in time and ∂D_{i_n} is the boundary to $D_{i,n}$. Here we have used the divergence theorem to obtain the third equality. To obtain the penultimate equality we have used the fact that $\bar{n}^t = 0$ on $\partial\Omega_0$. We have set $\bar{n}^t = \bar{n}_1^t$ and $[v] = v_1 - v_2$ on Γ_n to obtain the last equality. The second term in (6.5) is as we want it, so we move on to the third term. But first we note that for $i = 1, 2$ and $n = 1, \ldots, N$, we have

$$\begin{aligned}
&([u_h]_{n-1}, u^+_{h,n-1})_{\Omega_{i,n-1}} = ([u_h]_{n-1}, u^+_{h,n-1} + u^-_{h,n-1} - u^-_{h,n-1})_{\Omega_{i,n-1}} \\
&= ([u_h]_{n-1}, [u_h]_{n-1})_{\Omega_{i,n-1}} + ([u_h]_{n-1}, u^-_{h,n-1})_{\Omega_{i,n-1}} \\
&= ([u_h]_{n-1}, [u_h]_{n-1})_{\Omega_{i,n-1}} + (u^+_{h,n-1}, u^-_{h,n-1})_{\Omega_{i,n-1}} - (u^-_{h,n-1}, u^-_{h,n-1})_{\Omega_{i,n-1}} \\
&= \|[u_h]_{n-1}\|^2_{\Omega_{i,n-1}} + (u^+_{h,n-1}, u^-_{h,n-1})_{\Omega_{i,n-1}} - \|u^-_{h,n-1}\|^2_{\Omega_{i,n-1}},
\end{aligned} \tag{6.7}$$

and

$$\begin{aligned}
&([u_h]_{n-1}, u^+_{h,n-1})_{\Omega_{i,n-1}} = (u^+_{h,n-1} - u^-_{h,n-1}, u^+_{h,n-1})_{\Omega_{i,n-1}} \\
&= (u^+_{h,n-1}, u^+_{h,n-1})_{\Omega_{i,n-1}} - (u^-_{h,n-1}, u^+_{h,n-1})_{\Omega_{i,n-1}} \\
&= \|u^+_{h,n-1}\|^2_{\Omega_{i,n-1}} - (u^+_{h,n-1}, u^-_{h,n-1})_{\Omega_{i,n-1}}.
\end{aligned} \tag{6.8}$$

With (6.7) and (6.8), the third term in (6.5) is

$$\begin{aligned}
\mathrm{III} = \sum_{i=1}^{2}\sum_{n=1}^{N} 2([u_h]_{n-1}, u^+_{h,n-1})_{\Omega_{i,n-1}} &= \sum_{i=1}^{2}\sum_{n=1}^{N} \|[u_h]_{n-1}\|^2_{\Omega_{i,n-1}} \\
&+ \sum_{i=1}^{2}\sum_{n=1}^{N} \Big(\|u^+_{h,n-1}\|^2_{\Omega_{i,n-1}} - \|u^-_{h,n-1}\|^2_{\Omega_{i,n-1}} \Big).
\end{aligned} \tag{6.9}$$

Now we add the second term on the right-hand side of (6.9) to the first term in the last row of (6.6) to obtain

$$\begin{aligned}
&\sum_{i=1}^{2}\sum_{n=1}^{N} \Big(\|u^-_{h,n}\|^2_{\Omega_{i,n}} - \|u^+_{h,n-1}\|^2_{\Omega_{i,n-1}} \Big) + \sum_{i=1}^{2}\sum_{n=1}^{N} \Big(\|u^+_{h,n-1}\|^2_{\Omega_{i,n-1}} - \|u^-_{h,n-1}\|^2_{\Omega_{i,n-1}} \Big) \\
&= \sum_{i=1}^{2}\sum_{n=1}^{N} \Big(\|u^-_{h,n}\|^2_{\Omega_{i,n}} \underbrace{- \|u^+_{h,n-1}\|^2_{\Omega_{i,n-1}} + \|u^+_{h,n-1}\|^2_{\Omega_{i,n-1}}}_{=0} - \|u^-_{h,n-1}\|^2_{\Omega_{i,n-1}} \Big) \\
&= \sum_{i=1}^{2}\sum_{n=1}^{N-1} \Big(\underbrace{\|u^-_{h,n}\|^2_{\Omega_{i,n}} - \|u^-_{h,n}\|^2_{\Omega_{i,n}}}_{=0} \Big) + \sum_{i=1}^{2} \Big(\|u^-_{h,N}\|^2_{\Omega_{i,N}} - \|u^-_{h,0}\|^2_{\Omega_{i,0}} \Big) \\
&= \sum_{i=1}^{2} \|u^-_{h,N}\|^2_{\Omega_{i,N}} - \sum_{i=1}^{2} \|u_0\|^2_{\Omega_i(0)},
\end{aligned} \tag{6.10}$$

where we have used $u^-_{h,0} = u_0$ in the last equality. The fourth term in (6.5) is added to the second term in the last row of (6.6) to yield

$$
\begin{aligned}
&\sum_{n=1}^{N} \int_{\Gamma_n} \bar{n}^t [u_h^2] \,\mathrm{d}\bar{s} + \sum_{n=1}^{N} \int_{\Gamma_n} -2\bar{n}^t [u_h] u_{h,\sigma} \,\mathrm{d}\bar{s} = \sum_{n=1}^{N} \int_{\Gamma_n} \bar{n}^t ([u_h^2] - 2[u_h]u_{h,\sigma}) \,\mathrm{d}\bar{s} \\
&= \sum_{n=1}^{N} \int_{\Gamma_n} \bar{n}^t (u_{h,1}^2 - u_{h,2}^2 - 2u_{h,1}u_{h,\sigma} + 2u_{h,2}u_{h,\sigma}) \,\mathrm{d}\bar{s} \\
&= \sum_{n=1}^{N} \int_{\Gamma_n} \operatorname{sgn}(\bar{n}^t)\bar{n}^t (u_{h,1}^2 - 2u_{h,1}u_{h,2} + u_{h,2}^2) \,\mathrm{d}\bar{s} = \sum_{n=1}^{N} \int_{\Gamma_n} |\bar{n}^t| [u_h]^2 \,\mathrm{d}\bar{s} \\
&= \sum_{n=1}^{N} |\bar{n}^t| \|[u_h]\|_{\Gamma_n}^2 .
\end{aligned}
\tag{6.11}
$$

To obtain the third equality, we have noted that for $\sigma = 1$

$$
\begin{aligned}
&u_{h,1}^2 - u_{h,2}^2 - 2u_{h,1}u_{h,\sigma} + 2u_{h,2}u_{h,\sigma} = u_{h,1}^2 - u_{h,2}^2 - 2u_{h,1}^2 + 2u_{h,2}u_{h,1} \\
= &- (u_{h,1}^2 - 2u_{h,1}u_{h,2} + u_{h,2}^2),
\end{aligned}
$$

and for $\sigma = 2$,

$$
\begin{aligned}
&u_{h,1}^2 - u_{h,2}^2 - 2u_{h,1}u_{h,\sigma} + 2u_{h,2}u_{h,\sigma} = u_{h,1}^2 - u_{h,2}^2 - 2u_{h,1}u_{h,2} + 2u_{h,2}^2 \\
= &(u_{h,1}^2 - 2u_{h,1}u_{h,2} + u_{h,2}^2).
\end{aligned}
$$

The sign in last row of these two equalities depends on σ as $2\sigma - 3$. Since $\sigma = \frac{1}{2}(3 + \operatorname{sgn}(\bar{n}^t))$, the sign varies as $\operatorname{sgn}(\bar{n}^t)$.

To obtain (6.4), start by inserting (6.6) and (6.9) in (6.5), then use (6.10) and (6.11). Finally, move the second term in the last row of (6.10) to the right-hand side.

□

For $n = 1, \ldots, N$, and a real valued function v, we write

$$
\int_{I_n} \|v\|_{\Gamma(t)}^2 \,\mathrm{d}t = \int_{I_n} (v, v)_{\Gamma(t)} \,\mathrm{d}t = \int_{\Gamma_n} |v|^2 \,\mathrm{d}\bar{s} = \|v\|_{\Gamma_n}^2 \tag{6.12}
$$

We note that we may use (5.12) in the second term in (6.4). Doing this, rearranging the terms and using (6.12), gives us the following.

Corollary 6.2 (Using (5.12) in Lemma 6.1)**.** *Let u_h be the solution to (4.21) with $f \equiv 0$ and let u_0 be the initial value of the analytic solution to the problem presented in Section 2. We then have*

$$
\begin{aligned}
&\sum_{i=1}^{2} \|u_{h,N}^{-}\|^2_{\Omega_{i,N}} + \left(2 - \frac{4C_I}{\varepsilon}\right) \sum_{i=1}^{2}\sum_{n=1}^{N} \int_{I_n} \|\nabla u_h\|^2_{\Omega_i(t)} \,\mathrm{d}t + \sum_{i=1}^{2}\sum_{n=1}^{N} \|[u_h]_{n-1}\|^2_{\Omega_{i,n-1}} \\
&+ \sum_{n=1}^{N} \frac{2}{\varepsilon} \|\langle \partial_{\bar{n}^x} u_h \rangle\|^2_{-1/2,h,\Gamma_n} + \sum_{n=1}^{N} \left(|\bar{n}^t| + 2\left(\frac{|\bar{n}^x|\gamma - \varepsilon}{h} \right)\right) \|[u_h]\|^2_{\Gamma_n} \\
&+ \sum_{n=1}^{N} \int_{I_n} |\bar{n}^x| \lambda \|[\nabla u_h]\|^2_{\Omega_O(t)} \,\mathrm{d}t \\
\le &\sum_{i=1}^{2} \|u_0\|^2_{\Omega_i(0)}.
\end{aligned}
\tag{6.13}
$$

6.2 The second auxiliary stability estimate

Following the proof of Eriksson and Johnson [2, 3], we would start by taking $v = -\Delta_{h,t} u_h$ in (6.3) to try to obtain the second auxiliary stability estimate. But we cannot make that choice of v, since $-\Delta_{h,t} u_h \notin V_h$ in general. This is because $\Delta_{h,t}$ sends a function to $V_h(t)$, which is a mesh-dependent function space, and since $\mathcal{T}_G$ changes location when $|\mu| > 0$, the discrete Laplacian $\Delta_{h,t}$ is time-dependent. The nodal coefficients of $-\Delta_{h,t} u_h$ are therefore not necessarily piecewise polynomials of degree $\le q$ along the nodal trajectories, resulting in $-\Delta_{h,t} u_h \notin V_h$. The second of the two auxiliary stability estimates is therefore proposed as the following conjecture.

Conjecture 6.1 (The second auxiliary stability estimate)**.** *Let u_h be the solution to* (4.21) *with $f \equiv 0$ and let u_0 be the initial value of the analytic solution to the problem presented in Section 2. We then have*

$$
\begin{aligned}
&\sum_{i=1}^{2}\sum_{n=1}^{N} t_n \int_{I_n} \|\dot{u}_h\|^2_{\Omega_i(t)} + \|\nabla u_h\|^2_{\Omega_i(t)} \,\mathrm{d}t \\
&+ \sum_{i=1}^{2}\sum_{n=1}^{N} t_n \int_{I_n} \|\Delta_{h,t} u_h\|^2_{\Omega_i(t)} + |\mu|^{-2}_{I_n} \|\Delta_{h,t}\mathcal{E}_n^t u_h - \mathcal{E}_n^t \Delta_{h,n} u_h\|^2_{\Omega_i(t)} \,\mathrm{d}t \\
&+ \sum_{i=1}^{2}\sum_{n=1}^{N} \frac{t_n}{k_n} \|[u_h]_{n-1}\|^2_{\Omega_{i,n-1}} + \sum_{n=1}^{N} \frac{t_n}{k_n} \|[u_h]\|^2_{\Gamma_n} \\
\le\ & C \sum_{i=1}^{2} \|u_0\|^2_{\Omega_i(0)},
\end{aligned}
\tag{6.14}
$$

where $C > 0$ is a constant, and $|\mu|_{I_n} = \max_{t \in I_n} |\mu(t)|$.

6.3 Proof of the main stability estimate

For the proof of Theorem 6.1, we will use some additional inequalities. For $n = 1, \dots, N$ and $a_n, b_n \ge 0$, we have

$$\sum_{n=1}^{N} b_n^2 \geq \left(\sum_{n=1}^{N} a_n b_n\right)^2 \left(\sum_{n=1}^{N} a_n^2\right)^{-1}, \tag{6.15}$$

which comes from Cauchy-Schwarz inequality. For $a, b \in \mathbb{R}$,

$$a^2 + b^2 \geq \frac{1}{2}(a+b)^2. \tag{6.16}$$

Furthermore, noting that $k_1 = t_1$, since $t_0 = 0$, we have

$$\sum_{n=1}^{N} \frac{k_n}{t_n} = 1 + \sum_{n=2}^{N} \frac{k_n}{t_n} \leq 1 + \int_{t_1}^{t_N} \frac{1}{t}\,\mathrm{d}t = 1 + \log(t_N/k_1). \tag{6.17}$$

We are now ready to prove Theorem 6.1.

Proof. The proof idea is to derive lower bounds for separate terms on the left-hand sides of the auxiliary stability estimates. These lower bounds will then be used to obtain the main stability estimate. From the first auxiliary stability estimate (6.4), we have

$$\sum_{i=1}^{2} \|u_0\|_{\Omega_i(0)}^2 \geq \sum_{i=1}^{2} \|u_{h,N}^-\|_{\Omega_{i,N}}^2 \geq \frac{1}{2}\left(\sum_{i=1}^{2} \|u_{h,N}^-\|_{\Omega_{i,N}}\right)^2, \tag{6.18}$$

where we have used (6.16) to obtain the last inequality. From (6.18), we obtain

$$\sum_{i=1}^{2} \|u_{h,N}^-\|_{\Omega_{i,N}} \leq C\left(\sum_{i=1}^{2} \|u_0\|_{\Omega_i(0)}^2\right)^{1/2}, \tag{6.19}$$

where $C = \sqrt{2} > 0$. In the second auxiliary stability estimate (6.14), we may categorize the terms on the left-hand side as either *integral* terms or *jump* terms. Since the treatment will be the same for all the integral terms and very similar for both jump terms, we may consider a generic integral term and a generic jump term instead of treating all the terms on the left-hand side in (6.14). We start with the generic integral term:

$$\begin{aligned}
C\sum_{i=1}^{2} \|u_0\|_{\Omega_i(0)}^2 \geq& \sum_{i=1}^{2}\sum_{n=1}^{N} t_n \int_{I_n} \|w\|_{\Omega_i(t)}^2\,\mathrm{d}t \\
=& \sum_{i=1}^{2}\sum_{n=1}^{N} \frac{t_n}{k_n}\left(\int_{I_n} 1^2\,\mathrm{d}t\right)\left(\int_{I_n} \|w\|_{\Omega_i(t)}^2\,\mathrm{d}t\right) \\
\geq& \sum_{i=1}^{2}\sum_{n=1}^{N} \frac{t_n}{k_n}\left(\int_{I_n} \|w\|_{\Omega_i(t)}\,\mathrm{d}t\right)^2 \geq \sum_{n=1}^{N} \frac{t_n}{2k_n}\left(\sum_{i=1}^{2}\int_{I_n} \|w\|_{\Omega_i(t)}\,\mathrm{d}t\right)^2 \\
\geq& \left(\sum_{n=1}^{N} \frac{2k_n}{t_n}\right)^{-1}\left(\sum_{i=1}^{2}\sum_{n=1}^{N}\int_{I_n} \|w\|_{\Omega_i(t)}\,\mathrm{d}t\right)^2,
\end{aligned} \tag{6.20}$$

