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From the Donaldson-Uhlenbeck-Yau theorem to stability in mirror symmetry

An introduction to the deformed Hermitian-Yang-Mills equation and Bridgeland stability conditions in homological mirror symmetry

Master's thesis in Physics and Astronomy

Björn Eurenius

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Supervisor: Daniel Persson, Department of Mathematical Sciences

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Department of Physics
Chalmers University of Technology
SE-412 96 Gothenburg
Telephone +46 31 772 1000

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Abstract

We give an introduction to the mathematical formulation of Yang-Mills theory. In particular we derive the Hermitian-Yang-Mills equation and show that Hermitian-Yang-Mills connections can be described as the zeroes of the corresponding moment map. We then introduce deformed Hermitian-Yang-Mills equations by considering arbitrary moment maps.

Furthermore we introduce slope stability of holomorphic vector bundles and show that holomorphic vector bundles have a unique Harder-Narasimhan filtration. We then give a proof of the Donaldson-Uhlenbeck-Yau theorem in the case of algebraic surfaces, which states that there is a one-to-one correspondence between stable bundles and bundles that admit an irreducible Hermitian-Yang-Mills connection.

Finally we look at stability in the context of homological mirror symmetry. We discuss Bridgeland stability conditions on the bounded derived category of coherent sheafs $\mathcal{D}^b \operatorname{Coh}(M)$ over a Kähler manifold M and discuss how it connects to the deformed Hermitian-Yang-Mills equation.

Keywords: Donaldson-Uhlenbeck-Yau theorem, deformed Hermitian-Yang-Mills, Kobayashi-Hitchin correspondence, Mirror symmetry, Bridgeland stability.

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Contents

Chapter 1

Introduction

A recurring topic in mathematics is that of classification. Given some set of objects we want to parameterize them up to some notion of equality between the objects. Ideally we want this parametrization to preserve enough structure to allow the study of the original objects while also taking the form of a well behaved space, giving us a simpler object to work with. Particularly in geometry one asks if a collection of geometric objects itself can be given geometric structure, this is the concept of a moduli space. An example of this is the Riemann sphere, the space of all lines in $\mathbb{C}^2$ that goes through the origin.

An approach to construct moduli spaces in algebraic geometry is that of geometric invariant theory (GIT) which was introduced in 1965 by Mumford. Geometric invariant theory provides a way of taking quotients by a group action in algebraic geometry in such a way that the resulting space is in a certain sense "well-behaved". In particular we want the quotient to be Hausdorff, which commonly is not the case for the naive approach of taking the orbit space. To do this Mumford introduced the concept of stability, telling us that in certain cases the class of geometric objects satisfying a stability condition do form a "well-behaved" space.

1.1 The moduli space of bundles and the Hermitian-Yang-Mills equation

We will in this thesis study the moduli space of holomorphic vector bundles. This is a case that has been extensively studied and where a beautiful correspondence has been discovered between the moduli space of stable bundles and solutions to a geometric PDE, known as the Hermitian-Yang-Mills (HYM) equation. Mathematically this is interesting since it relates the algebraic condition of stability to that of solutions to a differential equation, thus relating different branches of mathematics.

There is also interest in this topic from a physics perspective. Modern physics is dominated by the study of symmetry, requiring that the equations of motion should be invariant under some symmetry group. An example of this is Yang-Mills theory from which the Yang-Mills equation and the Hermitian-Yang-Mills equation are derived. In particular the symmetry group $U(1)$ gives us electromagnetism in the form of Maxwell's equations while the symmetry group $SU(3) \times SU(2) \times U(1)$ describes the Standard model of particle physics. Yang-Mills theory has a particular class of solutions, to the equations of motion, known as instantons that in physics represents solutions to the classical equations of motion. The instantons will play a key part in our analysis since they give a lower bound on the energy functional that describes Yang-Mills theory.

The most well known result in the above discussed correspondence of the moduli spaces is the Donaldson-Uhlenbeck-Yau theorem which tells us that stable bundles have a one-to-one correspondence to irreducible Hermitian-Yang-Mills connections.

Theorem 1.1 (The Donaldson-Uhlenbeck-Yau theorem)**.** *A holomorphic vector bundle E over a compact Kähler manifold M is stable if and only if E admits an irreducible Hermitian-Yang-Mills connection. This connection is then unique.*

The correspondence is also known as Kobayashi-Hitchin correspondence, after Kobayashi and Hitchin who first conjectured the general relationship. In the case of algebraic curves, i.e. Riemannian manifolds, the theorem is known as the Narasimhan-Seshadri theorem. It should be noted that the above theorem is very much a collaborate effort by several mathematicians where the proof of the general theorem builds on techniques found when working with lower dimensional cases. In particular the if direction was first shown independently by Kobayashi in [1] and Lübke in [2]. Of particular importance to us for the only if direction is the proof in the case of Riemann surfaces by Atiyah and Bott in [3]. There they introduced the moment map perspective for Yang-Mills, showing that solutions to the Hermitian-Yang-Mills equation could be viewed as the zeroes of a moment map. The next step was done by Donaldson who in [4] extended the result to algebraic surfaces by using a heat equation method and results from Uhlenbeck in [5] and [6] on the choice of a "good gauge" and the removability of singularities for Yang-Mills connections. The general result over Kähler manifolds was then shown by Uhlenbeck and Yau in [7]. It is also possible to extend the theorem to Hermitian manifolds, dropping the Kähler requirement, which was done by Li and Yau in [8].

Much of our work will be to build up the background to and prove the Donaldson-Uhlenbeck-Yau theorem in the case of projective algebraic surfaces. We will devote chapter 3 to the introduction of Yang-Mills theory including the definition of Hermitian-Yang-Mills connections. These connections are also known as Hermitian-Einstein connections since the associated metric satisfy the Einstein condition,

$$R_{ij} = \alpha g_{ij}, \tag{1.1}$$

i.e. that the Ricci curvature tensor is proportional to the metric. In particular the Hermitian-Einstein metrics have constant scalar curvature, which can be seen by tracing the Einstein condition, and are thus examples of constant scalar curvature Kähler (cscK) metrics. In chapter 4 we will introduce the stability of vector bundles and their properties as well as proving the if direction using the approach of Lübke. Finally in chapter 5 we will show the only if direction in the case of projective algebraic surfaces, thus completing the proof of the theorem in this case. We will for our proof take the approach introduced by Donaldson, however subsequent work by others including Uhlenbeck, Yau, Li, and Kobayashi have allowed parts of the proof to be simplified which we will use when convenient. We will also provide some discussion on how the result was extended to the general case of Kähler manifolds.

It should be emphasised that the proof in itself is very much of interest where it can be said that the proof strategy is even more interesting than the theorem itself, having introduced techniques that now are part of the standard toolbox of geometric analysis. A key insight in the proof is the introduction of a functional whose associated gradient flow is given by the heat equation which is used to show convergence to a Hermitian-Yang-Mills connection. Another interesting approach is the use of Uhlenbeck's choice of good gauge which will allow of us to bound the functional for surfaces by the functional for curves, thereby reducing the case of algebraic surfaces to that of algebraic curves.

Let us finally note that while the Donaldson-Uhlenbeck-Yau theorem has been shown, there are other definitions of stability and conditions on the connections or metrics whose correspondence are being explored and this remains a very active area of research. In the next section we will look at such a case.

1.2 Mirror symmetry

The second part of our work will be to look at stability conditions in mirror symmetry, which is an active area of research both in mathematics and physics with its roots in string theory. We will here provide a brief introduction and motivation to mirror symmetry. For a detailed coverage of mirror symmetry we refer the reader to [9].

String theory is a ten-dimensional theory while the world we experience is four-dimensional. The approach taken in string theory is to require that the remaining six dimensions be compact and "small". Of particular interest due to physical considerations is the case when the six extra dimensions consists of a Calabi-Yau manifold of three complex dimensions. It was noticed that two different Calabi-Yau manifolds gave rise to the same four-dimensional physics, more specifically they give rise to the same scattering amplitudes. We say that such manifolds are mirror manifolds to each other.

It turned out that a manifold M and its mirror W could be quite different, in fact it was discovered that their Hodge numbers were related by

$$h^{p,q}(M) = h^{n-p,q}(W). \tag{1.2}$$

This relation between the Hodge numbers is known as topological/geometric mirror symmetry. It is known that there exists Calabi-Yau manifolds without a mirror satisfying this, so the conjecture is that it is possible to classify Calabi-Yau for which the relation of the Hodge numbers is valid.

We can arrange the Hodge numbers into what is known as a Hodge diamond. In dimension three the Hodge diamond is

$$\begin{array}{ccccccc}
 & & & h^{0,0} & & & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\
 & h^{3,1} & & h^{2,2} & & h^{1,3} & \\
 & & h^{3,2} & & h^{2,3} & & \\
 & & & h^{3,3} & & &
\end{array} \tag{1.3}$$

and for Calabi-Yau and its mirror it takes the forms

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & a & & 0 & \\
1 & & b & & b & & 1 \\
 & 0 & & a & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
\xleftrightarrow{Mirror\ map}
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & b & & 0 & \\
1 & & a & & a & & 1 \\
 & 0 & & b & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array} \tag{1.4}$$

for integers a and b. Empirical evidence for the geometric mirror conjecture can be seen in fig. 1.1 which displays a Hodge plot of known Calabi-Yau threefolds. With the advent of topological string theory, mirror symmetry took the form of an equivalence between the two main versions of topological string theory, the A-model and the B-model. The scattering amplitudes in the A-model depends only on the symplectic structure of the manifold while the scattering amplitudes in the B-model relies solely on the complex structure. Mirror symmetry then relates the complex geometry on the manifold M to the symplectic geometry of its mirror W and the other way around. The benefit of this relation is that complicated calculations in one model may be easier to perform in the other and vice versa.