where we have used Cauchy-Schwarz inequality to obtain the second inequality, (6.16) to obtain the third inequality. To obtain the last inequality, we have used (6.15) with $a_n = \sqrt{2k_n/t_n}$ and $b_n = \sqrt{t_n/2k_n}\sum_{i=1}^{2}\int_{I_n} \|w\|_{\Omega_i(t)}\,\mathrm{d}t$. From (6.20), we have

$$\sum_{i=1}^{2}\sum_{n=1}^{N}\int_{I_n}\|w\|_{\Omega_i(t)}\,\mathrm{d}t \le C\Big(\sum_{n=1}^{N}\frac{2k_n}{t_n}\Big)^{1/2}\Big(\sum_{i=1}^{2}\|u_0\|^2_{\Omega_i(0)}\Big)^{1/2}$$
$$\le C(1+\log(t_N/k_1))^{1/2}\Big(\sum_{i=1}^{2}\|u_0\|^2_{\Omega_i(0)}\Big)^{1/2}, \tag{6.21}$$

where we have used (6.17) to obtain the last inequality. The generic jump term is treated as follows:

$$C\sum_{i=1}^{2}\|u_0\|^2_{\Omega_i(0)} \ge \sum_{i=1}^{2}\sum_{n=1}^{N}\frac{t_n}{k_n}\|[w]\|^2 \ge \sum_{n=1}^{N}\frac{t_n}{2k_n}\Big(\sum_{i=1}^{2}\|[w]\|\Big)^2$$
$$\ge \Big(\sum_{n=1}^{N}\frac{2k_n}{t_n}\Big)^{-1}\Big(\sum_{i=1}^{2}\sum_{n=1}^{N}\|[w]\|\Big)^2, \tag{6.22}$$

where we have used (6.16) to obtain the second inequality, and (6.15) with $a_n = \sqrt{2k_n/t_n}$ and $b_n = \sqrt{t_n/2k_n}\sum_{i=1}^{2}\|[w]\|$, to obtain the last inequality. From (6.22), we have

$$\sum_{i=1}^{2}\sum_{n=1}^{N}\|[w]\| \le C\Big(\sum_{n=1}^{N}\frac{2k_n}{t_n}\Big)^{1/2}\Big(\sum_{i=1}^{2}\|u_0\|^2_{\Omega_i(0)}\Big)^{1/2}$$
$$\le C(1+\log(t_N/k_1))^{1/2}\Big(\sum_{i=1}^{2}\|u_0\|^2_{\Omega_i(0)}\Big)^{1/2}, \tag{6.23}$$

where we have used (6.17) to obtain the last inequality. By adding (6.19), (6.21) for all the integral terms in (6.14), (6.23) for the two jump terms in (6.14), and noting that $\log(t_N/k_1) \ge 0$, we may obtain the main stability estimate (6.1). This concludes the proof of Theorem 6.1. $\square$

7 A priori error analysis

To prove an a priori error estimate, we follow the methodology of [2, 3] and make suitable extensions to account for the cut mesh space-time formulation.

Theorem 7.1 (A priori error estimate)**.** *Let u be the solution to* (2.1)*, let u_h be the finite element solution defined by* (4.21)*, and for $n = 1, \dots, N$, let μ be constant on I_n. Then, for $q = 0, 1$,*

$$\begin{aligned}
&\|u(t_N) - u_{h,N}^-\|_{\Omega_0} \leq \\
&\leq C_N \max_{\substack{1\leq i\leq 2\\ 1\leq n\leq N}} \Bigg\{ k_n^{2q+1}\|\dot{u}^{(2q+1)}\|_{\Omega_i,I_n} + \min_{1\leq j\leq 2} \|h^j D^j u\|_{\Omega_i,I_n} + (1-q)E^0_{q,(\Omega_i,I_n)}(u) \\
&+ |\mu|_{I_n}\Bigg(k_n^{q+1}\big(\|\dot{u}^{(q+1)}\|_{\Omega_i,I_n} + \|\dot{u}^{(q+1)}\|_{\Gamma_n}\big) + E^0_{q,(\Omega_i,I_n)}(u) + E^0_{q,\Gamma_n}(u) \\
&+ \min_{1\leq j\leq 2}\big\{\|h^j D^j u\|_{\Omega_i,I_n} + \|h^j D^j u\|_{\Gamma_n}\big\}\Bigg)\Bigg\},
\end{aligned} \tag{7.1}$$

where $\|\cdot\|_{\Omega_0} = \|\cdot\|_{L_2(\Omega_0)}$, $C_N = C(\log(t_N/k_N)+1)^{1/2}$, where $C = C(\mu) > 0$ is a constant, $k_n = t_n - t_{n-1}$, $\|w\|_{\Omega_i,I_n} = \max_{t\in I_n}\|w\|_{\Omega_i(t)}$, $\dot{u}^{(2q+1)} = \partial^{2q+1}u/\partial t^{2q+1}$, h is the largest diameter of a simplex in $\mathcal{T}_0 \cup \mathcal{T}_G$, $D^j u = \max\{|D^\alpha u| : |\alpha| = j\}$, where $D^\alpha = \partial^{|\alpha|}/\partial x^{\alpha_1}\dots\partial x^{\alpha_d}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_d$, $|\mu|_{I_n} = \max_{t\in I_n}|\mu(t)|$, and $E^0_{q,(\Omega_i,I_n)}(u)$ is

$$E^0_{q,(\Omega_i,I_n)}(u) = |\mu|_{I_n} C k_n^{q+1}\Big(\|\nabla u\|_{\Omega_i,I_n} + q\big(\|H(u)\mu\|_{\Omega_i,I_n} + \|\nabla \dot{u}\|_{\Omega_i,I_n}\big)\Big), \tag{7.2}$$

where $\dot{\mu}^{(0)} = \mu$, $C = C(\mu) > 0$, and $H(u)$ is the Hessian matrix of u. The vector $H(u)\mu$ is the result of a matrix multiplication between matrix $H(u)$ and vector μ. The function $E^0_{q,\Gamma_n}(u)$ is the same as $E^0_{q,(\Omega_i,I_n)}(u)$ but with the norm $\|\cdot\|_{\Gamma_n}$ instead of $\|\cdot\|_{\Omega_i,I_n}$.

Proof. We split the error $e = u - u_h$ into the two parts $\eta = u - \tilde{u}$ and $\theta = \tilde{u} - u_h \in V_h$, where $\tilde{u} = \tilde{I}_n R_t u \in V_h$, and $\tilde{I}_n$ is the interpolation operator defined by (5.42). Note that from the definition of $\tilde{I}_n$, $\tilde{u} \in V_h$ only because μ is constant on I_n. With $e = \eta + \theta$, we have

$$\|u(t_N) - u_{h,N}^-\|_{\Omega_0} = \|e_N^-\|_{\Omega_0} = \|(\eta+\theta)_N^-\|_{\Omega_0} \leq \underbrace{\|\eta_N^-\|_{\Omega_0}}_{\text{The }\eta\text{-part}} + \underbrace{\|\theta_N^-\|_{\Omega_0}}_{\text{The }\theta\text{-part}}, \tag{7.3}$$

where $\|v\|_{\Omega_0} = \|v\|_{L_2(\Omega_0)}$. We may consider the η-part and the θ-part separately.

Estimation of the η-part

We first consider the term in (7.3) involving η. Since $\eta = u - \tilde{u} = u - \tilde{I}_n R_t u$, we have

$$
\begin{aligned}
\|\eta_N^-\|_{\Omega_0} &= \|(u-\tilde{I}_n R_t u)_N^-\|_{\Omega_0} = \|u(t_N) - (\tilde{I}_N R_N u)_N^-\|_{\Omega_0} = \|u(t_N) - (R_N u)_N^-\|_{\Omega_0} \\
&= \left(\sum_{i=1}^{2} \|u - R_N u\|^2_{\Omega_{i,N}} \right)^{1/2} \leq \left(\sum_{i=1}^{2} \|u - R_t u\|^2_{\Omega_{i,I_N}} \right)^{1/2} \\
&\leq \max_{\substack{1\leq i\leq 2 \\ 1\leq n\leq N}} \left\{ \|u - R_t u\|^2_{\Omega_{i,I_n}} \right\}^{1/2} \left(\sum_{i=1}^{2} 1 \right)^{1/2} \\
&= C \max_{\substack{1\leq i\leq 2 \\ 1\leq n\leq N}} \left\{ \|u - R_t u\|_{\Omega_{i,I_n}} \right\},
\end{aligned} \tag{7.4}
$$

where $\|w\|_{\Omega_i,I_n} = \max_{t\in I_n} \|w\|_{\Omega_i(t)}$, $R_N = R_{t_N}$ and $C = \sqrt{2} > 0$. Here we have used (5.43a) to obtain the third equality.

Estimation of the θ-part

We now consider the terms in (7.3) involving θ. We first note that from the Galerkin orthogonality (5.31), we have

$$
B_h(\theta, z_h) = -B_h(\eta, z_h), \tag{7.5}
$$

where we have used $e = \eta + \theta$ and chosen $v = z_h$. Since $\theta = \tilde{u} - u_h \in V_h$ is a permissible test function for the discrete dual problem (5.32), we may take $v = \theta$ in (5.32) and choose $z_{h,N}^+ = \theta_N^-$ to obtain

$$
B_h(\theta, z_h) = \sum_{i=1}^{2} \|\theta_N^-\|^2_{\Omega_{i,N}}. \tag{7.6}
$$

Combining (7.5) and (7.6), and using Lemma 5.2, we obtain the error representation

$$
\begin{aligned}
\sum_{i=1}^{2} \|\theta_N^-\|^2_{\Omega_{i,N}} &= - B_h(\eta, z_h) \\
&= \sum_{i=1}^{2} \sum_{n=1}^{N} \underbrace{\int_{I_n} (\eta, \dot{z}_h)_{\Omega_i(t)} \,\mathrm{d}t}_{=\,\mathrm{I}} - \sum_{n=1}^{N} \underbrace{\int_{I_n} A_{h,t}(\eta, z_h) \,\mathrm{d}t}_{=\,\mathrm{II}} \\
&\quad + \sum_{i=1}^{2} \sum_{n=1}^{N-1} \underbrace{(\eta_n^-, [z_h]_n)_{\Omega_{i,n}}}_{=\,\mathrm{III}} - \sum_{i=1}^{2} \underbrace{(\eta_N^-, z_{h,N}^-)_{\Omega_{i,N}}}_{=\,\mathrm{IV}} - \sum_{n=1}^{N} \underbrace{\int_{\Gamma_n} \bar{n}^t \eta_\rho [z_h] \,\mathrm{d}\bar{s}}_{=\,\mathrm{V}}.
\end{aligned} \tag{7.7}
$$

We consider the terms on the right-hand side of (7.7) separately, starting with the first term. Recall the partition of a function $\dot{v} \in \dot{V}_h^n$, given by (4.17). Since $\dot{z}_h \in \dot{V}_h^n$, we may write $\dot{z}_h = \delta_t z_h - \hat{\mu} \cdot \nabla z_h$. With this and noting that $\eta = u - \tilde{u} = u - \tilde{I}_n R_t u$, we have for $i = 1, 2$ and $n = 1, \ldots, N$,

$$
\begin{aligned}
\mathrm{I} &= \int_{I_n} (\eta, \dot{z}_h)_{\Omega_i(t)}\,\mathrm{d}t = \int_{I_n} (u - \tilde{I}_n R_t u, \delta_t z_h - \hat{\mu}\cdot\nabla z_h)_{\Omega_i(t)}\,\mathrm{d}t \\
&= \int_{I_n} (u, \delta_t z_h)_{\Omega_i(t)}\,\mathrm{d}t - \int_{I_n} (\tilde{I}_n R_t u, \delta_t z_h)_{\Omega_i(t)}\,\mathrm{d}t \\
&\quad + \int_{I_n} (u - \tilde{I}_n R_t u \underbrace{+ R_t u - R_t u}_{=0}, -\hat{\mu}\cdot\nabla z_h)_{\Omega_i(t)}\,\mathrm{d}t \\
&\overset{\text{4th}}{=} \int_{I_n} (u - R_t u, \delta_t z_h)_{\Omega_i(t)}\,\mathrm{d}t + \int_{I_n} (u - R_t u, -\hat{\mu}\cdot\nabla z_h)_{\Omega_i(t)}\,\mathrm{d}t \\
&\quad + \int_{I_n} (R_t u - \tilde{I}_n R_t u, -\hat{\mu}\cdot\nabla z_h)_{\Omega_i(t)}\,\mathrm{d}t \\
&= \int_{I_n} (u - R_t u, \dot{z}_h)_{\Omega_i(t)}\,\mathrm{d}t + \int_{I_n} (R_t u - \tilde{u}, -\hat{\mu}\cdot\nabla z_h)_{\Omega_i(t)}\,\mathrm{d}t \\
&\leq \|u - R_t u\|_{\Omega_i, I_n} \int_{I_n} \|\dot{z}_h\|_{\Omega_i(t)}\,\mathrm{d}t + \|R_t u - \tilde{u}\|_{\Omega_i, I_n} |\mu|_{I_n} \int_{I_n} \|\nabla z_h\|_{\Omega_i(t)}\,\mathrm{d}t,
\end{aligned}
\tag{7.8}
$$

where $|\mu|_{I_n} = \max_{t\in I_n} |\mu(t)|$. We have used the fact that $\delta_t z_h \in \delta_t V_h^n$ and applied (5.43b) to the second term in the second row to obtain the first term on the right-hand side in the fourth equality.