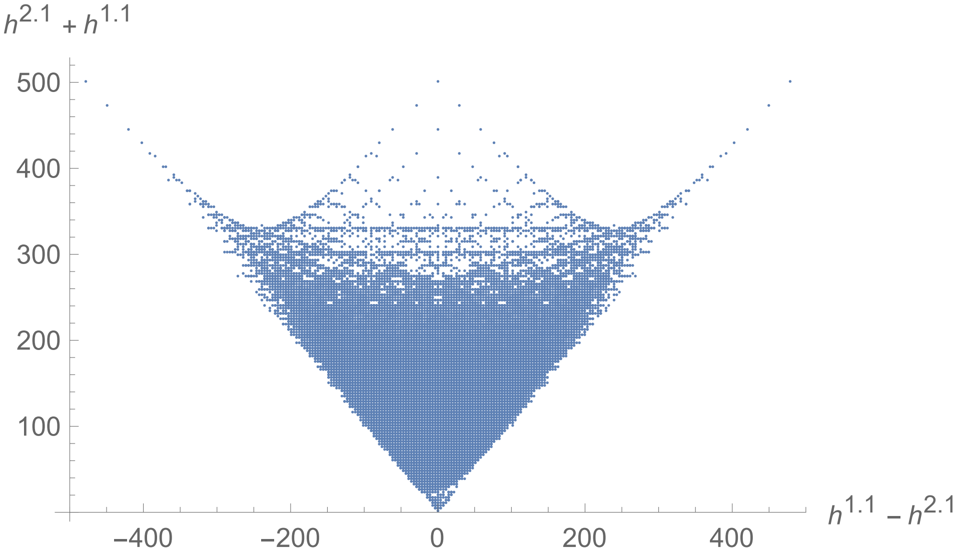

Figure 1.1: Hodge plot of known 3-dimensional Calabi-Yau manifolds. We see evidence for the existence of mirror manifolds by the symmetry around zero, which correspond to the symmetry $h^{1,1} \leftrightarrow h^{2,1}$ between mirror manifolds. The data is from [10].

A particular application that generated a lot of interest is that to enumerative geometry, where the counting of rational curves on a Calabi-Yau threefold turns into an easier calculation of period integrals on the mirror manifold. The first such calculation was done for quintic threefolds by Candelas, Ossa, Green, and Parkes in [11]. Since then some of these results have been proved rigorously by Givental in [12]. That it is possible to count the rational curves on a Calabi-Yau threefold in this way is known as the arithmetic/numerical mirror conjecture.

Another approach called homological mirror symmetry was proposed by Kontsevich in [13], resulting in the homological mirror conjecture. The conjecture states that for mirror manifolds M and W, mirror symmetry can be stated as the equivalence of two triangulated categories

$$\mathcal{D}^b \operatorname{Coh}(M) \simeq \mathcal{D}^{\mathrm{Fuk}}(W), \tag{1.5}$$

the left side coming from complex geometry and the right side from symplectic geometry. A fundamental set of objects in string theory are that of D-branes, in the A-model these roughly correspond to holomorphic vector bundles over M with holomorphic connections known as A-branes. Similarly in the B-model we have B-branes which roughly corresponds to Lagrangian submanifolds of W. The categories in Kontsevich conjecture are meant to correspond to the B- and A-branes respectively.

Some special cases of the homological mirror symmetry conjecture are known. The case of elliptic curves is done by by Polishchuk and Zaslow in [14] and the case of quartic surfaces by Seidel in [15]. Since we will work in the framework of homological mirror symmetry we may for our purposes treat the conjecture as a definition of what it means for two manifolds to be mirror to each other.

The D-branes in physics have their own notion of stability, corresponding to if they are decomposable or not. For A-branes the stability condition can be stated in terms of whether a class of Lagrangian submanifolds $[L]$ contain what is known as a special Lagrangian manifold (sLag). For B-branes stability corresponds to connections solving what is known as the MMMS equation, after Mariño, Minasian, Moore, and Strominger who derived the equation in [16]. The MMMS equation is also known as the deformed Hermitian-Yang-Mills (dHYM) equation as it can be seen as a generalization of the HYM equation. We can now summarise what we know of mirror symmetry in a table as

Mirror symmetry

B-model	A-model
• B-branes	• A-branes
• Complex geometry	• Symplectic geometry
• $\mathcal{D}^b \operatorname{Coh}(M)$	• $\mathcal{D}^{\text{Fuk}}(W)$
• $h^{p,q}(M)$	• $h^{n-p,q}(W)$
• Holomorphic connections A	• Lagrangian submanifolds L
• dHYM connections	• sLags in $[L]$.

Interestingly the stability conditions on D-branes coming from physics share properties with those found in mathematics as by geometric invariant theory. In particular in the context of homological mirror symmetry, a stability condition known as Π-stability was formulated on derived categories by Douglas, Fiol, and Römelsberger in [17], Douglas in [18], [19], [20], and Douglas and Aspinwall in [21]. This motivated Bridgeland in [22] to introduce a more general form of stability on triangulated categories, which we will refer to as Bridgeland stability.

Our aim in chapter 6 is to introduce Bridgeland stability and review some of the recent work on the deformed-Hermitian-Yang equation by Collins, Xie, and Yau in [23] and by Collins and Yau in [24]. In particular we are interested in if there is a theorem similar to the Donaldson-Uhlenbeck-Yau theorem for $\mathcal{D}^b \operatorname{Coh}(M)$ and the dHYM equation. This can be formulated as

Conjecture 1.2 (Conjecture 1.1 in [24]). *There is a Bridgeland stability condition on $\mathcal{D}^b \operatorname{Coh}(M)$ so that the holomorphic vector bundle E is stable if and only if it admits a deformed Hermitian-Yang-Mills metric.*

Correspondingly, given mirror symmetry, the conjecture on the symplectic side is

Conjecture 1.3 (Conjecture 1.1 in [24]). *There is a Bridgeland stability condition on $\mathcal{D}^{\mathrm{Fuk}}(W)$ so that the isomorphism class of a Lagrangian submanifold is stable if and only if it contains a special Lagrangian submanifold.*

1.3 Structure of the thesis

In chapter 2 we will go over some of the mathematical background we will rely on. Of particular importance for the following chapters are section 2.2.1 on principal bundles and gauge transformation, section 2.2.2 on sheaves, section 2.3 on moment maps in symplectic geometry, and section 2.4 on Kähler geometry. Since much of the motivation to our work comes from mirror symmetry we also provide some background on Calabi-Yau manifolds and special Lagrangian submanifolds in section 2.4.1 relating back to the introduction.

Chapter 3 introduces the Yang-Mills theory. We begin in section 3.1 by introducing the Yang-Mills functional and deriving the Yang-Mills equation. We then cover the self-dual and anti-self-dual solutions to the Yang-Mills equation and the gradient flow of the Yang-Mills functional. Then in section 3.2 we specialise to the case when the base manifold is Kähler which gives us the Hermitian-Yang-Mills equation. Next in section 3.3 we look at the Hermitian-Yang-Mills equation from the perspective of that of a moment map where the Hermitian-Yang-Mills connection correspond to zeroes of the moment map. Finally we use the moment map perspective to generalize the Hermitian-Yang-Mills equation to deformed-Hermitian-Yang-Mills equation by arbitrary moment maps.

In chapter 4 we begin by introducing the stability criteria for holomorphic vector bundles and proceed to show that every holomorphic vector bundle have a unique Harder-Narasimhan filtration. We then consider bundles that admit an irreducible Hermitian-Yang-Mills connection and show that such bundles are stable by the approach in [2], thus proving half of the Donaldson-Uhlenbeck-Yau theorem.

Chapter 5 deals with the other direction of the Donaldson-Uhlenbeck-Yau theorem. In the case of bundles over an algebraic surface we present a proof of the Donaldson-Uhlenbeck-Yau theorem originally found in [4]. We also discuss the general case, over an arbitrary Kähler surface, and why the proof of this in [7] needed a different approach.

Finally in Chapter 6 we connect our earlier work with stability in mirror symmetry. We begin in section 6.1 by introducing the derived category of coherent sheafs $D^b \operatorname{Coh}(M)$, over a Kähler manifold M. Then in section 6.2 we introduce Brigeland stability as a generalization of slope stability, and specifically look at Bridgeland stability conditions on $D^b \operatorname{Coh}(M)$. We conclude the chapter in section 6.3 by looking at the dHYM/MMMS equation with a focus on low-dimensional examples. We also review recent progress in [23] and [24] towards formulating a Bridgeland stability condition on $D^b \operatorname{Coh}(M)$ corresponding to the dHYM equation.

Chapter 2

Mathematical preliminaries

In this chapter we aim to introduce some of the tools, primarily from algebraic- and complex geometry, that will be used throughout our work. Not all material here is strictly necessarily for the remainder of the text and the reader is encouraged to pick and choose from this chapter, and use it as a first reference when encountering an unfamiliar concept in the main text.

We present the material with an emphasis through examples and caution the reader that the definitions and theorems stated will often not be in their most general form, rather we aim for simplicity. For a source giving a broad introduction to most of the topics here we encourage the reader look at [25], which also provides a well written introduction to mirror symmetry. Furthermore [9] covers most of the material. We also provide additional suggested reading in some of the sections, which the reader may find helpful.