For $n = 1, \ldots, N$, the second term on the right-hand side of (7.7) is

$$
\begin{aligned}
\mathrm{II} &= -\int_{I_n} A_{h,t}(\eta, z_h)\,\mathrm{d}t = -\int_{I_n} A_{h,t}(u, z_h) - A_{h,t}(\tilde{u}, z_h)\,\mathrm{d}t \\
&= \int_{I_n} -A_{h,t}(R_t u - \tilde{u}, z_h)\,\mathrm{d}t = \int_{I_n} \sum_{i=1}^{2} (R_t u - \tilde{u}, \Delta_{h,t} z_h)_{\Omega_i(t)}\,\mathrm{d}t,
\end{aligned}
\tag{7.9}
$$

where we have used the definition of the Ritz projector (5.35) to obtain the third equality and the definition of the discrete Laplacian (5.37), together with the symmetry of $A_{h,t}$, to obtain the fourth equality. The subsequent treatment of II is different for $q = 0$ and $q \geq 1$. For $q = 0$, we continue by writing

II for $q = 0$

$$
\begin{aligned}
\mathrm{II} &= \sum_{i=1}^{2} \int_{I_n} (R_t u - \tilde{u}, \Delta_{h,t} z_h)_{\Omega_i(t)}\,\mathrm{d}t \\
&\leq \sum_{i=1}^{2} \|R_t u - \tilde{u}\|_{\Omega_i, I_n} \int_{I_n} \|\Delta_{h,t} z_h\|_{\Omega_i(t)}\,\mathrm{d}t.
\end{aligned}
\tag{7.10}
$$

For $q \geq 1$, we may instead continue by writing

II for $q \geq 1$

$$\begin{aligned}
\mathrm{II} &= \sum_{i=1}^{2} \int_{I_n} (R_t u - \tilde{u}, \Delta_{h,t} \Big\{ \mathcal{E}_n^t z_h + \int_{t_n}^{t} \mathcal{E}_s^t \delta_t z_h \, \mathrm{d}s \Big\})_{\Omega_i(t)} \, \mathrm{d}t \\
&= \underbrace{\sum_{i=1}^{2} \int_{I_n} (R_t u - \tilde{u}, \Delta_{h,t} \mathcal{E}_n^t z_h)_{\Omega_i(t)} \, \mathrm{d}t}_{=\,\mathrm{IIa}} \\
&\quad + \underbrace{\sum_{i=1}^{2} \int_{I_n} (R_t u - \tilde{u}, \Delta_{h,t} \Big\{ \int_{t_n}^{t} \mathcal{E}_s^t \delta_t z_h \, \mathrm{d}s \Big\})_{\Omega_i(t)} \, \mathrm{d}t,,}_{=\,\mathrm{IIb}}
\end{aligned} \tag{7.11}$$

where we have partitioned z_h in the manner of (5.64). We consider IIa and IIb separately, starting with IIa:

$$\begin{aligned}
\mathrm{IIa} &= \sum_{i=1}^{2} \int_{I_n} (R_t u - \tilde{u}, \Delta_{h,t} \mathcal{E}_n^t z_h)_{\Omega_i(t)} \, \mathrm{d}t \\
&= \sum_{i=1}^{2} \int_{I_n} (R_t u - \tilde{u}, \Delta_{h,t} \mathcal{E}_n^t z_h - \mathcal{E}_n^t \Delta_{h,n} z_h)_{\Omega_i(t)} \, \mathrm{d}t \\
&\leq \sum_{i=1}^{2} \|R_t u - \tilde{u}\|_{\Omega_i, I_n} |\mu|_{I_n} \int_{I_n} |\mu|_{I_n}^{-1} \|\Delta_{h,t} \mathcal{E}_n^t z_h - \mathcal{E}_n^t \Delta_{h,n} z_h\|_{\Omega_i(t)} \, \mathrm{d}t,
\end{aligned} \tag{7.12}$$

where we have added $\mathcal{E}_n^t \Delta_{h,n} z_h$, which lies in $\delta_t V_h^n$ for $q \geq 1$, thus making it orthogonal to $R_t u - \tilde{u}$. We leave IIa like this and consider IIb. For $q = 1$, we may treat IIb in the following way:

$$\begin{aligned}
\mathrm{IIb} &= \sum_{i=1}^{2} \int_{I_n} (R_t u - \tilde{u}, \Delta_{h,t} \Big\{ \int_{t_n}^{t} \mathcal{E}_s^t \delta_t z_h \, \mathrm{d}s \Big\})_{\Omega_i(t)} \, \mathrm{d}t \\
&= \sum_{i=1}^{2} \int_{I_n} (\Delta_{h,t}(R_t u - \tilde{u}), (t - t_n) \delta_t z_h)_{\Omega_i(t)} \, \mathrm{d}t \\
&= \sum_{i=1}^{2} \int_{I_n} (\Delta_{h,t}(R_t u - \tilde{u}), (t - t_n)(\dot{z}_h + \hat{\mu} \cdot \nabla z_h))_{\Omega_i(t)} \, \mathrm{d}t \\
&\leq \sum_{i=1}^{2} \int_{I_n} \|\Delta_{h,t}(R_t u - \tilde{u})\|_{\Omega_i(t)} |t - t_n| \|\dot{z}_h + \hat{\mu} \cdot \nabla z_h\|_{\Omega_i(t)} \, \mathrm{d}t \\
&\leq \sum_{i=1}^{2} \|\Delta_{h,t}(R_t u - \tilde{u})\|_{\Omega_i, I_n} k_n \int_{I_n} \|\dot{z}_h\|_{\Omega_i(t)} + |\hat{\mu}| \|\nabla z_h\|_{\Omega_i(t)} \, \mathrm{d}t,
\end{aligned} \tag{7.13}$$

where we first have used (5.65) and then the definition of the discrete Laplacian (5.37) twice, since $(t - t_n)\delta_t z_h \in V_h(t)$, to obtain the second equality. To obtain the third equality, we have used (4.17) on $\dot{z}_h$. The first *inequality* comes from Cauchy-Schwarz's inequality and the k_n in the second *inequality* comes from $|t - t_n| \leq k_n$. We leave IIb like this.

For $i = 1, 2$ and $n = 1, \ldots, N-1$, the third term on the right-hand side of (7.7) can be estimated by

$$\begin{aligned} \text{III} &= (\eta_n^-, [z_h]_n)_{\Omega_{i,n}} = (u - (\tilde{I}_n R_t u)_n^-, [z_h]_n)_{\Omega_{i,n}} = (u - (R_t u)_n^-, [z_h]_n)_{\Omega_{i,n}} \\ &\leq \|u - (R_t u)_n^-\|_{\Omega_{i,n}} \|[z_h]_n\|_{\Omega_{i,n}} \leq \|u - R_t u\|_{\Omega_i, I_n} \|[z_h]_n\|_{\Omega_{i,n}}, \end{aligned} \tag{7.14}$$

where we have used (5.43a) to obtain the third equality. Similarly, for $i = 1, 2$, the fourth term on the right-hand side of (7.7) can be estimated by

$$\begin{aligned} \text{IV} &= -(\eta_N^-, z_{h,N}^-)_{\Omega_{i,N}} = -(u - (\tilde{I}_N R_t u)_N^-, z_{h,N}^-)_{\Omega_{i,N}} = -(u - (R_t u)_N^-, z_{h,N}^-)_{\Omega_{i,N}} \\ &\leq |-(u - (R_t u)_N^-, z_{h,N}^-)_{\Omega_{i,N}}| \leq \|u - (R_t u)_N^-\|_{\Omega_{i,N}} \|z_{h,N}^-\|_{\Omega_{i,N}} \\ &\leq \|u - R_t u\|_{\Omega_i, I_N} \|z_{h,N}^-\|_{\Omega_{i,N}}, \end{aligned} \tag{7.15}$$

where we again have used (5.43a) to obtain the third equality. Now we consider the fifth and last term in (7.7). For $n = 1, \ldots, N$, we have

$$\begin{aligned} \text{V} &= -\int_{\Gamma_n} \bar{n}^t \eta_\rho [z_h] \, \mathrm{d}\bar{s} = -\int_{\Gamma_n} \bar{n}^t (u - \tilde{u})_\rho [z_h] \, \mathrm{d}\bar{s} \\ &\leq |(\bar{n}^t (u - \tilde{u})_\rho, [z_h])_{\Gamma_n}| \leq |\bar{n}^t|_{I_n} \|(u - \tilde{u})_\rho\|_{\Gamma_n} \|[z_h]\|_{\Gamma_n} \\ &= |\bar{n}^t|_{I_n} \|(u - \tilde{u} \underbrace{+ R_t u - R_t u}_{=0})_\rho\|_{\Gamma_n} \|[z_h]\|_{\Gamma_n} \\ &\leq |\mu|_{I_n} \Big(\|(u - R_t u)_\rho\|_{\Gamma_n} + \|(R_t u - \tilde{u})_\rho\|_{\Gamma_n} \Big) \|[z_h]\|_{\Gamma_n}, \end{aligned} \tag{7.16}$$

where we have used (3.2) in $|\bar{n}^t| = |-1/(\sqrt{(n_1 \cdot \mu)^2 + 1})(n_1 \cdot \mu)| \leq |\mu|$, to obtain the last inequality.

Summing up what we have for $q = 0$, i.e. inserting (7.8), (7.10), (7.14), (7.15), and (7.16) in (7.7), we obtain

$q = 0$

$$\begin{aligned}
\sum_{i=1}^{2} \|\theta_N^-\|^2_{\Omega_{i,N}} \leq & \sum_{i=1}^{2}\sum_{n=1}^{N} \|u - R_t u\|_{\Omega_i,I_n} \int_{I_n} \|\dot{z}_h\|_{\Omega_i(t)} \,\mathrm{d}t \\
& + \sum_{i=1}^{2}\sum_{n=1}^{N} \|R_t u - \tilde{u}\|_{\Omega_i,I_n} |\mu|_{I_n} \int_{I_n} \|\nabla z_h\|_{\Omega_i(t)} \,\mathrm{d}t \\
& + \sum_{i=1}^{2}\sum_{n=1}^{N} \|R_t u - \tilde{u}\|_{\Omega_i,I_n} \int_{I_n} \|\Delta_{h,t} z_h\|_{\Omega_i(t)} \,\mathrm{d}t \\
& + \sum_{i=1}^{2}\sum_{n=1}^{N} \|u - R_t u\|_{\Omega_i,I_n} \|[z_h]_n\|_{\Omega_{i,n}} \\
& + \sum_{i=1}^{2} \|u - R_t u\|_{\Omega_i,I_N} \|z_{h,N}^-\|_{\Omega_{i,N}} \\
& + \sum_{n=1}^{N} |\mu|_{I_n} \Big(\|(u - R_t u)_\rho\|_{\Gamma_n} + \|(R_t u - \tilde{u})_\rho\|_{\Gamma_n} \Big) \|[z_h]\|_{\Gamma_n}.
\end{aligned} \tag{7.17}$$

By taking the max over $1 \leq i \leq 2$ and $1 \leq n \leq N$ for all the factors on the left in every term in (7.17), we get

$$\begin{aligned}
\sum_{i=1}^{2} \|\theta_N^-\|^2_{\Omega_{i,N}} \leq & \max_{\substack{1\leq i\leq 2\\ 1\leq n\leq N}} \Big\{ \|u - R_t u\|_{\Omega_i,I_n} + (|\mu|_{I_n} + 1)\|R_t u - \tilde{u}\|_{\Omega_i,I_n} \\
& + |\mu|_{I_n} \Big(\|(u - R_t u)_\rho\|_{\Gamma_n} + \|(R_t u - \tilde{u})_\rho\|_{\Gamma_n} \Big) \Big\} \times \\
& \times \Big(\sum_{i=1}^{2}\sum_{n=1}^{N} \int_{I_n} \|\dot{z}_h\|_{\Omega_i(t)} + \|\nabla z_h\|_{\Omega_i(t)} + \|\Delta_{h,t} z_h\|_{\Omega_i(t)} \,\mathrm{d}t \\
& + \sum_{i=1}^{2}\sum_{n=1}^{N} \|[z_h]_n\|_{\Omega_{i,n}} + \sum_{i=1}^{2} \|z_{h,N}^-\|_{\Omega_{i,N}} + \sum_{n=1}^{N} \|[z_h]\|_{\Gamma_n} \Big) \\
\leq & C_N F_0(u) \Big(\sum_{i=1}^{2} \|\theta_N^-\|^2_{\Omega_{i,N}} \Big)^{1/2},
\end{aligned} \tag{7.18}$$

where $F_0(u)$ is the factor with the max-function. To obtain the last inequality, we have used the stability estimate (6.2) with $z_{h,N}^+ = \theta_N^-$. Analogously, summing up what we have for $q = 1$, i.e. inserting (7.8), (7.11), (7.14), (7.15), and (7.16) in (7.7), where we have inserted (7.12) and (7.13) in (7.11), we obtain

$q = 1$

$$
\begin{aligned}
\sum_{i=1}^{2} \|\theta_N^-\|^2_{\Omega_{i,N}} \leq & \sum_{i=1}^{2}\sum_{n=1}^{N} \|u - R_t u\|_{\Omega_i, I_n} \int_{I_n} \|\dot{z}_h\|_{\Omega_i(t)}\,\mathrm{d}t \\
& + \sum_{i=1}^{2}\sum_{n=1}^{N} \|R_t u - \tilde{u}\|_{\Omega_i, I_n} |\mu|_{I_n} \int_{I_n} \|\nabla z_h\|_{\Omega_i(t)}\,\mathrm{d}t \\
& + \sum_{i=1}^{2}\sum_{n=1}^{N} \|R_t u - \tilde{u}\|_{\Omega_i, I_n} |\mu|_{I_n} \int_{I_n} |\mu|^{-1}_{I_n} \|\Delta_{h,t}\mathcal{E}_n^t z_h - \mathcal{E}_n^t \Delta_{h,n} z_h\|_{\Omega_i(t)}\,\mathrm{d}t \\
& + \sum_{i=1}^{2}\sum_{n=1}^{N} \|\Delta_{h,t}(R_t u - \tilde{u})\|_{\Omega_i, I_n} k_n \int_{I_n} \|\dot{z}_h\|_{\Omega_i(t)} + |\hat{\mu}| \|\nabla z_h\|_{\Omega_i(t)}\,\mathrm{d}t \qquad (7.19) \\
& + \sum_{i=1}^{2}\sum_{n=1}^{N} \|u - R_t u\|_{\Omega_i, I_n} \|[z_h]_n\|_{\Omega_{i,n}} \\
& + \sum_{i=1}^{2} \|u - R_t u\|_{\Omega_i, I_N} \|z_{h,N}^-\|_{\Omega_{i,N}} \\
& + \sum_{n=1}^{N} |\mu|_{I_n} \Big(\|(u - R_t u)_\rho\|_{\Gamma_n} + \|(R_t u - \tilde{u})_\rho\|_{\Gamma_n} \Big) \|[z_h]\|_{\Gamma_n}.
\end{aligned}
$$

By taking the max over $1 \leq i \leq 2$ and $1 \leq n \leq N$ for all the factors on the left in every term in (7.19), we get

$$
\begin{aligned}
\sum_{i=1}^{2} \|\theta_N^-\|^2_{\Omega_{i,N}} \leq & \max_{\substack{1\leq i\leq 2\\ 1\leq n\leq N}} \Big\{ \|u - R_t u\|_{\Omega_i, I_n} + |\mu|_{I_n} \|R_t u - \tilde{u}\|_{\Omega_i, I_n} \\
& + k_n (1 + |\mu|_{I_n}) \|\Delta_{h,t}(R_t u - \tilde{u})\|_{\Omega_i, I_n} \\
& + |\mu|_{I_n} \Big(\|(u - R_t u)_\rho\|_{\Gamma_n} + \|(R_t u - \tilde{u})_\rho\|_{\Gamma_n} \Big) \Big\} \times \\
& \times 2 \Big(\sum_{i=1}^{2}\sum_{n=1}^{N} \int_{I_n} \|\dot{z}_h\|_{\Omega_i(t)} + \|\nabla z_h\|_{\Omega_i(t)}\,\mathrm{d}t \\
& + \sum_{i=1}^{2}\sum_{n=1}^{N} \int_{I_n} |\mu|^{-1}_{I_n} \|\Delta_{h,t}\mathcal{E}_n^t z_h - \mathcal{E}_n^t \Delta_{h,n} z_h\|_{\Omega_i(t)}\,\mathrm{d}t \qquad (7.20) \\
& + \sum_{i=1}^{2}\sum_{n=1}^{N} \|[z_h]_n\|_{\Omega_{i,n}} + \sum_{i=1}^{2} \|z_{h,N}^-\|_{\Omega_{i,N}} + \sum_{n=1}^{N} \|[z_h]\|_{\Gamma_n} \Big) \\
\leq & C_N F_1(u) \Big(\sum_{i=1}^{2} \|\theta_N^-\|^2_{\Omega_{i,N}} \Big)^{1/2},
\end{aligned}
$$

where $F_1(u)$ is the factor with the max-function. To obtain the last inequality, we have used the stability estimate (6.2) with $z_{h,N}^+ = \theta_N^-$.