2.1 Algebraic varieties

In this section we give a short overview of algebraic varieties over $\mathbb{C}$.

Affine varieties

We say that a set $X \subset \mathbb{C}^n$ is algebraic if it is the zero locus $Z(S)$ of a set of polynomials $S \subset \mathbb{C}[X_1, \dots X_n]$. That is

$$X = Z(S) = \{x \in \mathbb{C}^n | f(x) = 0 \ \ \forall f \in S\}. \tag{2.1}$$

An example of an algebraic set is $\{(x,0)\} \cup \{(0,y)\}$ since it is the zero locus to the polynomial $f(x,y) = xy$. An algebraic set X that is irreducible, meaning it is not the the union of two proper algebraic subsets of X, is known as an affine variety.

Every algebraic set is the zero set of some ideal in the polynomial ring. Given an algebraic set X we have that X is the zero set of the ideal

$$I(X) = \{f \in \mathbb{C}[X_1, \dots X_n] \mid f(x) = 0 \, \forall x \in X\}. \tag{2.2}$$

There is a one-to-one correspondence between algebraic sets and radical ideals, i.e. ideals satisfying

$$I = \sqrt{I} = \{x \in \mathbb{C}[X_1, \dots, X_n] \mid x^r \in I \text{ for some } r \in \mathbb{Z}^+\}, \tag{2.3}$$

by

Theorem 2.1 (Hilbert's Nullstellensatz)**.** *For every ideal $J \subset \mathbb{C}[X_1, \dots X_n]$*

$$I(Z(J)) = \sqrt{J}. \tag{2.4}$$

This allows us to express the irreducibility of an algebraic set in terms of its ideal.

Proposition 2.2. *An algebraic set X is irreducible if and only if the ideal $I(X)$ is prime.*

Thus we have a correspondence between affine varieties and $\mathrm{Spec}(\mathbb{C}[X_1, \dots, X_n])$, the set of prime ideals of the polynomial ring. Relating back to our example of the algebraic set $\{(x,0)\} \cup \{(0,y)\}$ we see that $I(\{x,0\})$ consists of all polynomials of the form $f(x,y) = p(x,y)y$ and is thus a prime in $\mathbb{C}[X,Y]$ and similarly for $I(\{0,y\})$. Thus the sets $\{(x,0)\}$ and $\{(0,y)\}$ are affine varieties.

Every algebraic set can be written as a finite union of affine varieties. This follows from

Theorem 2.3 (Hilbert basis theorem)**.** *If a ring R is Noetherian then so is $R[X]$.*

By induction if R is Noetherian then so is the polynomial ring $R[X_1, \dots X_n]$ and in particular since any field is Noetherian so is $\mathbb{C}[X_1, \dots X_n]$. We remind ourself that a ring being Noetherian is equivalent to each of its ideals being finitely generated. Thus we conclude that any algebraic set is the zero locus of a finite set of polynomials.

Projective varieties

Consider the vector space $\mathbb{C}^n$ for $n \geq 2$ and denote by $\mathbb{C}^*$ the multiplicative group of $\mathbb{C}$. Introduce the equivalence relation $x \sim y$ by $x = \lambda y$ for some $\lambda \in C^*$. The projective space of $\mathbb{C}^n$ is then defined to be the quotient space

$$\mathbb{CP}^{n-1} = \mathbb{P}(\mathbb{C}^n) = (\mathbb{C}^n - \mathbf{0})/\sim . \tag{2.5}$$

Geometrically the projective space is the space of lines through the origin of the vector space. Inductively we have $\mathbb{CP}^n \cong \mathbb{C}^n \cup \mathbb{CP}^{n-1}$. In particular $\mathbb{CP}^1 \cong \mathbb{C} \cup \infty$ is known as the complex projective line or Riemann sphere. We denote the group of projective transformations of $\mathbb{P}(\mathbb{C}^n)$ as $PGL(\mathbb{C}^n)$ i.e. the quotient of $GL(\mathbb{C}^n)$ under the action of scalar transformations λI, the center of $GL(\mathbb{C}^n)$.

A point $p \in \mathbb{CP}^n$ is described by any $(n+1)$-tuple in the equivalence class of p, called homogeneous coordinates. Analogously to before a set $X \subset \mathbb{CP}^n$ is algebraic if it is the zero locus of a set of homogeneous polynomials S, that is

$$X = Z(S) = \{x \in \mathbb{CP}^n | f(x) = 0 \ , \ \forall f \in S\}. \tag{2.6}$$

A projective variety is defined as the zero locus of an irreducible set of homogeneous polynomials, or equivalently as the zero locus of a set of homogeneous polynomials that generate a prime ideal.

2.2 Fiber bundles

We will here introduce the concepts of fiber bundles and the gauge theory. There are excellent lecture notes on gauge theory by Figueroa-O'Farrill at [26] and Evans [27] which treat these topics in more detail and covers some of the derivations omitted here.

A fibre bundle consists of three topological spaces E,B,F and a continuous surjective map $\pi : E \to B$, known as the projection. The spaces E,B,F are known as total space, base space and the fiber respectively. Furthermore each point $b \in B$ is required to have an open neighbourhood U and a homeomorphism

$$\phi : \pi^{-1}(U) \to U \times F, \tag{2.7}$$

such that $\pi = \mathrm{proj}_1 \circ \phi$, where $\mathrm{proj}_1 : (u,f) \mapsto u$ is the natural projection. The set of all (U_i, ϕ_i) is called the local trivialisation. This can also expressed as that the diagram

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\mathrm{proj}_1"] \\
& U &
\end{tikzcd}$$

commutes.

A fiber bundle is a way of taking products so that locally the total space E looks like the product space of $B \times F$. In particular a bundle where $E = B \times F$ with $\pi = \mathrm{proj}_1$ is known as a trivial bundle. See fig. 2.1 for an illustration of the difference between a trivial and non trivial bundle.

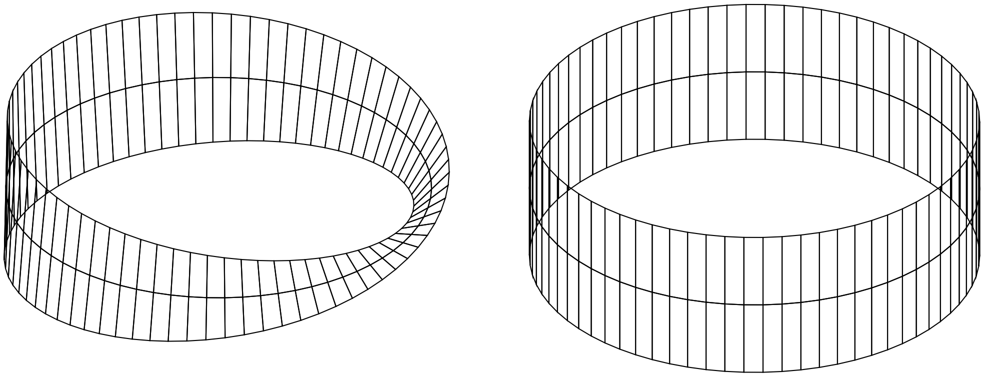

Figure 2.1: A Möbius strip is an example of a non-trivial vector bundle with the corresponding trivial bundle being the cylinder $S^1 \times \mathbb{R}$.

We are primarily interested in vector bundles $E \to M$, whose fiber F is a vector space $\mathbb{R}^n$ or $\mathbb{C}^n$. We further will assume that the base space is a real or complex manifold M and that the transition functions are smooth respective holomorphic. These bundles are then known as smooth and holomorphic vector bundles respectively.

With a fiber bundle $\pi : E \to M$ comes the idea of local sections. These are morphisms from an open set $U \subset M$ of the base space

$$\sigma : U \to E \tag{2.8}$$

such that they are a local right inverse to the projection

$$\pi \circ \sigma = \mathrm{id}. \tag{2.9}$$

A global right inverse $\sigma : M \to E$ is known as a global section. We can form sections from the trivialization maps $\phi_i^{-1} : U_i \times F \to \pi^{-1}$ by considering morphisms $U \to F$.

Remark 2.4. *In the context of smooth bundles we talk about smooth sections* $C^\infty(U,E)$ *and correspondingly holomorphic sections for holomorphic bundles.*

Example 2.5. Let $\pi : E \to M$ be a vector bundle. By mapping every point to the zero vector we obtain a global section, the zero section

$$\sigma : m \mapsto (m,0). \tag{2.10}$$

2.2.1 Principal bundles

A principal G-bundle $\pi : P \to M$ is a fiber bundle where the fiber is a Lie group G, known as the structure group, that acts freely on the right on the total space P. For our purposes we will assume that the base space $M = P/G$ is a manifold with smooth transition functions. Note that $\pi(pg) = \pi(p)$ by definition since π maps into P/G.

Given a representation of G we can construct an associated bundle with a different fiber but with induced transition functions ϕ_i. In particular we can form a vector bundle, the adjoint bundle $\pi' : \operatorname{ad} P \to M$ with the fiber $\mathfrak{g} = \mathrm{Lie}(G)$ using the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(G)$. The total space is

$$\operatorname{ad} P = P \times_{\mathrm{Ad}} \mathfrak{g} = (P \times \mathfrak{g})/\sim, \tag{2.11}$$

with equivalence under the action

$$(p,X)g = (pg,\mathrm{Ad}_{g^{-1}}(X)) = (pg,g^{-1}Xg), \tag{2.12}$$

assuming G is a matrix group. We set $\pi'(p,X) = \pi(p)$ and the transition functions $\phi' : U \times \mathfrak{g} \to \pi^{-1}(U) \times \mathfrak{g}$.