With (7.18) and (7.20), the estimation of the θ-part for $q = 0, 1$, finally becomes

$$\|\theta_N^-\|_{\Omega_0} = \left(\sum_{i=1}^{2} \|\theta_N^-\|_{\Omega_{i,N}}^2 \right)^{1/2} \leq C_N F_q(u). \tag{7.21}$$

Estimation of $F_q(u)$

Now we need an estimate for $F_q(u)$. From (7.18) and (7.20), we note that we may write $F_q(u)$ for $q = 0, 1$, as

$$\begin{aligned} F_q(u) = \max_{\substack{1\leq i\leq 2\\ 1\leq n\leq N}} \Big\{ & \underbrace{\|u - R_t u\|_{\Omega_i,I_n}}_{=\,\mathrm{I}} + (|\mu|_{I_n} + 1 - q) \underbrace{\|R_t u - \tilde{u}\|_{\Omega_i,I_n}}_{=\,\mathrm{II}} \\ & + q k_n (1 + |\mu|_{I_n}) \underbrace{\|\Delta_{h,t}(R_t u - \tilde{u})\|_{\Omega_i,I_n}}_{=\,\mathrm{III}} \\ & + |\mu|_{I_n} \Big(\underbrace{\|(u - R_t u)_\rho\|_{\Gamma_n}}_{=\,\mathrm{IV}} + \underbrace{\|(R_t u - \tilde{u})_\rho\|_{\Gamma_n}}_{=\,\mathrm{V}} \Big) \Big\}. \end{aligned} \tag{7.22}$$

We treat the differences in the norms separately, starting with the first. To estimate the first term, we use (5.36) from Conjecture 5.1:

$$\mathrm{I} = \|u - R_t u\|_{\Omega_i,I_n} \leq C_{\mathrm{I}} \min_{1\leq j\leq 2} \|h^j D^j u\|_{\Omega_i,I_n}, \tag{7.23}$$

where $C_{\mathrm{I}} > 0$ is a constant. The second term on the right-hand side of (7.22) is treated as follows:

$$\begin{aligned} \mathrm{II} = \|R_t u - \tilde{u}\|_{\Omega_i,I_n} &= \|R_t u - \tilde{I}_n R_t u + \tilde{I}_n u - \tilde{I}_n u + u - u\|_{\Omega_i,I_n} \\ &\leq \|u - R_t u\|_{\Omega_i,I_n} + \|\tilde{I}_n (u - R_t u)\|_{\Omega_i,I_n} + \|u - \tilde{I}_n u\|_{\Omega_i,I_n} \\ &\leq C_{\mathrm{II}} \Big(\min_{1\leq j\leq 2} \|h^j D^j u\|_{\Omega_i,I_n} + k_n^{q+1} \|\dot{u}^{(q+1)}\|_{\Omega_i,I_n} + E^0_{q,(\Omega_i,I_n)}(u) \Big), \end{aligned} \tag{7.24}$$

where $C_{\mathrm{II}} = C_{\mathrm{II}}(\mu) > 0$ is a constant, and $E^0_{q,(\Omega_i,I_n)}(u)$ is the same as $E_{q,(\Omega_i,I_n)}(u)$, defined by (5.45), but with $|\dot{\mu}| = 0$, since μ is constant on I_n. We have used (7.23) on the first term in the second row of (7.24). On the second term, we have first used the boundedness of $\tilde{I}_n$ from Lemma 5.4, and then applied (7.23). On the last term in the second row of (7.24), we have used (5.44) from Lemma 5.4. Now we move on to the third term in (7.22). Note that this term is only present for $q = 1$. Based on an estimate for $\Delta_{h,t}(R_t w - \tilde{I}_n R_t w)$, for the case with only a background mesh, presented in [3], we conjecture a corresponding estimate for our model.

Conjecture 7.1 (An estimate for $\Delta_{h,t}(R_t w - \tilde{I}_n R_t w)$)**.** *For $t \in (0, T]$, $n = 1, \ldots, N$, $i = 1, 2$, a function $w \in H^1(\Omega_1(t), \Omega_2(t))$, the discrete Laplacian $\Delta_{h,t}$ defined by (5.37), the Ritz projection operator R_t defined by (5.35), and the interpolation operator $\tilde{I}_n$ defined by (5.42), we have for $q = 1$,*

$$\|\Delta_{h,t}(R_t w - \tilde{I}_n R_t w)\|_{\Omega_i(t)} \leq C k_n^2 \|\dot{w}^{(3)}\|_{\Omega_i(t)}, \tag{7.25}$$

where $\|w\|_{\Omega_i(t)} = \|w\|_{L_2(\Omega_i(t))}$, $C > 0$ is a constant, $k_n = t_n - t_{n-1}$, and $\dot{w}^{(3)} = \partial^3 w / \partial t^3$.

With Conjecture 7.1, the third term in (7.22) is estimated as follows

$$\mathrm{III} = \|\Delta_{h,t}(R_t u - \tilde{u})\|_{\Omega_i,I_n} \leq C_{\mathrm{III}} k_n^2 \|\dot{u}^{(3)}\|_{\Omega_i,I_n}, \tag{7.26}$$

where $C_{\mathrm{III}} > 0$ is a constant. For the treatment of the forth and fifth term in (7.22), we propose the following conjecture, which corresponds to Conjecture 5.1 and Lemma 5.4, but for the norm $\|(\cdot)_\rho\|_{\Gamma_n}$.

Conjecture 7.2 (Estimates on Γ_n)**.** *For $n = 1,\dots,N$, a function $w : D_{0,n} \to \mathbb{R}$ with sufficient regularity, and the Ritz projection operator R_t defined by* (5.35)*, we have*

$$\|(w - R_t w)_\rho\|_{\Gamma_n} \leq C_1 \min_{1\leq j\leq 2} \|h^j (D^j w)_\rho\|_{\Gamma_n}, \tag{7.27}$$

and for μ and its time derivative bounded on I_n, and $\tilde{I}_n$ defined by (5.42)*, we have for $q = 0,1$,*

$$\|(w - \tilde{I}_n w)_\rho\|_{\Gamma_n} \leq C_2 k_n^{q+1} \|(\dot{w}^{(q+1)})_\rho\|_{\Gamma_n} + E_{q,\Gamma_n}(w), \tag{7.28}$$

where $\rho = \frac{1}{2}(3 - \mathrm{sgn}(\bar{n}^t))$, $C_1, C_2 = C_2(\mu) > 0$ are constants, h and D^j are given in Conjecture 5.1*, $\dot{w}^{(q+1)}$ is given in* Lemma 5.4*, and $E_{q,\Gamma_n}(w)$ is the same as $E_{q,(\Omega_i,I_n)}(w)$, defined by* (5.45)*, but with the norm $\|(\cdot)_\rho\|_{\Gamma_n}$ instead of $\|\cdot\|_{\Omega_i,I_n}$.*

By applying (7.27) from Conjecture 7.2 on the fourth term in (7.22), and using a treatment of the fifth term in (7.22), analogous to (7.24), but with (7.27) and (7.28) from Conjecture 7.2, we get the following estimate for the fourth and fifth terms in (7.22),

$$\begin{aligned}\mathrm{IV} + \mathrm{V} &= \|(u - R_t u)_\rho\|_{\Gamma_n} + \|(R_t u - \tilde{u})_\rho\|_{\Gamma_n} \\ &\leq C_{\mathrm{IVV}} \Big(\min_{1\leq j\leq 2} \|h^j D^j u\|_{\Gamma_n} + k_n^{q+1} \|\dot{u}^{(q+1)}\|_{\Gamma_n} + E^0_{q,\Gamma_n}(u) \Big),\end{aligned} \tag{7.29}$$

where $C_{\mathrm{IVV}} = C_{\mathrm{IVV}}(\mu) > 0$ is a constant, $E^0_{q,\Gamma_n}(u)$ is the same as $E_{q,\Gamma_n}(u)$, but with $|\dot{\mu}| = 0$, since μ is constant on I_n. The index ρ has been omitted in the last row of (7.29) due to the regularity of u. With the insertion of (7.23), (7.24), (7.26) and (7.29) in (7.22), we get the following estimate for $F_q(u)$ for $q = 0,1$:

$$
\begin{aligned}
&F_q(u) \leq \dots \\
&\leq \max_{\substack{1\leq i\leq 2\\ 1\leq n\leq N}} \Bigg\{ C_{\mathrm{I}} \min_{1\leq j\leq 2} \|h^j D^j u\|_{\Omega_i, I_n} \\
&\quad + (|\mu|_{I_n} + 1 - q) C_{\mathrm{II}} \bigg(\min_{1\leq j\leq 2} \|h^j D^j u\|_{\Omega_i, I_n} + k_n^{q+1} \|\dot{u}^{(q+1)}\|_{\Omega_i, I_n} + E^0_{q,(\Omega_i, I_n)}(u) \bigg) \\
&\quad + q k_n (1 + |\mu|_{I_n}) C_{\mathrm{III}} k_n^2 \|\dot{u}^{(3)}\|_{\Omega_i, I_n} \\
&\quad + |\mu|_{I_n} C_{\mathrm{IVV}} \bigg(\min_{1\leq j\leq 2} \|h^j D^j u\|_{\Gamma_n} + k_n^{q+1} \|\dot{u}^{(q+1)}\|_{\Gamma_n} + E^0_{q,\Gamma_n}(u) \bigg) \Bigg\} \qquad (7.30) \\
&\leq C \max_{\substack{1\leq i\leq 2\\ 1\leq n\leq N}} \Bigg\{ k_n^{2q+1} \|\dot{u}^{(2q+1)}\|_{\Omega_i, I_n} + \min_{1\leq j\leq 2} \|h^j D^j u\|_{\Omega_i, I_n} + (1-q) E^0_{q,(\Omega_i, I_n)}(u) \\
&\quad + |\mu|_{I_n} \bigg(k_n^{q+1} \big(\|\dot{u}^{(q+1)}\|_{\Omega_i, I_n} + \|\dot{u}^{(q+1)}\|_{\Gamma_n} \big) + E^0_{q,(\Omega_i, I_n)}(u) + E^0_{q,\Gamma_n}(u) \\
&\quad + \min_{1\leq j\leq 2} \big\{ \|h^j D^j u\|_{\Omega_i, I_n} + \|h^j D^j u\|_{\Gamma_n} \big\} \bigg) \Bigg\},
\end{aligned}
$$

where $C = C(\mu) > 0$ is a constant. We have used the fact that μ is bounded for all $t \in (0, T]$, which comes from μ being constant on I_n, to estimate the third term on the left-hand side in the last inequality.

The final step

To obtain the desired error estimate, we insert the estimations of the η-part (7.4) and the θ-part (7.21) in (7.3) to obtain

$$
\begin{aligned}
&\|u(t_N) - u^-_{h,N}\|_{\Omega_0} \leq \dots \\
&\leq C \max_{\substack{1\leq i\leq 2\\ 1\leq n\leq N}} \Big\{ \|u - R_t u\|_{\Omega_i, I_n} \Big\} + C_N F_q(u) \leq C_N F_q(u) \\
&\leq C_N \max_{\substack{1\leq i\leq 2\\ 1\leq n\leq N}} \Bigg\{ k_n^{2q+1} \|\dot{u}^{(2q+1)}\|_{\Omega_i, I_n} + \min_{1\leq j\leq 2} \|h^j D^j u\|_{\Omega_i, I_n} + (1-q) E^0_{q,(\Omega_i, I_n)}(u) \qquad (7.31) \\
&\quad + |\mu|_{I_n} \bigg(k_n^{q+1} \big(\|\dot{u}^{(q+1)}\|_{\Omega_i, I_n} + \|\dot{u}^{(q+1)}\|_{\Gamma_n} \big) + E^0_{q,(\Omega_i, I_n)}(u) + E^0_{q,\Gamma_n}(u) \\
&\quad + \min_{1\leq j\leq 2} \big\{ \|h^j D^j u\|_{\Omega_i, I_n} + \|h^j D^j u\|_{\Gamma_n} \big\} \bigg) \Bigg\},
\end{aligned}
$$

where we have used the estimation of $F_q(u)$, given by (7.30). This concludes the proof of Theorem 7.1.

□

8 Numerical results

Here we present numerical results from the implementation of (4.21) for the following model problem in one spatial dimension:

$$\begin{cases} \dot{u} - u_{xx} = f & \text{in } (0,1)\times(0,3], \\ u = 0 & \text{on } \{0,1\}\times[0,3], \\ u = \sin^2(\pi x) & \text{in } (0,1)\times\{0\}, \end{cases} \tag{8.1a}$$

where

$$f(x,t) = -(\frac{1}{2}\sin^2(\pi x) + 2\pi^2\cos(2\pi x))e^{-t/2}. \tag{8.1b}$$

The exact solution to (8.1) is

$$u = \sin^2(\pi x)e^{-t/2}. \tag{8.2}$$

8.1 Implementation and simulation settings

To obtain the finite element solution u_h, we have used piecewise linear basis functions in space, and in time we have used the discontinuous Galerkin methods dG(0) and dG(1).

For the implementation of dG(1), three-point Lobatto quadrature was used to approximate the time integrals over I_n and the integrals over the space-time boundary Γ_n in (4.21). The reason for choosing Lobatto quadrature is that the implementation gets somewhat easier compared to, e.g., Gauss and Radau quadrature. The reason for choosing *three*-point Lobatto quadrature is to ensure that the quadrature error will be of a higher order than the finite element error. The quadrature error, that arises from applying three-point Lobatto quadrature in the aforementioned way, is of the fourth order with respect to the time step, i.e., *quadrature error* $\propto k_n^4$. Whereas, in the case of dG(1), the highest possible order that one can hope to obtain for the finite element error, with respect to the time step, is of the third order, i.e., *error* $\propto k_n^3$.

The numerical results consist of solution plots of the finite element solution u_h and error convergence plots, where we compare u_h with the exact solution u given by (8.2). The velocity μ of the overlapping mesh has been constant at the value $\mu(t_n)$ on every subinterval $I_n = (t_{n-1}, t_n]$. The stabilization parameters have been $\gamma = \lambda = 10$ in all simulations used to obtain the numerical results presented in this section.

Settings for the solution plots

The solution plots are presented for two different equidistant space-time meshes, where G is immersed in Ω_0 for all $t \in [0,3]$, and the length of G is 0.25 for both meshes. Firstly, we have *the coarse mesh*: $(22+7)\times 10$, i.e., 22 nodes for $\mathcal{T}_0$, 7 nodes for $\mathcal{T}_G$, and 10 time steps on the interval $(0,3]$. Secondly, we have *the fine mesh*: $(44+14)\times 30$, i.e., 44 nodes for $\mathcal{T}_0$, 14 nodes for $\mathcal{T}_G$, and 30 time steps on the interval $(0,3]$. We also present the solution plots on these two space-time meshes for three different velocities (μ's) of the overlapping mesh: $\mu = 0$, $\mu = 0.1$, and $\mu = \frac{1}{2}\sin(\frac{2\pi t}{3})$, which is $\frac{1}{2}\sin(\frac{2\pi t_n}{3})$ on I_n in the implementation. We thus have six different space-time meshes, shown in Figure 11 − 13 below.

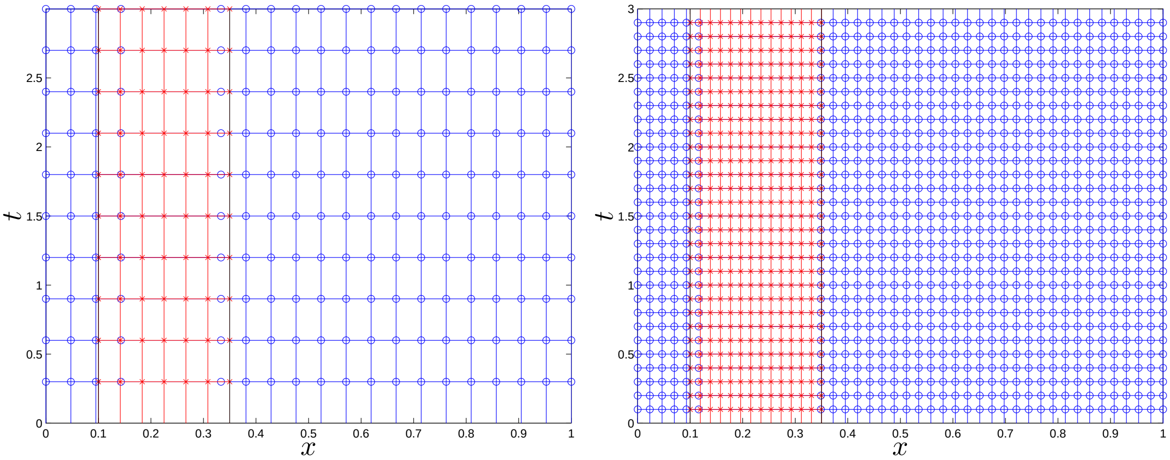

Figure 11: The coarse mesh and the fine mesh for $\mu = 0$. The background mesh $\mathcal{T}_0$ is blue and its nodes are marked with small blue circles. The overlapping mesh $\mathcal{T}_G$ is red and its nodes are marked with small red crosses. The space-time boundary Γ_n between the two meshes is black.

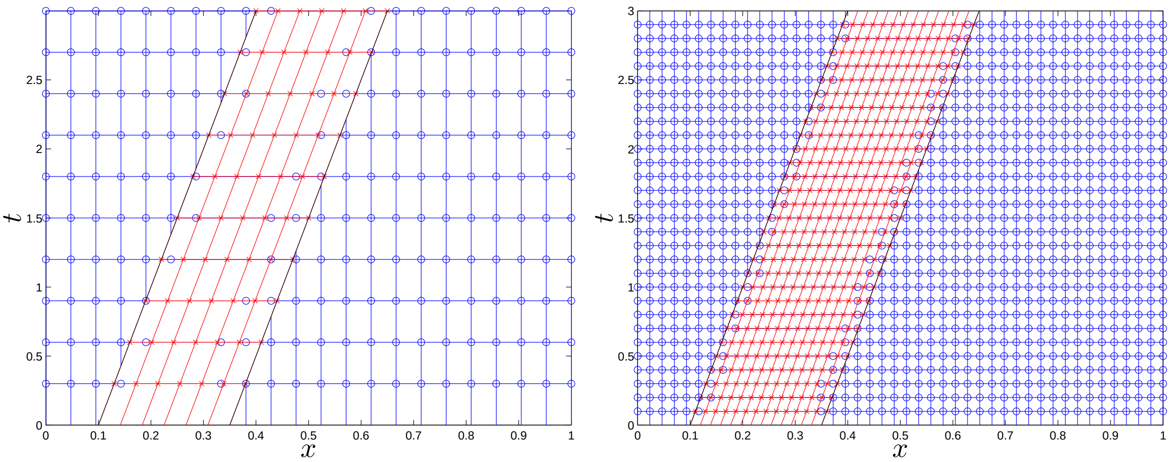

Figure 12: The coarse mesh and the fine mesh for $\mu = 0.1$. The background mesh $\mathcal{T}_0$ is blue and its nodes are marked with small blue circles. The overlapping mesh $\mathcal{T}_G$ is red and its nodes are marked with small red crosses. The space-time boundary Γ_n between the two meshes is black.

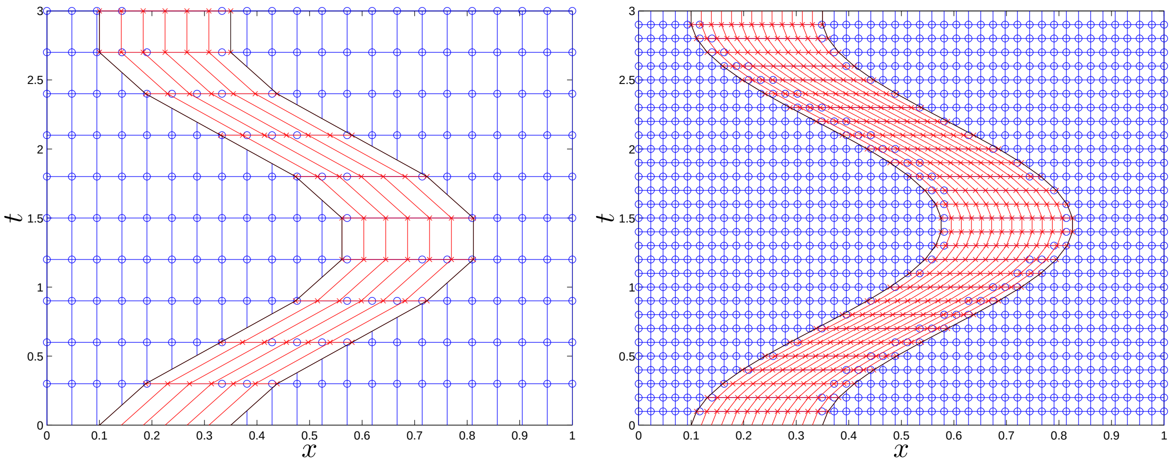

Figure 13: The coarse mesh and the fine mesh for $\mu = \frac{1}{2}\sin(\frac{2\pi t}{3})$. The background mesh $\mathcal{T}_0$ is blue and its nodes are marked with small blue circles. The overlapping mesh $\mathcal{T}_G$ is red and its nodes are marked with small red crosses. The space-time boundary Γ_n between the two meshes is black.

Settings for the error convergence plots

In the error convergence plots, the *error* is the L_2-norm of the difference between the exact and the finite element solution at the final time, i.e., $\|u(T) - u_{h,N}^{-}\|_{\Omega_0}$. We present error convergence plots displaying the *error*'s dependence on both the time step k_n and the step size h, separately, for different constant values of μ. Besides the computed *error*, each error convergence plot contains a line segment that has been computed with the linear least squares method to fit the error data. This line segment is referred to as the lls of the *error*. The slope of the lls of the *error* is given in the caption beneath each error convergence figure. Slope triangles have also been added for reference. For dG(1), we also present a plot that shows how the order of convergence of the *error* on a k_n-interval depends on μ. In the computations of the error convergence plots, both $\mathcal{T}_0$ and $\mathcal{T}_G$ have been uniform meshes, with step sizes h_0 and h_G, respectively. The time step k_n has also been constant for each instance. Furthermore, in all computations for the error convergence plots, the final time $T = 1$, the length of the overlapping mesh $\mathcal{T}_G$ has been 0.25 and $\mathcal{T}_G$ has started at the space interval $[0.125, 0.125 + 0.25]$. In the plots with the *error* versus k_n, the step sizes $h = h_0 = h_G$ have been fixed at a sufficiently small value so that the *error*'s dependence on h has been negligible in comparison with its dependence on k_n. Analogously, in the plots with the *error* versus $h = h_0 \approx h_G$, the time step k_n has been fixed at a sufficiently small value so that the *error*'s dependence on k_n has been negligible in comparison with its dependence on h. The fixed values for the step size and the time step have been obtained by trial and error.

8.2 dG(0) plots

dG(0) *solution plots*

Figure 14 – 19 display the dG(0) finite element solution u_h on the six different space-time meshes, shown in Figure 11 – 13, from two different angles.

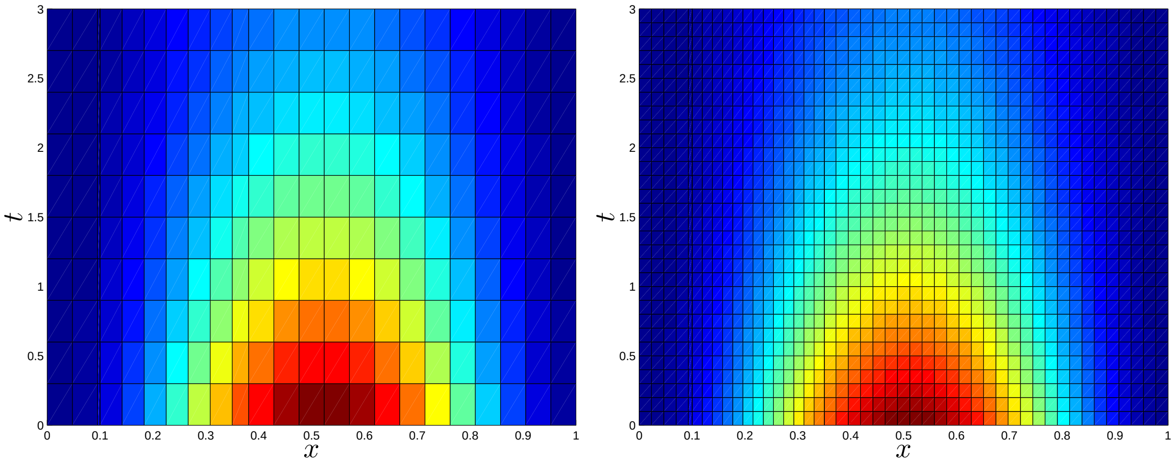

Figure 14: The dG(0) finite element solution u_h on the coarse mesh and on the fine mesh for $\mu = 0$.

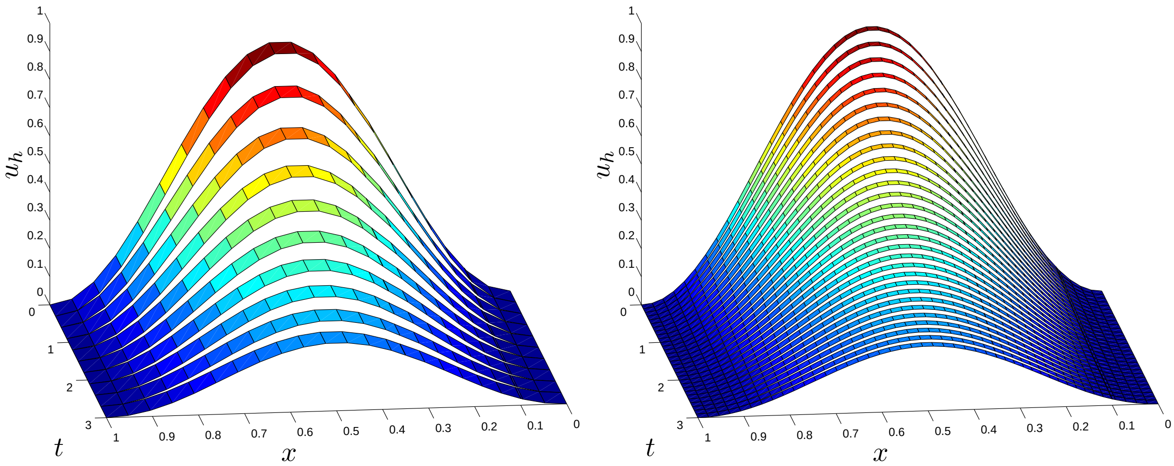

Figure 15: The dG(0) finite element solution u_h on the coarse mesh and on the fine mesh for $\mu = 0$ from a different angle.

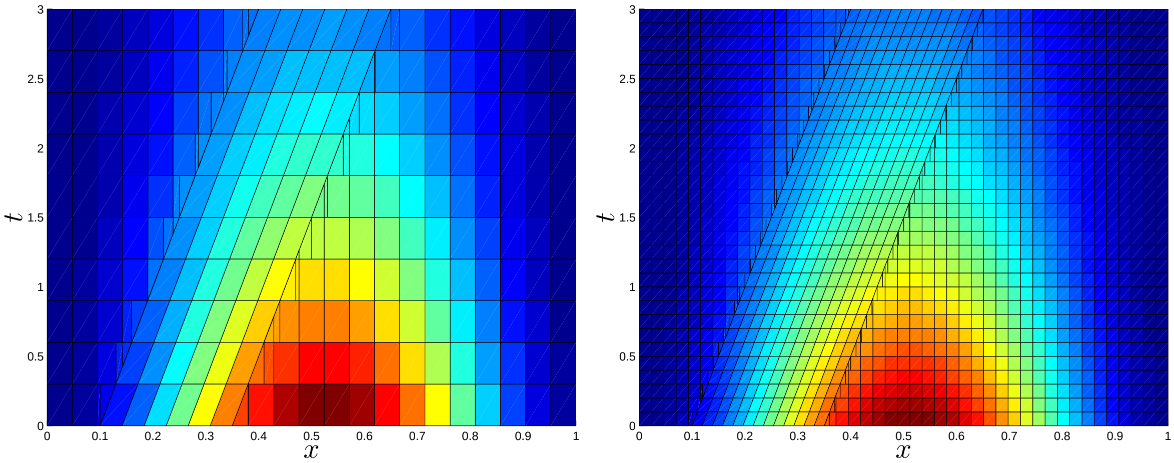

Figure 16: The dG(0) finite element solution u_h on the coarse mesh and on the fine mesh for $\mu = 0.1$.

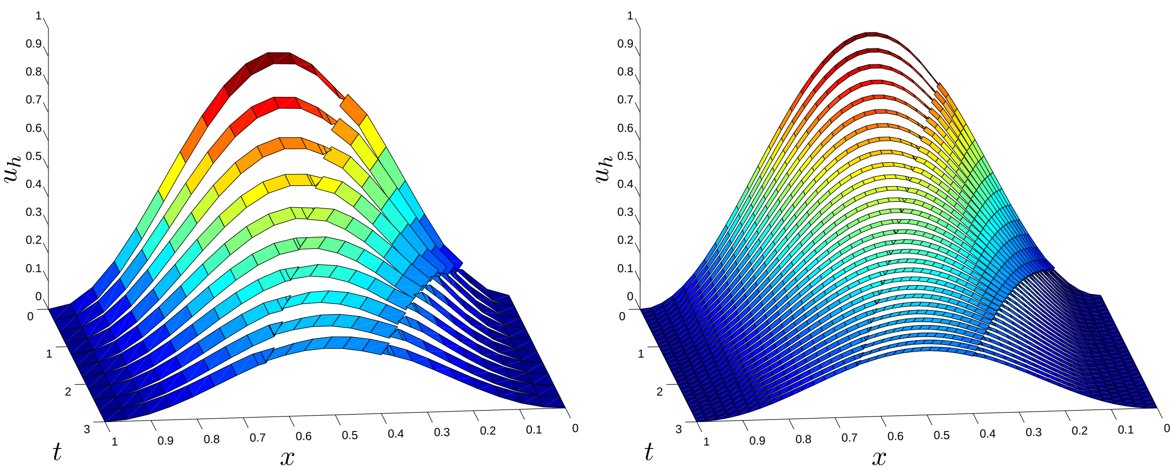

Figure 17: The dG(0) finite element solution u_h on the coarse mesh and on the fine mesh for $\mu = 0.1$ from a different angle.

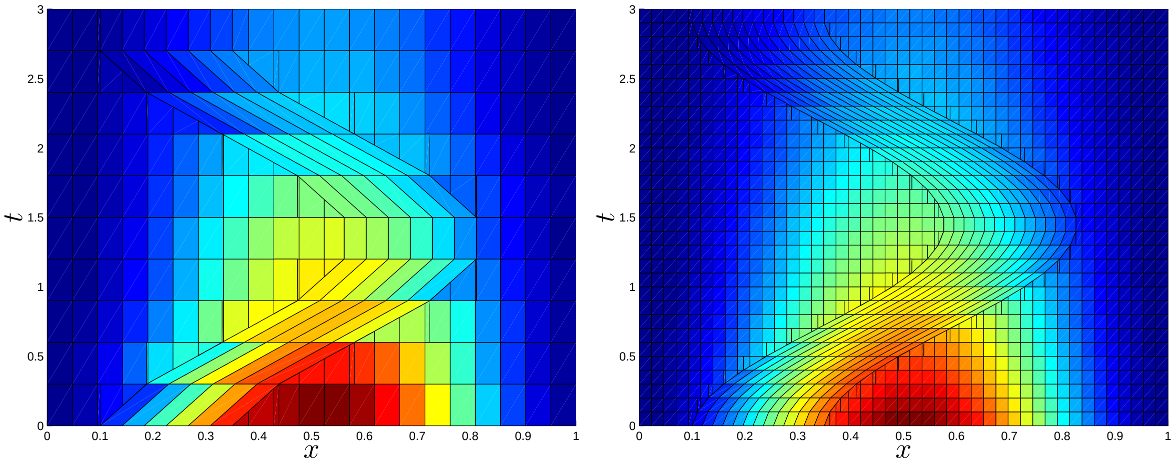

Figure 18: The dG(0) finite element solution u_h on the coarse mesh and on the fine mesh for $\mu = \frac{1}{2}\sin(\frac{2\pi t}{3})$.