Conversely given a vector bundle with fiber V there is an associated Principal bundle known as the frame bundle by letting G be the automorphism group $GL(V)$. This allows us to move between between vector bundles and principal bundles. Since we will work with bundles with additional structure in the form of a metric we also want this relation to hold. The structure groups of a Riemannian manifold is $O(n)$ giving an orthogonal frame bundle. Similarly with an Hermitian metric we will work with the associated $U(n)$-bundles.

A connection on a principal G-bundle P is a horizontal G-equivariant Lie-algebra valued 1-form $A \in \Omega^1(M,\mathfrak{g})$. The connection forms are naturally identified with forms in $\Omega^1(M, \operatorname{ad} P)$. For a connection A we have the exterior covariant derivative $\mathrm{d}_A : \Omega^p(M, \operatorname{ad} P) \to \Omega^{p+1}(M, \operatorname{ad} P)$ given by

$$\mathrm{d}_A = \mathrm{d} + A \tag{2.13}$$

and the curvature

$$F_A = \mathrm{d}_A^2 = \mathrm{d}_A A = \mathrm{d}A + A \wedge A, \tag{2.14}$$

which measures the exactness of the connection.

Gauge transformations

Principal bundles comes with a set of gauge transformation which are transformation between local sections of the bundle describing the redundancy of the bundle structure. In Physics this freedom is commonly used to represent symmetries. A particular useful technique is that of gauge fixing, where a particular gauge with nice properties is chosen.

Definition 2.6. *Let P be a principal bundle. A gauge transformation on P is a G-equivariant diffeomorphism $\Phi : P \to P$ that preserves the bundle structure.*

G-equivariance means that $\Phi(pg) = \Phi(p)g$ and the preservation of bundle structure meaning that $\pi \circ \Phi = \pi$, i.e. that the diagram

$$\begin{tikzcd}
P \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & P \arrow[dl, "\pi"] \\
& M &
\end{tikzcd}$$

commutes. We call the group of gauge transformations $\mathscr{G}$ the gauge group of P. By considering their action on the transition function we can view a gauge transformation as a choice of a section on the associated bundle Ad P. This is the bundle generated by the representation $\mathrm{Ad} : G \to \mathrm{Diff}(G)$ which is constructed the same way as the adjoint bundle. Then the gauge group is $\mathscr{G} = \Omega^0(M,\mathrm{Ad}P)$. Acting on a connection with the gauge transformation $g \in \mathscr{G}$ the connection transforms as

$$A \mapsto gAg^{-1} - \mathrm{d}gg^{-1} \tag{2.15}$$

while the curvature transforms as

$$F_A \mapsto gF_Ag^{-1}. \tag{2.16}$$

That is the curvature transforms equivariantly.

Example 2.7 (Maxwell theory)**.** Consider a $U(1)$-bundle over some complex manifold M. A gauge transformation is then of the form

$$(m,e^{i\phi}) \mapsto (m,e^{i(\phi+f)}). \tag{2.17}$$

In this case the connection form is an element $A \in \Omega^1(P,\mathfrak{g})$, meaning it takes values in $\mathfrak{u}(1)$, the set of imaginary numbers. Applying a gauge transformation then results in the transformation

$$A \mapsto A + i\,\mathrm{d}f, \tag{2.18}$$

that is adding an exact imaginary 1-form. In particular connections in the same equivalence class have the same curvature and the flat connections are precisely those that are equivalent to the trivial connection.

2.2.2 Sheaves

In order to introduce stability we need to introduce coherent sheafs, which we will view as a generalisation of vector bundles. In the context we are interested in, that of holomorphic vector bundles, this is covered [25] and [28].

Definition 2.8. *A sheaf of abelian groups $\mathscr{E}$ on a topological space X is a map that assigns to every open set $U \subset X$ an abelian group $\mathscr{E}(U)$. The elements of $\mathscr{E}(U)$ are known as sections over U. Additionally we require that for every open subsets $V \subset U$ there exists a group homomorphism $\rho_{UV} : \mathscr{E}(U) \to \mathscr{E}(V)$ known as the restriction map satisfying:*
(i) $\rho_{UU} = id$.
(ii) For $V \subset U \subset W$.

$$\rho_{UV} \circ \rho_{WU} = \rho_{WV}.$$

(iii) Let U_i be a open covering of U. If $f,g \in \mathscr{F}(U)$ and $f|_{U_i} = g|_{U_i}$ for all i then $f = g$.
(iv) If a family $f_i \in \mathscr{F}(U_i)$ satisfies

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j},$$

then there exists a $f \in \mathscr{F}(U)$ with $f|_{U_i} = f_i$.

Remark 2.9. *The conditions (i) and (ii) makes $\mathscr{E}$ into a presheaf. Property (iii) adds locality and (iv) is known as gluing, telling us that compatible sections combine to a gluing section which is unique by (iii).*

One consequence of the sheaf axioms are that that $\mathscr{F}(\emptyset)$ is the trivial group. This follows from (iii) by considering $f,g \in \mathscr{F}(\emptyset)$ and noting that the only cover of $\emptyset$ is itself and thus $f = g$ by (iii).

Relating this to vector bundles we note that the sections on a vector bundle E over an open set U forms a module. Furthermore there is a natural restriction map satisfying the axioms making the sections of the bundle into a sheaf, the sheaf of sections which we denote $\mathcal{O}(E)$. We remind the reader that depending if we work with C^∞ or holomorphic bundles we consider respective sections. Since we are interested in the holomorphic case we will proceed to work in this setting while noting that the case of smooth bundles are completely analogous.

For a complex manifold M we can form the structure sheaf $\mathcal{O}_M$ consisting of holomorphic functions. We have that this is the sheaf of the trivial line bundle $E = M \times \mathbb{C}$ while $\mathcal{O}_M^k$ correspond to the trivial bundle of rank k, $E = M \times \mathbb{C}^k$. The property we need a sheaf to satisfy to be a vector bundle is that it locally looks like a trivial bundle, such sheafs are known as locally free.

Definition 2.10. *A sheaf $\mathscr{E}$ over M is locally free of rank k if for every open subset $U \subset M$*

$$\mathcal{O}_M^k|_U = \mathscr{E}|_U. \tag{2.19}$$

Since the sections determine the vector bundle there is a one-to-one correspondence between locally free sheaves and vector bundles. Thus we will move between these viewpoints and treat locally free sheaves as vector bundles.

Locally free sheafs do not in general form an abelian category, since if we take the quotient of two locally free sheafs the result may not be locally free. What we need is the category of coherent sheafs, which do form an abelian category.

Definition 2.11. *We say that a sheaf $\mathscr{E}$ of $\mathcal{O}$-modules is coherent if there exists an exact sequence*

$$\mathcal{O}^r|U \to \mathcal{O}^s|U \to \mathscr{E}|U \to 0. \tag{2.20}$$

Remark 2.12. *We caution the reader that the above definition in algebraic geometry texts usually is for $\mathscr{E}$ to be finitely presentable. However in our case, when considering sheafs of finite rank and $\mathcal{O}$ being locally free, the notion of coherence and being finitely presented coincide.*

We see that a locally free sheaf is coherent since $\mathcal{O}^k \xrightarrow{f} \mathscr{E} \to 0$ is exact if and only if f is surjective and thus the sequence $\mathcal{O}^k \xrightarrow{\mathrm{id}} \mathcal{O}^k \xrightarrow{f} \mathscr{E} \to 0$ is exact. Coherent sheafs are free outside of a singularity set.

Definition 2.13. *The singularity set S of a coherent sheaf $\mathscr{E}$ is the the set of points where $\mathscr{E}$ is not locally free*

$$S = \{x \in M | \mathscr{E}_x \text{ is not free}\}. \tag{2.21}$$

The rank of a coherent sheaf $\mathscr{E}$ is defined as the rank outside the singularity set $S(\mathscr{E})$,

$$\operatorname{rk}\mathscr{E} = \operatorname{rk}\mathscr{E}_x, \quad x \in M - S. \tag{2.22}$$

In the case of algebraic curves we have

Proposition 2.14. *A subsheaf of a locally free sheaf over a projective curve X is locally free.*

For the proof of the Donaldson-Uhlenbeck-Yau theorem we will rely on

Proposition 2.15 (Proposition 5.6.13 in [28])**.** *A monomorphism $\mathscr{S} \to \mathscr{S}'$ between torsion-free coherent sheaves of the same rank induces a monomorphism $\det \mathscr{S} \to \mathscr{S}'$.*

2.3 Symplectic geometry & moment maps

Symplectic geometry has its roots in classical mechanics, with a symplectic manifold being the generalization of phase space. Moment maps are similarly derived from the concept of generalized momenta and correspond to conserved quantities. An excellent source for learning about symplectic geometry and in particular moment maps is the lectures notes by Berline and Vergne in [29]. Furthermore the classical work of Arnold in [30] also provides a very readable introduction to the subject.