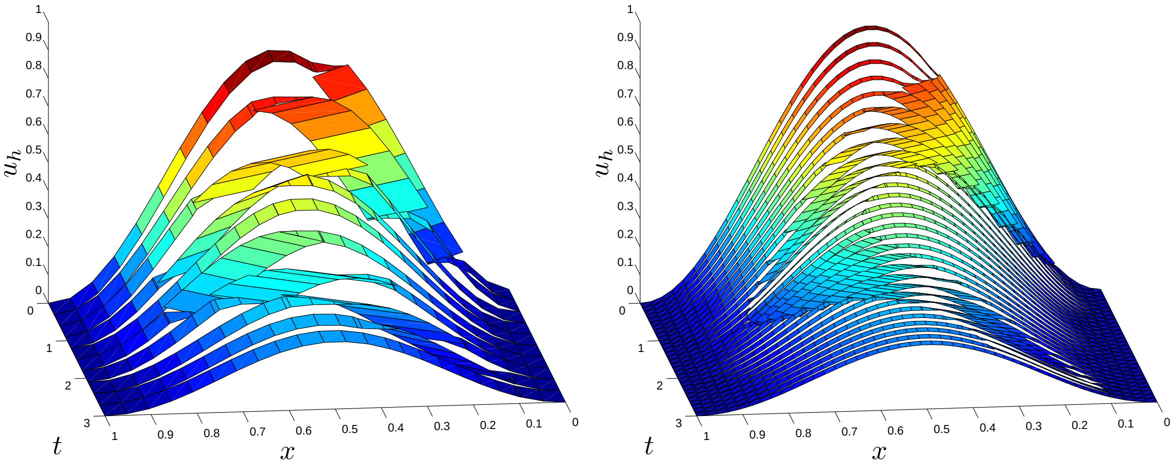

Figure 19: The dG(0) finite element solution u_h on the coarse mesh and on the fine mesh for $\mu = \frac{1}{2}\sin(\frac{2\pi t}{3})$ from a different angle.

dG(0) *error convergence plots*

Figure 20 and Figure 21 display two error convergence plots each. The left plots show the *error* versus k_n, and the right plots show the *error* versus $h = h_0 \approx h_G$. The velocity is $\mu = 0$ in Figure 20 and $\mu = 0.1$ in Figure 21. In the plots displaying the *error* versus k_n, the step sizes have been fixed at $h = h_0 = h_G = 10^{-3}$. Analogously, in the plots with the *error* versus h, the time step has been fixed at $k_n = 10^{-4}$.

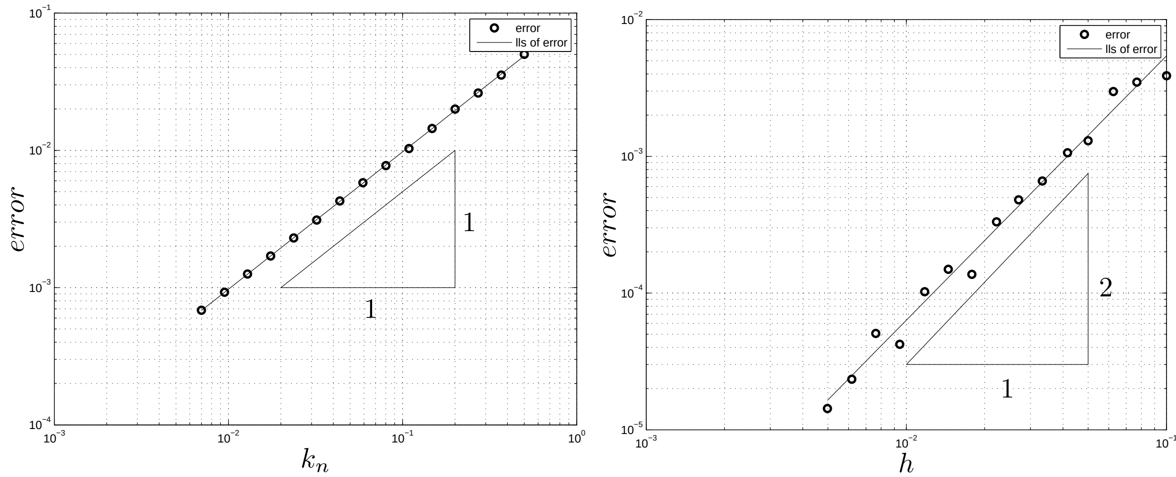

Figure 20: Error convergence for dG(0) when $\mu = 0$. *Left*: The *error* versus k_n. The slope of the lls of the *error* is 1.001. *Right*: The *error* versus h. The slope of the lls of the *error* is 1.935.

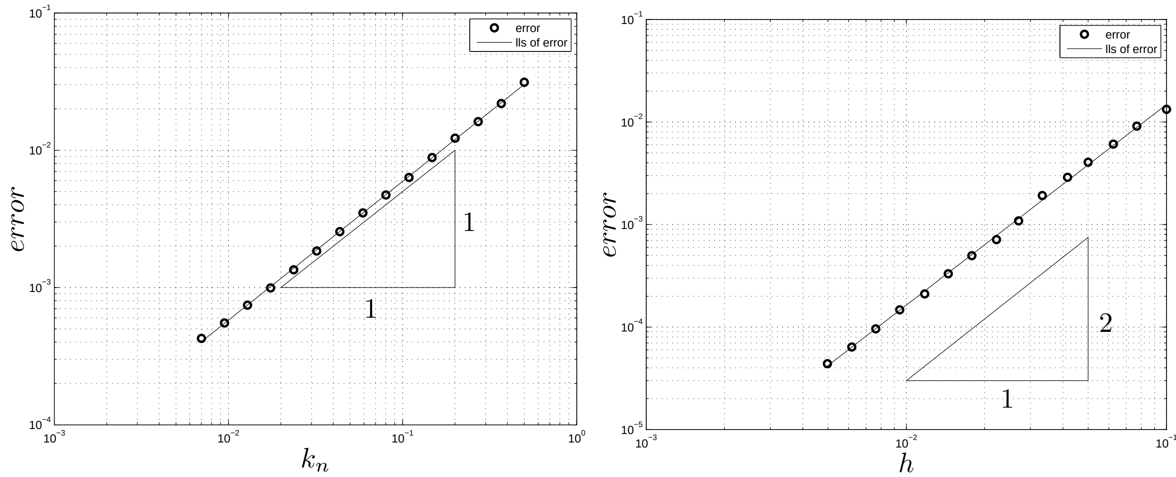

Figure 21: Error convergence for dG(0) when $\mu = 0.1$. *Left*: The *error* versus k_n. The slope of the lls of the *error* is 1.011. *Right*: The *error* versus h. The slope of the lls of the *error* is 1.957.

The slopes of the lls of the *error*, given in the captions of Figure 20 and Figure 21, are summarized in Table 1.

	Slope of the lls of the *error*	
μ	*error* versus k_n	*error* versus h
0	1.001	1.935
0.1	1.011	1.957

Table 1: The slope of the lls of the *error* versus k_n and h for different values of μ for dG(0).

8.3 dG(1) plots

dG(1) *solution plots*

Figure 22 – 27 display the dG(1) finite element solution u_h on the six different space-time meshes, shown in Figure 11 – 13, from two different angles.

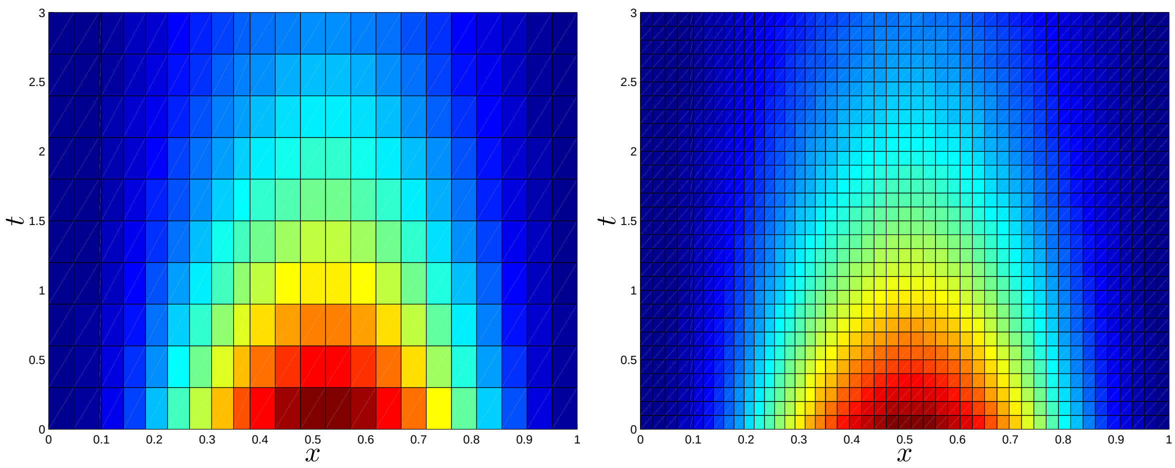

Figure 22: The dG(1) finite element solution u_h on the coarse mesh and on the fine mesh for $\mu = 0$.

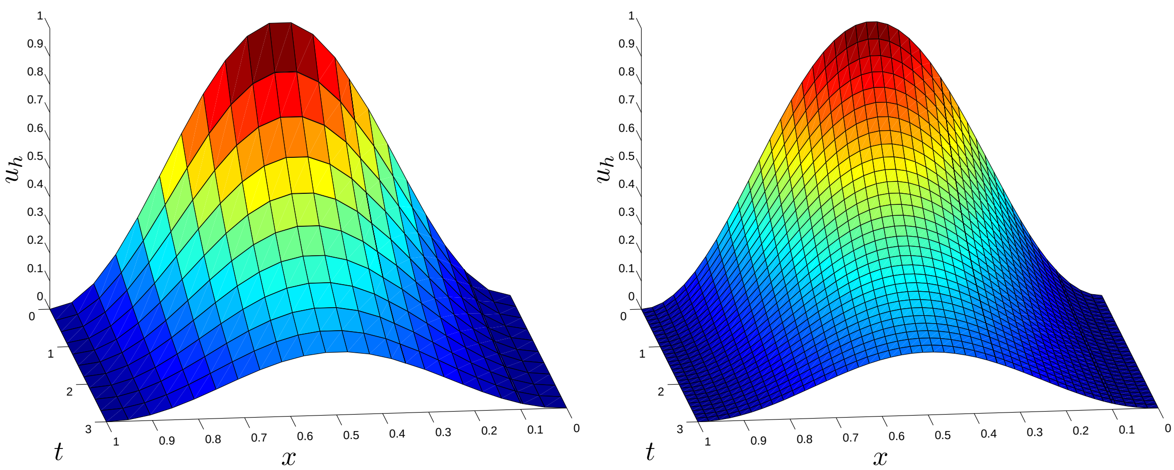

Figure 23: The dG(1) finite element solution u_h on the coarse mesh and on the fine mesh for $\mu = 0$ from a different angle.

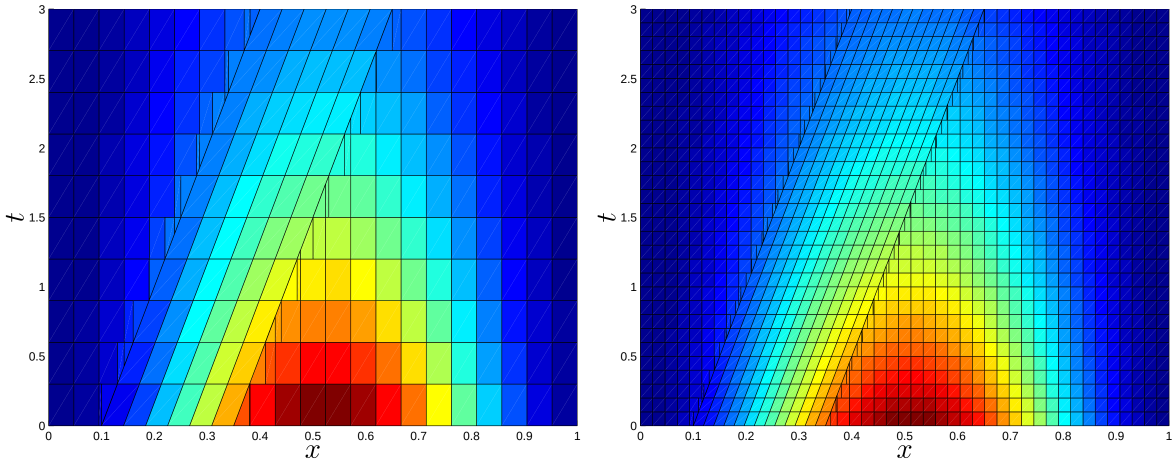

Figure 24: The dG(1) finite element solution u_h on the coarse mesh and on the fine mesh for $\mu = 0.1$.

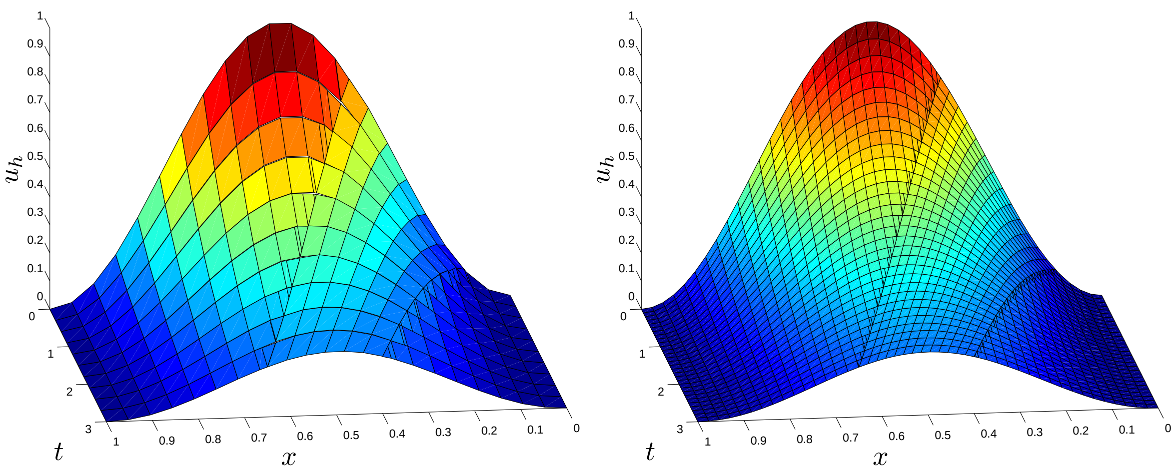

Figure 25: The dG(1) finite element solution u_h on the coarse mesh and on the fine mesh for $\mu = 0.1$ from a different angle.