Definition 2.16. *A symplectic manifold (M,ω) of dimension $2n$ is a manifold M together with a closed non-degenerative 2-form ω on M, known as a symplectic form.*

Remark 2.17. *A symplectic manifold is always even dimensional in order to have a symplectic form. The prototype of a symplectic manifold is $\mathbb{R}^{2n}$ with symplectic form $\omega = \sum \mathrm{d}x_i \wedge \mathrm{d}y_i$.*

Let us look at some of the properties of symplectic manifolds and in particular their context in Hamiltonian mechanics. A function $H \in C^\infty(M)$ give rise to a vector field X_H by

$$\mathrm{d}H(Y) = \omega(X_H, Y) \tag{2.23}$$

or restated

$$\mathrm{d}H = \iota_{X_H}\omega, \tag{2.24}$$

where $\iota : \Omega^k(M) \to \Omega^{k-1}(M)$ is the contraction of a vector field X and form α defined by

$$(\iota_X\alpha)(X_1, \dots, X_{k-1}) = \alpha(X, X_1, \dots, X_{k-1}). \tag{2.25}$$

Note that X_H is unique and exists for all H. Conversely a vector field X specify H up to a constant. X_H is known as a Hamiltonian vector field and H a Hamiltonian function.

Every symplectic manifold have a natural Poisson structure with Poisson bracket

$$\{-,-\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M) : \{f,g\} \mapsto \omega(X_f, X_g). \tag{2.26}$$

The 1-form $\mathrm{d}H$ acting on a Hamiltonian vector field X_G can then be expressed in terms of the Poisson bracket as

$$\mathrm{d}H(X_G) = \omega(X_H, X_G) = \{H, G\}. \tag{2.27}$$

The flow defined by the vector field X_H is known as Hamiltonian flow. In particular the time evolution of the coordinates $\mathbf{x}$ under Hamiltonian flow is

$$\dot{\mathbf{x}} = X_H(\mathbf{x}). \tag{2.28}$$

More generally the flow of a tensor field f under X_H is given by the Lie derivative as

$$\frac{\partial f}{\partial t} = \mathcal{L}_{X_H}(f). \tag{2.29}$$

In the context of Hamiltonian mechanics the Hamiltonian flow will correspond to equations of motion.

There is a canonical choice of local coordinates for a symplectic manifold. In classical mechanics these coordinates are known as generalised coordinates and generalised momenta $\mathbf{x} = (\mathbf{q},\mathbf{p})$. That such a choice is possible in general is a consequence of Darboux's theorem.

Theorem 2.18 (Darboux's theorem)**.** *Let (M, ω) be a symplectic manifold of dimension $2n$. Then around every point $m \in M$ there exists coordinates*

$$q_1, \dots q_n, p_1, \dots p_n \tag{2.30}$$

such that

$$\omega = \mathrm{d}q_1 \wedge \mathrm{d}p_1 + \cdots + \mathrm{d}q_n \wedge \mathrm{d}p_n. \tag{2.31}$$

These coordinates are known as canonical coordinates or Darboux coordinates.

A proof can be found in [30]. Using these coordinates we can now derive the properties of Hamiltonian mechanics as a consequence of symplectic geometry.

Example 2.19 (Hamiltonian mechanics)**.** Let (M,ω) be a symplectic manifold of dimension $2n$ with canonical coordinates (q_i,p_i). For $H \in C^\infty(M)$ we have

$$\mathrm{d}H = \sum_{i=1}^{n} \frac{\partial H}{\partial q_i}\,\mathrm{d}q_i + \frac{\partial H}{\partial p_i}\,\mathrm{d}p_i. \tag{2.32}$$

Additionally

$$\omega(X_H,Y) = \sum_{i=1}^{n} \mathrm{d}q_i \wedge \mathrm{d}p_i(X_H,Y) = \sum_{i=1}^{n} \mathrm{d}q_i(X_H)\,\mathrm{d}p_i(Y) - \mathrm{d}q_i(Y)\,\mathrm{d}p_i(X_H). \tag{2.33}$$

Matching the condition that $\mathrm{d}H = \omega(X_H,-)$ gives the conditions

$$\begin{cases} \mathrm{d}q_i(X_H) = \frac{\partial H}{\partial p_i} \\ \mathrm{d}p_i(X_H) = -\frac{\partial H}{\partial q_i}, \end{cases} \tag{2.34}$$

and thus the Hamiltonian vector field X_H is given by

$$X_H = \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}, \tag{2.35}$$

and the Poisson bracket of $H,G \in C^\infty(M)$ is then

$$\{H,G\} = \mathrm{d}H(X_G) = \sum_{i=1}^{n} \frac{\partial H}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial G}{\partial q_i}. \tag{2.36}$$

Explicitly the Hamiltonian flow $\dot{\mathbf{x}} = X_H(\mathbf{x})$ is

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial q}, \end{cases} \tag{2.37}$$

which can be recognised as Hamilton's equation of motion from classical mechanics. The flow lines are then the trajectories in phase space.

We are now ready to introduce moment maps on symplectic manifolds. Let G be a Lie group that acts on the left on a symplectic manifold (M,ω). There is then a corresponding infinitesimal action by the Lie algebra $\mathfrak{g}$.

$$\mathfrak{g} \times M \to TM : (X,m) \mapsto X^M, \tag{2.38}$$

where for every $X \in \mathfrak{g}$ the vector field X^M is defined at $m \in M$ by exponentiation of the Lie algebra

$$X^M = \frac{\mathrm{d}}{\mathrm{d}t}\exp(tX)m\bigg|_{t=0}. \tag{2.39}$$

Definition 2.20. *A moment map is a G-equivariant map*

$$\mu : M \to \mathfrak{g}^*, \tag{2.40}$$

such that for every $X \in \mathfrak{g}$ the map X^M is the Hamiltonian vector field of $m \mapsto \langle \mu(m), X\rangle$. That is

$$\mathrm{d}\langle \mu, X\rangle = \iota(X^M)\omega = \omega(X^M,-). \tag{2.41}$$

Proposition 2.21. *The moment map of an element $X \in \mathfrak{g}$ is unique up to a constant.*

Proof. Suppose that μ_1,μ_2 are two moment maps for $X \in \mathfrak{g}$. Then

$$\mathrm{d}(\mu_1 - \mu_2) = 0. \tag{2.42}$$

□

Let us illustrate the concept of moment maps as conserved quantities with the two following examples from classical mechanics.

Example 2.22 (Linear momentum)**.** Consider $\mathbb{R}^2$ with a group action of $\mathbb{R}$ given by $(q,p) \mapsto (q+t,p)$ and let $\omega = \mathrm{d}q \wedge \mathrm{d}p$ be the symplectic form on $\mathbb{R}^2$. The vector field is then corresponding to the generator X of the Lie algebra is then

$$X^M = \frac{\partial}{\partial q}. \tag{2.43}$$

We have that

$$\iota(X^M)\omega = \mathrm{d}q \wedge \mathrm{d}p\left(\frac{\partial}{\partial q},-\right) = \mathrm{d}p, \tag{2.44}$$

and hence up to a constant

$$\langle \mu(q,p), - \rangle = p, \tag{2.45}$$

giving us the moment map

$$\mu(q,p) = p. \tag{2.46}$$

Example 2.23 (Angular momentum)**.** Consider an action by the rotation group $SO(3)$ on $\mathbb{R}^6$ with the standard symplectic form. The vector field then takes the form for the three generators $X_i \in \mathfrak{so}(3)$ as

$$X_i^M = \epsilon_{ijk} \left(q_j \frac{\partial}{\partial q_k} + p_j \frac{\partial}{\partial p_k} \right), \tag{2.47}$$

or in vector notation

$$X^M = \mathbf{q} \times \nabla_{\mathbf{q}} + \mathbf{p} \times \nabla_{\mathbf{p}}. \tag{2.48}$$

Then

$$\begin{cases} \omega(\frac{\partial}{\partial q_i}, -) = \mathrm{d}p_i \\ \omega(\frac{\partial}{\partial p_i}, -) = -\,\mathrm{d}q_i, \end{cases} \tag{2.49}$$

which gives

$$\iota(X_i^M)\omega = \epsilon_{ijk}(q_j\,\mathrm{d}p_k - p_j\,\mathrm{d}q_k). \tag{2.50}$$

Again putting this in vector notation

$$\iota(X_M)\omega = q \times \mathrm{d}p - p \times \mathrm{d}q = q \times \mathrm{d}p + \mathrm{d}q \times p = \mathrm{d}(q \times p). \tag{2.51}$$

The moment map is then the angular momentum

$$\mu(p,q) = p \times q. \tag{2.52}$$

Note that the result of the conservation of the antisymmetric tensor $q_i p_j - q_j p_i$ is valid in any dimension under rotation $SO(n)$, while in $n = 3$ dimensions the angular momentum takes the familiar form of a cross product.

2.4 Kähler geometry

A Kähler manifold is a manifold with three compatible structures, a complex structure, a symplectic structure, and a Riemannian structure. The prototype for Kähler manifolds are $\mathbb{C}^n$, which we will often return to as an example. In the last section we introduced symplectic manifolds and the reader is likely already familiar with Riemannian manifolds. There are several well written sources on Kähler geometry however we especially would like to refer to Foundations of differential geometry by Kobayashi and Nomizu in [31] and [32], which provide the necessary background for [28] who cover the material in the context of complex vector bundles.

Let us thus begin by saying a few words about the complex structure.

Definition 2.24. *An almost complex structure J on a manifold M is a smooth $(1,1)$-tensor field $J \in \Gamma(TM \otimes T^*M)$ such that*

$$J_p^2 = -id. \tag{2.53}$$

We say that the pair (M, J) forms an almost complex manifold.

Since $\Gamma(TM \otimes T^*M) \cong \mathrm{End}(TM)$, what the above definition says is that J assigns smoothly a linear endomorphism $J_p : T_pM \to T_pM$ for each $p \in M$ such that $J_p^2 = -\mathrm{id}$.