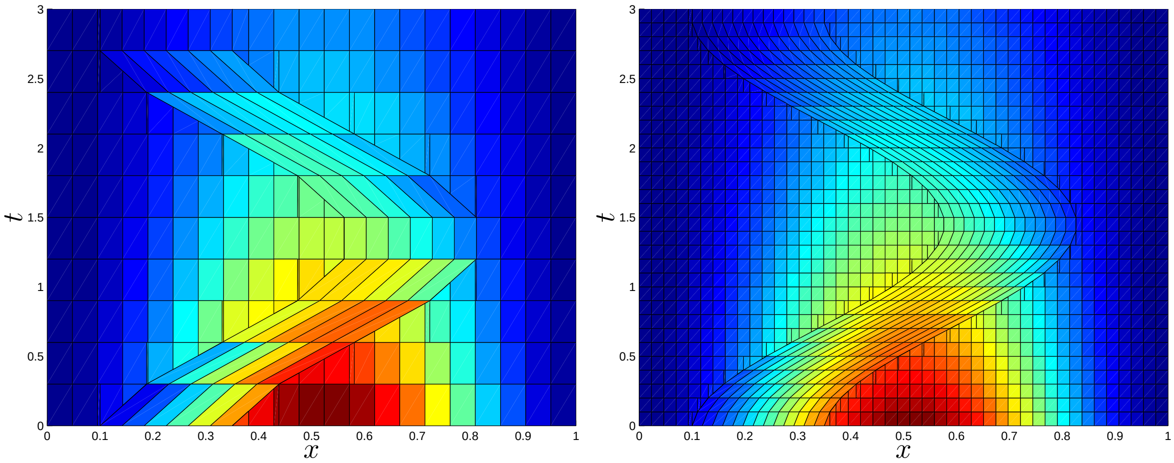

Figure 26: The dG(1) finite element solution u_h on the coarse mesh and on the fine mesh for $\mu = \frac{1}{2}\sin(\frac{2\pi t}{3})$.

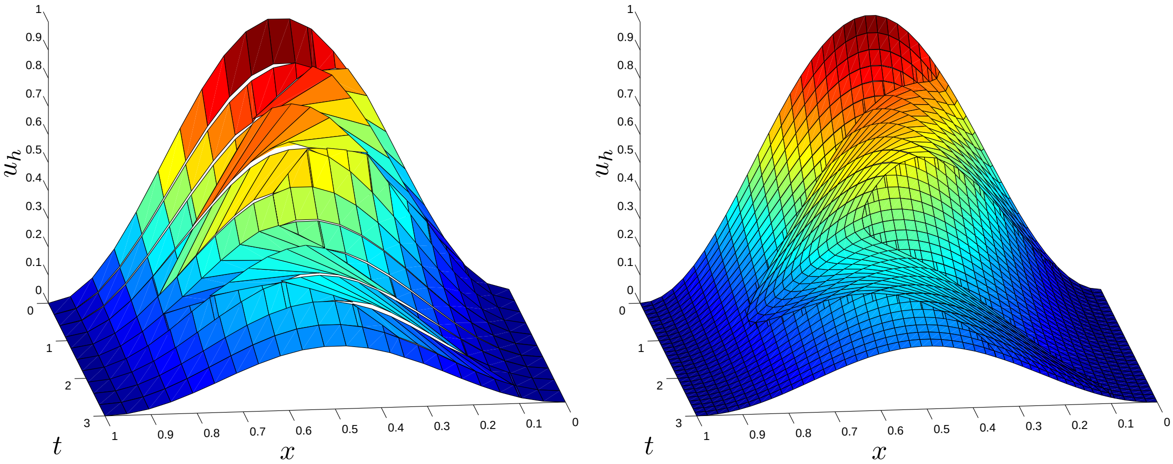

Figure 27: The dG(1) finite element solution u_h on the coarse mesh and on the fine mesh for $\mu = \frac{1}{2}\sin(\frac{2\pi t}{3})$ from a different angle.

dG(1) *error convergence plots*

Figure 28 – 31 display two error convergence plots each. The left plots show the *error* versus k_n, and the right plots show the *error* versus $h = h_0 \approx h_G$. The velocity is $\mu = 0$ in Figure 28, $\mu = 0.001$ in Figure 29, $\mu = 0.01$ in Figure 30, and $\mu = 0.1$ in Figure 31. In the plots displaying the *error* versus k_n, the step sizes have been fixed at $h = h_0 = h_G = 5 \cdot 10^{-4}$. Analogously, in the plots with the *error* versus h, the time step has been fixed at $k_n = 2 \cdot 10^{-2}$.

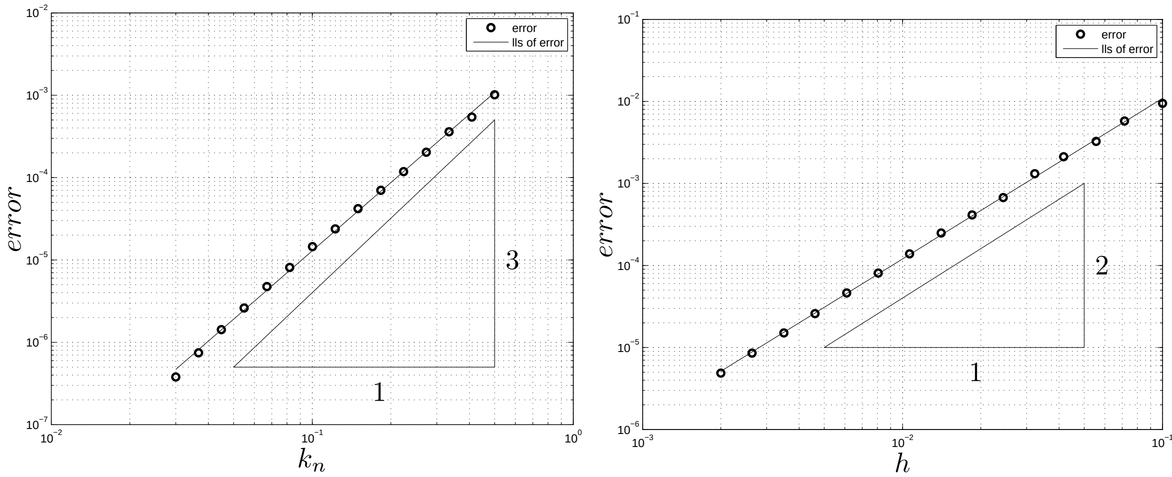

Figure 28: Error convergence for dG(1) when $\mu = 0$. *Left*: The *error* versus k_n. The slope of the lls of the *error* is 2.752. *Right*: The *error* versus h. The slope of the lls of the *error* is 1.959.

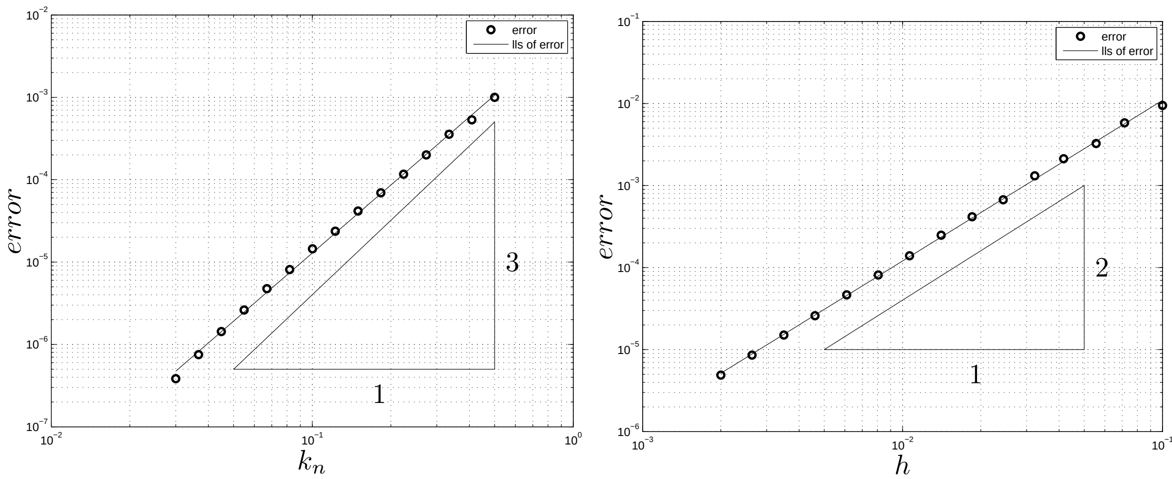

Figure 29: Error convergence for dG(1) when $\mu = 0.001$. *Left*: The *error* versus k_n. The slope of the lls of the *error* is 2.742. *Right*: The *error* versus h. The slope of the lls of the *error* is 1.959.

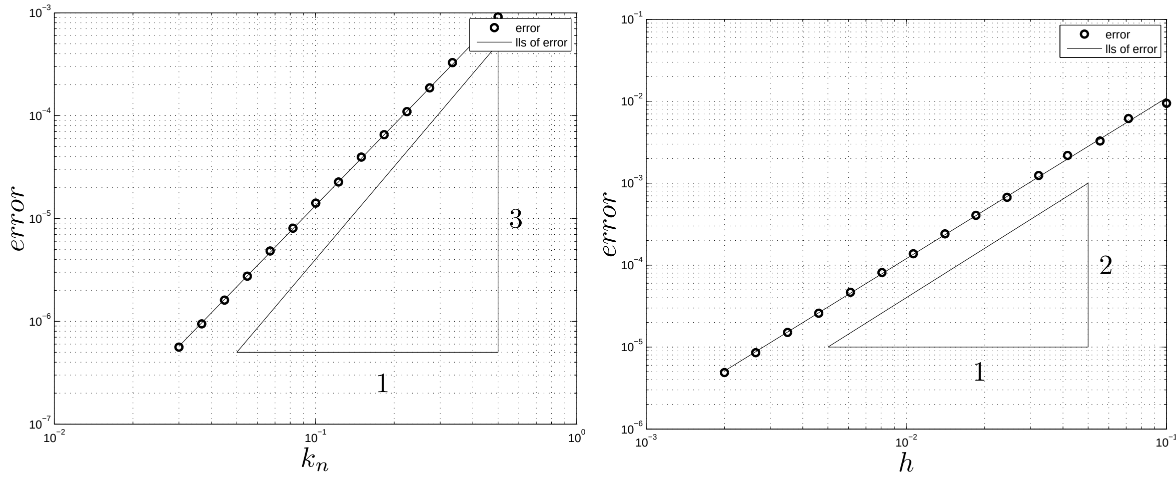

Figure 30: Error convergence for dG(1) when $\mu = 0.01$. *Left*: The *error* versus k_n. The slope of the lls of the *error* is 2.618. *Right*: The *error* versus h. The slope of the lls of the *error* is 1.962.

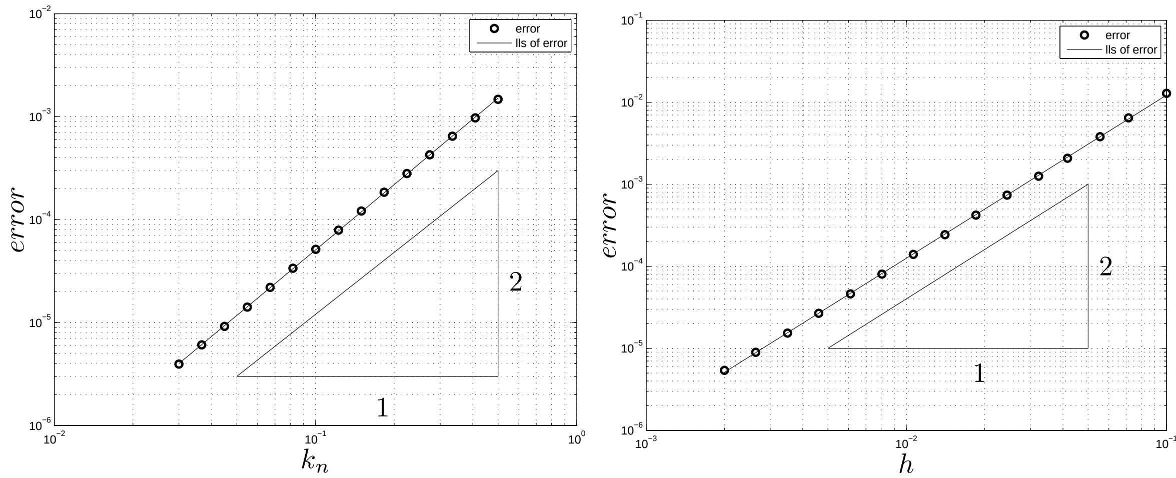

Figure 31: Error convergence for dG(1) when $\mu = 0.1$. *Left*: The *error* versus k_n. The slope of the lls of the *error* is 2.11. *Right*: The *error* versus h. The slope of the lls of the *error* is 1.989.

The slopes of the lls of the *error*, given in the captions of Figure 28 − 31, are summarized in Table 2.

	Slope of the lls of the *error*	
μ	*error* versus k_n	*error* versus h
0	2.752	1.959
0.001	2.742	1.959
0.01	2.618	1.962
0.1	2.11	1.989

Table 2: The slope of the lls of the *error* versus k_n and h for different values of μ for dG(1).

Figure 32 displays the order of convergence of the *error* on the k_n-interval $[0.05, 0.5]$ versus μ for dG(1). The order of convergence is computed as the slope of the lls of the *error* for 12 logarithmically equidistributed evaluation points on the k_n-interval. In all instances the step sizes have been fixed at $h = h_0 = h_G = 0.001$. The figure also shows the location of the aforementioned k_n-interval on the x-axis and μ_{sweep}, which is the speed at which the mesh $\mathcal{T}_G$ sweeps over a distance of the step size h_0 in one time step, i.e., $\mu_{sweep} = h_0/k_{n,max} = 0.001/0.5 = 0.002$.

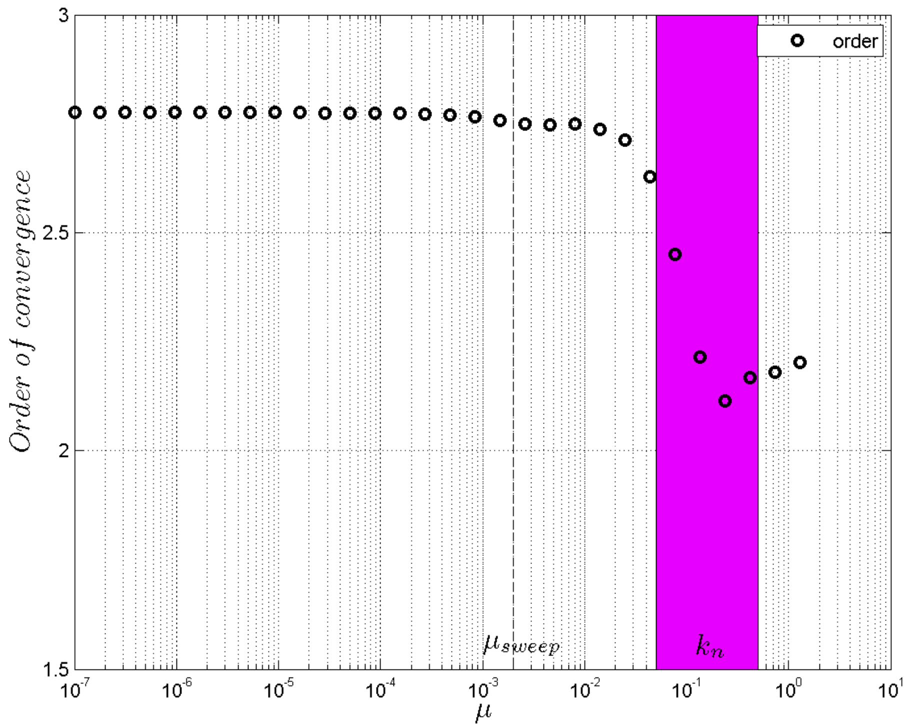

Figure 32: Order of convergence of the *error* on the k_n-interval $[0.05, 0.5]$ versus μ for dG(1). The location of the k_n-interval on the x-axis is marked with a purple bar. The speed μ_{sweep} is marked with a dotted vertical line.

From Figure 32, we note that there is a reduction in the order of convergence when the value of μ approaches and lies in the k_n-interval. We also note that the order of convergence is barley affected when $\mu \approx \mu_{sweep}$.

9 Conclusions

From the solution plots in Section 8 it can be observed how the finite element solution u_h propagates through the combined space-time meshes. The propagation is naturally much smoother in the dG(1) solution plots than in the dG(0) solution plots, since there are two additional degrees of freedom for u_h on each space-time simplex for dG(1) compared with dG(0). This makes the dG(1) solution a closer approximation, than the dG(0) solution, to the smooth exact solution u. The superior approximation capacity of dG(1) becomes even more evident when $|\mu| \neq 0$. By comparing the left plot in Figure 19 with the left plot in Figure 27, it can easily be seen how much smoother the dG(1) solution can be compared with the dG(0) solution for the same simulation settings.

In the error convergence plots, we may observe how the order of convergence of the $error \approx u - u_h$ depends on the time step k_n and the step size h, for different velocities μ of the moving mesh. Intuitively, the *error*'s dependence on k_n and h should be $error \propto k_n^{q+1} + h^{p+1}$, since we use polynomials of degree q in time, and polynomials of degree p in space, to approximate u with u_h. For the case, with only a background mesh, presented in [2, 3], the *error* depends on k_n and h as

$$error \propto k_n^{2q+1} + h^{p+1}. \tag{9.1}$$

So the intuitive idea of the *error*'s dependence on k_n and h holds true for $q = 0$, but for $q = 1$, we get *superconvergence* with respect to k_n. So we are interested in, whether the order of convergence of the error is preserved or if it is affected by using overlapping meshes. Recall that $p = 1$, $q = 0$ for dG(0), and $q = 1$ for dG(1) in the simulations for the numerical results.

For dG(0), we can see in Figure 20, where $\mu = 0$, and in Figure 21, where $\mu = 0.1$, how the *error* converges with respect to k_n and h. From these figures we conclude, that the convergence is of the first order, with respect to k_n, and of the second order, with respect to h, for both $\mu = 0$ and $\mu = 0.1$. Based on these figures, it is thus reasonable to assume that the method preserves the order of convergence of the *error* for dG(0) and that the movement of the moving mesh does not affect the order of convergence of the *error* for dG(0).

For dG(1), we can see in Figure 28 − 31, where $\mu = 0, 0.001, 0.01, 0.1$, respectively, how the *error* converges with respect to k_n and h. From the plots to the right in these figures, we draw the conclusion that the convergence is of the second order, with respect to h, for all the aforementioned velocities. It is thus reasonable to assume that the order of convergence of the *error*, with respect to h, is preserved, and unaffected by μ for dG(1). In the plots to the left in Figure 28 − 30, the convergence with respect to k_n, seems to be of the third order, or close to it. But in the left plot of Figure 31, where $\mu = 0.1$, the convergence with respect to k_n, seems to be closer to the second order. From the plots to the left in Figure 28 − 31, we thus conclude that the order of convergence for dG(1) is affected by the movement of the moving mesh. In Figure 32, it becomes apparent how μ influences the convergence of the dG(1) *error*, by decreasing the convergence with respect to k_n, from the third to the second order, when $|\mu|$ becomes big enough. From Figure 32, this change in convergence order seems to occur when $|\mu|$ and k_n are of the same magnitude.

Let us shift our focus to the analysis of the method, so that we may compare theoretical results to the previously discussed numerical ones. The a priori error estimate, presented in Theorem 7.1, may be expressed in a more simplified way, to explicitly clarify the *error*'s theoretical dependence on k_n, h and μ:

$$error \propto k_n^{2q+1} + h^{p+1} + |\mu|(k_n^{q+1} + h^{p+1}) \tag{9.2}$$

From (9.2), we first note that the *error*'s theoretical behaviour is the same as in the case with only a background mesh (9.1), when $\mu = 0$. This is something that holds intuitively. Furthermore, from (9.2) it is clear that μ does not affect the order of convergence with respect to h. For $q = 0$, there is also no interference from μ on the convergence with respect to k_n. But for $q = 1$, the *error*'s dependence on k_n becomes $error \propto k_n^3 + |\mu|k_n^2$. From this it is apparent that μ decreases the convergence with respect to k_n, from the third to the second order, when $|\mu|$ is of the same magnitude as k_n or larger.

The *error*'s theoretical behaviour (9.2) coincides well with its numerical behaviour, seen in the error convergence plots, i.e. Figure 20 – 21, and Figure 28 – 32. Although we have used the four conjectures: Conjecture 5.1 (An estimate for $w - R_t w$), Conjecture 6.1 (The second auxiliary stability estimate), Conjecture 7.1 (An estimate for $\Delta_{h,t}(R_t w - \tilde{I}_n R_t w)$), and Conjecture 7.2 (Estimates on Γ_n), in the proof of Theorem 7.1, which of course could weaken the a priori error estimate's credibility, there is still a strong indication from both the analysis and the numerical results that (9.2) holds.

10 Outlook and future work

The work on the space-time cut FEM for the heat equation presented in this thesis has started from scratch and taken the method a notable distance on the way, but as always, there is more that can be done. Future work on this method, that lies close at hand, includes the implementation of the method in two and three spatial dimensions and a complete proof of the a priori error estimate. One way to obtain a complete proof could of course be to prove the aforementioned conjectures. Other ways are to start with a completely new proof idea or perhaps use some parts of the proof presented in this thesis, but try a different approach at some point in the proof. The future work on space-time cut FEMs for *other* time-dependent PDEs, includes the derivation, analysis and implementation of methods for the time-dependent Stokes problem and the Navier-Stokes equations. The time-dependent Stokes problem could be seen as a bridge between the heat equation and the Navier-Stokes equations, even though the distance between the two might seem great.

A Mathematical tools

Lemma A.1 (A jump lemma)**.** *Let* $\omega_+, \omega_- \in \mathbb{R}$ *and* $\omega_+ + \omega_- = 1$*, let* $[A] := A_+ - A_-$*, and* $\langle A \rangle := \omega_+ A_+ + \omega_- A_-$*. We then have*

$$[AB] = [A]\langle B \rangle + \langle A \rangle [B] + (\omega_- - \omega_+)[A][B]. \tag{A.1}$$

Proof. The three terms on the right-hand side of (A.1) are

$$\begin{aligned} [A]\langle B \rangle &= (A_+ - A_-)(\omega_+ B_+ + \omega_- B_-) \\ &= \omega_+ A_+ B_+ + \omega_- A_+ B_- - \omega_+ A_- B_+ - \omega_- A_- B_-, \end{aligned}$$

$$\begin{aligned} \langle A \rangle [B] &= (\omega_+ A_+ + \omega_- A_-)(B_+ - B_-) \\ &= \omega_+ A_+ B_+ - \omega_+ A_+ B_- + \omega_- A_- B_+ - \omega_- A_- B_-, \end{aligned}$$

$$\begin{aligned} (\omega_- - \omega_+)[A][B] &= (\omega_- - \omega_+)(A_+ - A_-)(B_+ - B_-) \\ &= (\omega_- - \omega_+)(A_+ B_+ - A_+ B_- - A_- B_+ + A_- B_-). \end{aligned}$$

Adding these three expressions gives

$$\begin{aligned} [A]\langle B \rangle + \langle A \rangle [B] + (\omega_- - \omega_+)[A][B] =& \omega_+ A_+ B_+ + \omega_- A_+ B_- - \omega_+ A_- B_+ - \omega_- A_- B_- \\ &+ \omega_+ A_+ B_+ - \omega_+ A_+ B_- + \omega_- A_- B_+ - \omega_- A_- B_- \\ &+ \omega_- A_+ B_+ - \omega_- A_+ B_- - \omega_- A_- B_+ + \omega_- A_- B_- \\ &- \omega_+ A_+ B_+ + \omega_+ A_+ B_- + \omega_+ A_- B_+ - \omega_+ A_- B_-, \end{aligned}$$

which after cancellation of most of the terms yields

$$(\omega_+ + \omega_-) A_+ B_+ - (\omega_+ + \omega_-) A_- B_- = A_+ B_+ - A_- B_- = [AB].$$

□

Lemma A.2 (The Dotphi Lemma)**.** *Consider an arbitrary linear basis function* φ_j*, belonging to either* $\mathcal{T}_0$ *or* $\mathcal{T}_G$*, for the finite element problem presented in Section 4. Let the space vector* $\hat{\mu} \in \mathbb{R}^d$ *be defined by*

$$\hat{\mu} = \hat{\mu}(x,t) = \begin{cases} 0, & x \in \Omega_1(t), \\ \mu(t), & x \in \Omega_2(t), \end{cases} \tag{A.2}$$

we then have

$$\dot{\varphi}_j(x,t) = -\hat{\mu} \cdot \nabla \varphi_j(x,t). \tag{A.3}$$

Proof. Let $s \in \mathbb{R}^{d+1}$ be the space-time vector defined by $s = (\hat{\mu}, 1)$. Noting that the derivative of φ_j with respect to s is zero, we have

$$\begin{aligned} 0 &= \frac{d\varphi_j}{ds}(x(s), t(s)) = \frac{\partial \varphi_j}{\partial x_1}\frac{dx_1}{ds} + \cdots + \frac{\partial \varphi_j}{\partial x_d}\frac{dx_d}{ds} + \frac{\partial \varphi_j}{\partial t}\frac{dt}{ds} \\ &= \nabla \varphi_j \cdot \frac{dx}{ds} + \dot{\varphi}_j \frac{dt}{ds}. \end{aligned}$$

Moving over the last term in the last row to the left-hand side and multiplying both sides with $-\frac{ds}{dt}$, gives us

$$\dot{\varphi}_j \frac{dt}{ds}\frac{ds}{dt} = -\nabla \varphi_j \cdot \frac{dx}{ds}\frac{ds}{dt} = -\nabla \varphi_j \cdot \frac{dx}{dt}.$$

Since $\frac{dt}{ds}\frac{ds}{dt} = 1$ and $\frac{dx}{dt} = \hat{\mu}$, this proves (A.3).

□

Lemma A.3 (An inverse inequality on $\Gamma(t)$)**.** *For $v \in V_h(t)$, and $t \in (0, T]$, there is a constant $C_I > 0$, such that*

$$\|\langle \partial_{\bar{n}^x} v \rangle\|^2_{-1/2,h,\Gamma(t)} \leq C_I \sum_{i=1}^{2} \|\nabla v\|^2_{\Omega_i(t)}. \tag{A.4}$$

Proof. The proof follows Hansbo and Hansbo [5], but with some additional modifications. Recall that $\Gamma_K(t) = \Gamma_{K_0}(t) = K_0 \cap \Gamma(t)$ and $\mathcal{T}_{0,\Gamma}(t) = \{K_0 \in \mathcal{T}_0 : K_0 \cap \Gamma(t) \neq \emptyset\}$. Analogously, we define $\Gamma_{K_G}(t) := K_G \cap \Gamma(t)$ and $\mathcal{T}_{G,\Gamma}(t) := \{K_G \in \mathcal{T}_G : K_G \cap \Gamma(t) \neq \emptyset\}$. Note that

$$\sum_{K_0 \in \mathcal{T}_{0,\Gamma}(t)} \|v\|^2_{\Gamma_{K_0}(t)} = \sum_{K_G \in \mathcal{T}_{G,\Gamma}(t)} \|v\|^2_{\Gamma_{K_G}(t)}, \tag{A.5}$$

since $\cup_{K_0 \in \mathcal{T}_{0,\Gamma}(t)} \Gamma_{K_0}(t) = \cup_{K_G \in \mathcal{T}_{G,\Gamma}(t)} \Gamma_{K_G}(t)$. Recalling that $\langle v \rangle = \omega_1 v_1 + \omega_2 v_2$, we have

$$\begin{aligned} \|\langle \partial_{\bar{n}^x} v \rangle\|^2_{-1/2,h,\Gamma(t)} &= \sum_{K_0 \in \mathcal{T}_{0,\Gamma}(t)} h_K \|\langle \partial_{\bar{n}^x} v \rangle\|^2_{\Gamma_{K_0}(t)} \\ &\leq \sum_{K_0 \in \mathcal{T}_{0,\Gamma}(t)} h \Big(\|\omega_1 (\partial_{\bar{n}^x} v)_1\|_{\Gamma_{K_0}(t)} + \|\omega_2 (\partial_{\bar{n}^x} v)_2\|_{\Gamma_{K_0}(t)} \Big)^2 \\ &\leq \sum_{K_0 \in \mathcal{T}_{0,\Gamma}(t)} 2h \|\omega_1 (\partial_{\bar{n}^x} v)_1\|^2_{\Gamma_{K_0}(t)} + \sum_{K_G \in \mathcal{T}_{G,\Gamma}(t)} 2h \|\omega_2 (\partial_{\bar{n}^x} v)_2\|^2_{\Gamma_{K_G}(t)}, \end{aligned} \tag{A.6}$$

where $h = \max_{K_l \in \mathcal{T}_0 \cup \mathcal{T}_G}(h_{K_l})$. To obtain the last inequality, we have used (A.5). For index $j \in \{0, G\}$, such that, if $j = 0$, then $i = 1$ and if $j = G$, then $i = 2$, we may write both terms in the last row of (A.6) as $\|\omega_i (\partial_{\bar{n}^x} v)_i\|^2_{\Gamma_{K_j}(t)}$. Following the proof of Hansbo and Hansbo [5], we have

$$\begin{aligned}
\|\omega_i(\partial_{\bar{n}^x} v)_i\|^2_{\Gamma_{K_j}(t)} = & \int_{\Gamma_{K_j}(t)} |\omega_i|^2 |\bar{n}^x \cdot (\nabla v)_i|^2 \,\mathrm{d}s \leq \max_{\Gamma_{K_j}(t)} (|\omega_i|^2) \int_{\Gamma_{K_j}(t)} |(\nabla v)_i|^2 \,\mathrm{d}s \\
= & \max_{\Gamma_{K_j}(t)} (|\omega_i|^2) |(\nabla v)_i|^2 |\Gamma_{K_j}(t)| \frac{\int_{K_j \cap \Omega_i(t)} \,\mathrm{d}x}{|K_j \cap \Omega_i(t)|} \\
= & \, C_{K_j} \int_{K_j \cap \Omega_i(t)} |(\nabla v)_i|^2 \,\mathrm{d}x = C_{K_j} \|\nabla v\|^2_{K_j \cap \Omega_i(t)},
\end{aligned} \tag{A.7}$$

where $|D|$ denotes the Lebesgue measure of a set D and $C_{K_j} = \max_{\Gamma_{K_j}(t)}(|\omega_i|^2)|\Gamma_{K_j}(t)|/|K_j \cap \Omega_i(t)|$. The inequality is obtained by noting that $|\bar{n}^x| \leq 1$. To obtain the second, third and fourth equality, we note that $(\nabla v)_i = \nabla v$ is constant on K_j, since v is linear on each K_j. Inserting (A.7) in (A.6), gives

$$\begin{aligned}
\|\langle \partial_{\bar{n}^x} v \rangle\|^2_{-1/2,h,\Gamma(t)} \leq & \sum_{K_0 \in \mathcal{T}_{0,\Gamma}(t)} 2hC_{K_0} \|\nabla v\|^2_{K_0 \cap \Omega_1(t)} + \sum_{K_G \in \mathcal{T}_{G,\Gamma}(t)} 2hC_{K_G} \|\nabla v\|^2_{K_G \cap \Omega_2(t)} \\
\leq & \, 2h\hat{C}_{K_0} \|\nabla v\|^2_{\Omega_1(t)} + 2h\hat{C}_{K_G} \|\nabla v\|^2_{\Omega_2(t)} \\
\leq & \, C_I \sum_{i=1}^{2} \|\nabla v\|^2_{\Omega_i(t)},
\end{aligned} \tag{A.8}$$

where $\hat{C}_{K_j} = \max_{K_j \in \mathcal{T}_{j,\Gamma}(t)}(C_{K_j})$ and $C_I = 2h \max(\hat{C}_{K_0}, \hat{C}_{K_G})$. This concludes the proof.

□

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