Example 2.25. In the case of $\mathbb{C}^n$ with coordinates $z_i = x_i + iy_i$ we then have the natural complex structure given by

$$J : (x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto (-y_1, \ldots, -y_n, x_1, \ldots, x_n). \tag{2.54}$$

That is J work as a multiplication with i on z_k. Note that a similar construction can be made for any even dimensional vector space.

Every complex manifold have an almost complex structure however the converse is not generally true. What is missing is an integrability requirement which if satisfied gives us an integrable almost complex structure. For an almost complex structure J this integrability requirement is given by the Newlander-Niremberg theorem which tells us an almost complex structure J is integrable, and thus defines a complex structure, if and only if the Nijenhuis tensor of J given by

$$N_J(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY], \tag{2.55}$$

vanishes for all vector fields $X, Y \in \Gamma(TM)$. A proof of this theorem can be found in most textbooks on complex geometry or see [33] for the original proof by Newlander-Niremberg.

We are now ready to give a definition of Kähler manifolds.

Definition 2.26. *A Kähler manifold* (M, J, ω) *is a symplectic manifold* (M, ω) *with an integrable almost complex structure* J *such that*

$$g(X,Y) = \omega(X,JY), \tag{2.56}$$

is a symmetric positive definite bilinear form, i.e. a Riemannian metric. We say that ω *is a Kähler form.*

Remark 2.27. *If the integrability requirement of* J *is dropped, we say that* (M, J, Ω) *is an almost Kähler manifold.*

In the above definition we gave a Kähler in terms of the symplectic form and the complex structure while checking that g is a Riemannian metric, however one could specify a Kähler manifold in terms of any pair of (ω, J, g) while checking that the third is compatible. Compare this to to the two out of three property of the unitary group. Given the Riemannian metric g and the complex structure J we can form the symplectic form by

$$\omega(X,Y) = g(JX,Y), \tag{2.57}$$

where we would have to check that ω is closed. Similarly given g and ω we can form the complex structure by

$$J(X) = g^{-1} \circ (\omega(X,-)). \tag{2.58}$$

One can also specify an Hermitian metric by

$$h(X,Y) = g(X,Y) - i\omega(X,Y), \tag{2.59}$$

which can be used to recover the Riemannian metric as $g = \mathrm{Re}\, h$ and the symplectic form as $\omega = -\mathrm{Im}\, h$.

Let us also note that for a Kähler manifold the complex structure preserves ω, g and, h i.e.

$$\omega(JX,JY) = \omega(X,Y), \quad g(JX,JY) = g(X,Y), \quad \text{and} \quad h(JX,JY) = h(X,Y). \tag{2.60}$$

Example 2.28. Let us return to our example of $\mathbb{C}^n$. In standard coordinates the symplectic form is given by

$$\omega = \frac{i}{2}\sum_{k=1}^{n} \mathrm{d}z_k \wedge \mathrm{d}\bar{z}_k = \sum_{k=1}^{n} \mathrm{d}x_k \wedge \mathrm{d}y_k. \tag{2.61}$$

If we then consider two vector fields locally given by

$$X = \sum_{k=1}^{n}\left(A_k\frac{\partial}{\partial x_k} + B_k\frac{\partial}{\partial y_k}\right), \quad \text{and} \quad Y = \sum_{k=1}^{n}\left(C_k\frac{\partial}{\partial x_k} + D_k\frac{\partial}{\partial y_k}\right), \tag{2.62}$$

we see that with the complex structure introduced in example 2.25 that

$$g(X,Y) = \omega(X,JY) = \sum_{k=1}^{n}\left(A_kD_k + B_kC_k\right), \tag{2.63}$$

which the dot product of Euclidean space. We can also write the metric in the perhaps more familiar form of the line element as

$$\mathrm{d}s^2 = \sum_{k=1}^{n}\left(\mathrm{d}x_k^2 + \mathrm{d}y_k^2\right). \tag{2.64}$$

In addition to $\mathbb{C}^n$ being the simplest example of a Kähler manifold we will often work in local coordinates where the manifold will look like $\mathbb{C}^n$. Let us finally note that the Nijenhuis tensor trivially vanishes in the above case since second derivatives commute and thus by working in local coordinates the same is true for any complex manifold.

On complex manifolds the exterior derivative can be written in terms of the Dolbeault operators

$$\mathrm{d} = \partial + \bar{\partial}, \tag{2.65}$$

where $\partial : \Omega^{p,q} \to \Omega^{p+1,q}$ and $\bar{\partial} : \Omega^{p,q} \to \Omega^{p,q+1}$. These operators satisfy

$$\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0. \tag{2.66}$$

Note that we have forms of total degree r as

$$\Omega^r(M) = \bigoplus_{p+q=r} \Omega^{p,q}(M). \tag{2.67}$$

which the exterior derivative acts on.

Specifically on Kähler manifolds the Dolbeault cohomology can be related to the de Rham cohomology as

$$H^r(M) = \bigoplus_{p+q=r} H^{p,q}(M). \tag{2.68}$$

We also have the Kähler version of the Poincare lemma, the $\partial\bar{\partial}$ lemma. Locally we can write a closed (1,1)-form α as

$$\alpha = \partial\bar{\partial} f \tag{2.69}$$

for a smooth function f. In particular one often write the Kähler form in form of a Kähler potential f

$$\omega = i\partial\bar{\partial} f, \tag{2.70}$$

where the factor of i is often included by convention. In $\mathbb{C}^n$ or in the standard local coordinates the Kähler potential would be

$$f = \frac{1}{2}\sum_{k=1}^{n} \mathrm{d}z_k \, \mathrm{d}\bar{z}_k. \tag{2.71}$$

We define the Lefschetz operator $L : \Omega^{p,q} \to \Omega^{p+1,q+1}$ by

$$L\alpha = \omega \wedge \alpha. \tag{2.72}$$

Denote the adjoint operator by $\Lambda = L^* : \Omega^{p,q} \to \Omega^{p-1,q-1}$. We will commonly refer to Λ as the trace operator. In particular when acting on a (1,1)-form α and assuming the standard volume form on (M,ω),

$$\mathrm{dvol} = \frac{\omega^n}{n!}, \tag{2.73}$$

Λ acts by

$$\frac{\Lambda\alpha\omega^n}{n!} = \frac{1}{(n-1)!}\alpha \wedge \omega^{n-1}, \tag{2.74}$$

or rearranged

$$\Lambda\alpha\omega^n = n\alpha \wedge \omega^{n-1}. \tag{2.75}$$

One can show show the identities

$$\partial^* = i[\Lambda, \bar{\partial}], \qquad \bar{\partial}^* = -i[\Lambda, \partial], \tag{2.76}$$

which together with similar identities for L are known as Kähler identities.

These identities prove useful for showing that the three natural Laplacians we can form on a Kähler manifold defined as

$$\Delta_d = \mathrm{d}^*\,\mathrm{d} + \mathrm{d}\,\mathrm{d}^*, \quad \Delta_\partial = \partial^*\partial + \partial\partial^*, \quad \Delta_{\bar{\partial}} = \bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*, \tag{2.77}$$

are equal up to a constant.

Proposition 2.29. *On a Kähler manifold the Laplacians are related as*

$$\Delta_d = 2\Delta_\partial = 2\Delta_{\bar{\partial}}, \tag{2.78}$$

i.e. they are equal up to a constant.

Proof. By the Kähler identities we have that

$$\partial\bar{\partial}^* + \bar{\partial}^*\partial = -i(\partial[\Lambda,\partial] + [\Lambda,\partial]\partial) = -i\left(\partial\Lambda\partial + \cancel{\partial^2\Lambda} + \cancel{\Lambda\partial^2} - \partial\Lambda\partial\right) = 0, \tag{2.79}$$

and by also considering the adjoint it follows that

$$\partial^*\bar{\partial} + \bar{\partial}\partial^* = \partial\bar{\partial}^* + \bar{\partial}^*\partial = 0. \tag{2.80}$$

As a consequence

$$\Delta_\mathrm{d} = \mathrm{d}^*\,\mathrm{d} + \mathrm{d}\,\mathrm{d}^* = \{\partial + \bar{\partial}, \partial^* + \bar{\partial}^*\} = \{\partial, \partial^*\} + \cancel{\{\partial, \bar{\partial}^*\}} + \cancel{\{\bar{\partial}, \partial^*\}} + \{\bar{\partial}, \bar{\partial}^*\} = \Delta_\partial + \Delta_{\bar{\partial}}, \tag{2.81}$$

where we use the brackets to denote the anticommutator. W also have that

$$\Delta_\partial = \partial\partial^* + \partial^*\partial = i\left(\partial[\Lambda,\bar{\partial}] + [\Lambda,\bar{\partial}]\partial\right) = i\left(\partial\Lambda\bar{\partial} - \partial\bar{\partial}\Lambda + \Lambda\bar{\partial}\partial - \bar{\partial}\Lambda\partial\right), \tag{2.82}$$

and taking the adjoint we have

$$\Delta_{\bar{\partial}} = -i\left(\bar{\partial}\Lambda\partial - \bar{\partial}\partial\Lambda + \Lambda\partial\bar{\partial} - \partial\Lambda\bar{\partial}\right). \tag{2.83}$$

Since $\partial\bar{\partial} = -\bar{\partial}\partial$ we see that $\Delta_\partial = \Delta_{\bar{\partial}}$. □

2.4.1 Calabi-Yau manifolds & Special Lagrangian submanifolds

An important class of Kähler manifolds is that of Calabi-Yau manifolds.

Definition 2.30. *A Calabi-Yau manifold* (M, J, ω, Ω) *is a compact Kähler manifold with a nowhere vanishing holomorphic volume form* Ω.

Remark 2.31. *We will on occasion drop the requirement of compactness. It is also common to see requirement that a Calabi-Yau should also be simply connected or algebraic. Both these properties will be true in all the examples we consider.*

We remind ourselves that Ω being a holomorphic volume form means that it is a $(n, 0)$-form satisfying

$$\bar{\partial}\Omega = 0. \tag{2.84}$$

Note that while Ω is not a volume form on the underlying real manifold, $\Omega \wedge \bar{\Omega}$ is. On $\mathbb{C}^n$ a holomorphic volume form is $\Omega = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n$.

The Calabi-Yau manifolds of dimension $n = 1$ are elliptic curves, i.e. projective varieties. In dimension $n = 2$ the simply connected Calabi-Yau manifolds are known as $K3$ surfaces.

One important property of Calabi-Yau manifolds is that their first Chern class vanish. This is often taken as the definition of Calabi-Yau manifold, as Kähler manifolds with vanishing first Chern class, and is equivalent for simply connected Calabi-Yau. We can further note that the vanishing of the first Chern class on a Calabi-Yau manifold M corresponds to that the curvature form F takes values in $\mathfrak{su}(n)$ by

$$c_1(M) = \frac{i}{2\pi}\,\mathrm{tr}(F) = 0. \tag{2.85}$$

In the introduction we discussed special Lagrangians as an important object on the symplectic side of mirror symmetry, in the form of branes in the A-model. Let us begin with Lagrangian submanifold in the context of symplectic geometry which in classical mechanics can be thought of as corresponding to the set of generalized momenta.

Definition 2.32. *Let* (M,ω) *be a symplectic manifold of dimension* $2n$. *A Lagrangian submanifold* L *is a submanifold to* M *of dimension* n *where the restriction of the symplectic form vanish, i.e.*

$$\omega|_L = 0. \tag{2.86}$$

Remark 2.33. *Submanifolds where the symplectic form vanishes are more generally known as isotropic. Lagrangian submanifolds are those that have maximal dimension, i.e. half that of* M.

Special Lagrangian arise as submanifolds in the context of Calabi-Yau manifolds where in addition to the restriction on the Kähler form ω, there is one on the holomorphic volume form Ω.

Definition 2.34. *Let* (M,J,ω,Ω) *be a (not necessarily compact) Calabi-Yau manifold. We say that a Lagrangian submanifold* L *of* M *is a special Lagrangian (sLag) if*

$$\mathrm{Im}(e^{-i\theta}\Omega|_L) = 0, \tag{2.87}$$

for $\theta \in \mathbb{R}$. *The angle* θ *is known as the Lagrangian angle.*

Remark 2.35. *It is common to define sLags without the phase factor* $e^{-i\theta}$. *We will instead refer to sLags with* $\theta = 0$ *as simple sLags.*

Special Lagrangians are examples of minimal submanifolds, that is they minimize volume locally. More specifically they are calibrated manifolds with calibration

$$\eta = \mathrm{Re}(e^{-i\theta}\Omega). \tag{2.88}$$

For the details of this we refer the reader to [34]. Let us illustrate sLags in the simplest possible example with $M = \mathbb{C}$, where the sLags are geodesics.

Example 2.36. Consider $\mathbb{C}$ with holomorphic volume form $\Omega = \mathrm{d}z = \mathrm{d}x + i\,\mathrm{d}y$ and symplectic form $\omega = \frac{i}{2}\Omega \wedge \bar{\Omega} = \frac{i}{2}\,\mathrm{d}z \wedge \mathrm{d}\bar{z} = \mathrm{d}x \wedge \mathrm{d}y$. Take note that ω is the volume form of the underlying real manifold while Ω is the holomorphic volume form on $\mathbb{C}$.

We first note that every curve in $\mathbb{C}$ is a Lagrangian submanifold since ω is a 2-form while a curve is of dimension one. We have that

$$\mathrm{Re}(e^{-i\theta}\Omega) = \cos\theta\,\mathrm{d}x + \sin\theta\,\mathrm{d}y, \tag{2.89}$$

and

$$\mathrm{Im}(e^{-i\theta}\Omega) = \cos\theta\,\mathrm{d}x - \sin\theta\,\mathrm{d}y. \tag{2.90}$$

Let i denote the inclusion map. Let $i : t \mapsto (x(t),y(t))$ be the inclusion map of a curve L. The pullback

$$i^*(\cos\theta\,\mathrm{d}x - \sin\theta\,\mathrm{d}y) = \left(\cos\theta\frac{\partial x}{\partial t} - \sin\theta\frac{\partial y}{\partial t}\right)\mathrm{d}t \tag{2.91}$$

is zero when $\sin\theta x(t) = \cos\theta y(t) + c$ for any constant c. Thus the curve can be written

$$\begin{cases} x(t) = \sin(\theta)t + x_0 \\ y(t) = \cos(\theta)t + y_0. \end{cases} \tag{2.92}$$

These are all the straight lines in $\mathbb{R}^2$ i.e. the geodesics in $\mathbb{R}^2$ so the sLags are minimal in volume as expected.

Chapter 3

Yang-Mills theory

We will in this chapter give an introduction to Yang-Mills (YM) theory which will be needed for chapter 5 and chapter 6. For a more extensive discussion of the material we refer the reader to [35]. There are also lecture notes by Figueroa [26] and Evans [27] on Yang-Mills theory which the reader may find useful.

3.1 The Yang-Mills functional

Let $P \to M$ be a principal G-bundle where M is an compact oriented pseudo-Riemannian manifold. We will further demand that the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ has a G-invariant inner product. By the definition of the Hodge dual the corresponding norm of the curvature F_A can be written

$$|F_A|^2 \,\mathrm{dvol} = \mathrm{tr}(F_A \wedge *F_A), \tag{3.1}$$

where dvol is the volume form/measure on M.

Remark 3.1. *In the case of $\mathfrak{g}$ being semisimple and compact such a metric on $\mathfrak{g}$ would be given by the killing form. This is the case in the standard examples with G as the unitary or special unitary group.*

Definition 3.2. *The Yang-Mills functional for connection $A \in \Omega^1(P,\mathfrak{g})$ is defined as*

$$S_{YM}(A) = ||F_A||^2_{L^2} = \int_M |F_A|^2 \,\mathrm{dvol}, \tag{3.2}$$

We say that a connection is a Yang-Mills connection if

$$\frac{\mathrm{d}}{\mathrm{d}t} S_{YM}(A + t\alpha)\bigg|_{t=0} = 0, \quad \forall \alpha \in \Omega^1(P,\mathfrak{g}). \tag{3.3}$$

Taking the first variation of the YM-functional gives

$$|F_{A+t\alpha}|^2 = |F_A + t\,\mathrm{d}_A\alpha|^2 + O(t^2) = |F_A|^2 + 2t\langle F_A, \mathrm{d}_A\alpha\rangle + O(t^2) = |F_A|^2 + 2t\langle \mathrm{d}_A^* F_A, \alpha\rangle + O(t^2). \tag{3.4}$$

We have shown

Proposition 3.3. *A connection A is Yang-Mills if and only if*

$$\mathrm{d}_A^* F_A = 0. \tag{3.5}$$

This equation is known as the Yang-Mills equation and can be stated equivalently

Corollary 3.4. *A connection A is Yang-Mills if and only if* $\mathrm{d}_A * F_A = 0$.

Proof. On a manifold of dimension m the adjoint operator acting on a k-form can be written

$$\mathrm{d}_A^* = (-1)^{m(k+1)+1} * \mathrm{d}_A *, \tag{3.6}$$

which implies

$$\mathrm{d}_A^* F_A = 0 \Leftrightarrow \mathrm{d}_A * F_A = 0. \tag{3.7}$$

□

In particular in four dimensions with a positive signature of the metric it follows that self-dual and anti-self-dual (ASD) connections, i.e. connections satisfying $*F_A = \pm F_A$, are by the Bianchi identity $\mathrm{d}_A F_A = 0$ a solution to the YM-equation. These solutions are known as instantons.

Proposition 3.5. *One of the instantons corresponds to a global minimum of the Yang-Mills functional.*

Proof. Since in 4-dimensions with positive signature $*^2 = 1$ there is a splitting of the 2-forms by the ± 1 eigenspaces of the Hodge dual

$$\Omega^2 = \Omega^2_+ \oplus \Omega^2_-, \tag{3.8}$$

resulting in that a curvature form F_A can be we decomposed into self-dual and anti-self-dual parts

$$F_A = F_A^+ \oplus F_A^-. \tag{3.9}$$

In particular a connection A is ASD if and only if $F_A^+ = 0$ or equivalently $\Lambda F_A = 0$. Since the decomposition is orthogonal the Yang-Mills functional can be written

$$||F_A||^2 = \int_M |F_A|^2 \,\mathrm{dvol} = \int_M |F_A^+|^2 \,\mathrm{dvol} + \int_M |F_A^-|^2 \,\mathrm{dvol}\,. \tag{3.10}$$

Setting

$$C = \int_M \mathrm{tr}(F_A^2) = \int_M |F_A^-|^2 \,\mathrm{dvol} - \int_M |F_A^+|^2 \,\mathrm{dvol}, \tag{3.11}$$

now gives the lower bound on the Yang-Mills functional by

$$||F_A||^2 \geq |C|, \tag{3.12}$$

with equality only when F_A is self-dual or anti-self-dual for C negative or positive respectively. □

Remark 3.6. *The integer*

$$C = \int_M \mathrm{tr}(F_A^2) = 8\pi^2(c_2(\mathrm{ad}\,P) - \frac{1}{2}c_1(\mathrm{ad}\,P)^2), \tag{3.13}$$

is known as the instanton number. In the case when $G = SU(n)$ the constant C is given by the second Chern form

$$C = c_2(\mathrm{ad}\,P), \tag{3.14}$$

since $\mathfrak{su}(n)$ consists of traceless matrices making the $\mathrm{tr}(F_A)^2$ term vanish.

Remark 3.7. *If the metric had negative signature as in the Lorentzian case we would instead have instantons with* $*F_A = \pm iF_A$ *since* $*^2 = -1$ *in this case.*

It is natural to consider the gradient flow of the Yang-Mills functional as a way of finding a YM connection. Thus we introduce the Yang-Mills flow

$$\frac{\partial}{\partial t}A = -\,\mathrm{d}_A^* F_A, \tag{3.15}$$

for a collection of connections A_t which we say satisfy Yang-Mills flow. It can be seen that the critical points of the YM flow are YM connections. Looking at how F_A transforms under YM flow we see

$$\frac{\partial}{\partial t}F_A = \frac{\partial}{\partial t}\left(\mathrm{d}A + A \wedge A\right) = -\,\mathrm{d}\,\mathrm{d}_A^* F_A - \mathrm{d}_A^* F_A \wedge A - A \wedge \mathrm{d}_A^* F_A = -\,\mathrm{d}_A\,\mathrm{d}_A^* F_A. \tag{3.16}$$

Using the Bianchi identity $\mathrm{d}_A F_A = 0$ this can be written

$$\frac{\partial}{\partial t}F_A = -(\mathrm{d}_A\,\mathrm{d}_A^* + \mathrm{d}_A^*\,\mathrm{d}_A)F_A = -\Delta_A F_A, \tag{3.17}$$

which we recognise as a heat equation. It also follows that

$$\frac{\partial}{\partial t}|F_A|^2 = \frac{\partial}{\partial t}\langle F_A, F_A\rangle = 2\langle\frac{\partial}{\partial t}F_A, F_A\rangle = -2\langle \mathrm{d}_A\,\mathrm{d}_A^* F_A, F_A\rangle = -2\langle \mathrm{d}_A^* F_A, \mathrm{d}_A^* F_A\rangle = -2|\,\mathrm{d}_A^* F_A|^2, \tag{3.18}$$

and thus

$$\frac{\partial}{\partial t}S_{YM}(A) = -2||\,\mathrm{d}_A^* F_A||^2 = -2S_{YM}(\mathrm{d}_A^* F_A). \tag{3.19}$$

3.2 The Hermitian-Yang-Mills equation

We will now specialise to the case when the base space is a Kähler manifold (M, ω). By the Kähler identities

$$\mathrm{d}_A^* F_A = (\partial_A^* + \bar{\partial}_A^*)F_A = i([\Lambda, \bar{\partial}_A] - [\Lambda, \partial_A])F_A = -i(\bar{\partial}_A - \partial_A)\Lambda F_A. \tag{3.20}$$

The Hermitian-Yang-Mills (HYM) connections are those with the mean curvature ΛF_A being constant, that is satisfying the Einstein condition.

Definition 3.8. *A connection* A *is Hermitian-Einstein or Hermitian-Yang-Mills if the curvature form satisfy*

$$\Lambda F_A = \lambda I \tag{3.21}$$

for some scalar λ*, where* I *is the identity operator on the bundle. This equation is known as the Hermitian-Einstein equation or the Hermitian-Yang-Mills equation.*

By the definition of the trace operator Λ the Hermitian-Yang-Mills equation can also be expressed as

$$F_A \wedge \omega^{n-1} = \frac{\lambda}{n}\omega^n. \tag{3.22}$$

One can also define a weak Hermitian-Einstein condition

$$\Lambda F = \varphi I_E, \tag{3.23}$$

for φ a differentiable scalar function.

Proposition 3.9 (Proposition 4.2.4 in [28])**.** *Given a weak Hermitian-Einstein connection there is a conformal change giving an Hermitian-Einstein connection.*

Proof. Consider the conformal change of the metric

$$h' = fh, \tag{3.24}$$

for f a real and positive function on M. The corresponding change in the mean curvature is

$$\varphi' = \varphi - \Delta \log f. \tag{3.25}$$

A suitable choice of f then makes φ' constant. $\square$

The Yang-Mills flow equation is in this case

$$\frac{\partial}{\partial t}A = i(\bar{\partial}_A - \partial_A)\Lambda F_A. \tag{3.26}$$

Instead looking at the flow of the corresponding family of Hermitian metric H_t we have

$$\frac{\partial H_t}{\partial t} = -2iH_t(\hat{F}_t - \lambda I_E). \tag{3.27}$$

3.3 The moment map perspective

Atiyah and Bott showed in [3] how the curvature can be viewed as a moment map, a result that is also covered in [35]. This will provide us with a way of generalizing the HYM equation and also find connections to stability in the GIT sense.

We begin by viewing the space of connections $\mathcal{A}$ as an infinite dimensional manifold modelled as an affine space on $\Omega^1(M, \mathrm{ad}(P))$. As such the tangent space at every point is also $\Omega^1(M, \mathrm{ad}(P))$. The natural inner product on $\Omega^*(M, \mathrm{ad}(P))$ is given by

$$\langle \alpha, \beta \rangle = \int_M \mathrm{tr}(\alpha \wedge *\beta). \tag{3.28}$$

The Lie coalgebra $\mathfrak{g}^*$ can be associated to $\Omega^{2n}(M, \mathrm{ad}(P))$ and we have the natural pairing $\langle -, - \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ given by

$$\langle \alpha, \beta \rangle = \int_M \mathrm{tr}(\alpha\beta). \tag{3.29}$$

Proposition 3.10. *There is a canonical symplectic form on the space of connections $\mathcal{A}$, given by*

$$\omega_{\mathcal{A}}(\alpha, \beta) = \int_M \mathrm{tr}(\alpha \wedge \beta \wedge \omega^{n-1}). \tag{3.30}$$

Proof. By design we have that $\omega_{\mathcal{A}}$ is alternating and thus a 2-form. Since the integrand is a top form it is also closed an so is $\omega_{\mathcal{A}}$. The map $x \mapsto \omega_{\mathcal{A}}(x, -)$ is one-to-one so it is an isomorphism. Thus $\omega_{\mathcal{A}}$ is non-degenerative. $\square$

An element $\phi \in \mathscr{G} = \Omega^0(M, \mathrm{ad}(P))$ induces a vector field $V_\phi = \mathrm{d}\phi \in \Omega^1(M, \mathrm{ad}(P))$ by exponentiation

$$\phi^{\mathcal{A}} = \frac{\mathrm{d}}{\mathrm{d}t} \exp(t\phi)A \bigg|_{t=0} = -\,\mathrm{d}_A\phi = \mathrm{d}_A\phi^{-1}. \tag{3.31}$$

Proposition 3.11 (Proposition 6.5.8 in [35])**.** *The map* $m : \mathcal{A} \to \Omega^{2n}(M, \mathrm{ad}(P))$ *given by*

$$A \mapsto F_A \wedge \omega^{n-1} \tag{3.32}$$

is a moment map for the $\mathcal{G}$*-action on* $\mathcal{A}$*. That is the map* $F_\phi : \mathcal{A} \to \mathbb{R}$ *given by*

$$A \mapsto \int_M \mathrm{tr}(\phi F_A \wedge \omega^{n-1}) \tag{3.33}$$

is a Hamiltonian function for the vector field $-\,\mathrm{d}_A\phi$*.*

Proof. Let δ be the exterior derivative on $\mathcal{A}$ to distinguish it from d_A on M. The exterior derivative of m at the point A for $\psi \in \mathfrak{g}$ is given by

$$\delta m(A)(\psi) = \mathrm{d}_A\psi \wedge \omega^{n-1}. \tag{3.34}$$

Then by integration by parts

$$\langle \mathrm{d}_A\psi \wedge \omega^{n-1}, \phi\rangle = \int_M \mathrm{tr}(\phi\, \mathrm{d}_A\psi \wedge \omega^{n-1}) = -\int_M \mathrm{tr}(\psi\, \mathrm{d}_A\phi \wedge \omega^{n-1}) = \omega_{\mathcal{A}}(-\,\mathrm{d}_A\phi, \psi). \tag{3.35}$$

Note that we do not have a term with $\mathrm{d}_A\omega^{n-1}$ since ω is closed. Finally we note that m is equivariant since under a gauge transformation g the curvature transforms as $F_{gA} = gF_Ag^{-1}$. $\square$

The Yang-Mills functional can then be viewed as the L^2-norm of the moment map. Furthermore note that we are free to add a constant of integration to the moment map, in particular the HYM equation can be expressed as the moment map $m : \mathcal{A} \to \Omega^{2n}(M, \mathrm{ad}(P))$ given by

$$m(A) = F_A \wedge \omega^{n-1} - \frac{\lambda}{n}\omega^n. \tag{3.36}$$

The zeroes of the moment map are then the HYM connections. The moment map perspective may be seen as a first motivation to that the HYM equation should relate to some stability condition by symplectic reduction.

3.3.1 The deformed Hermitian Yang-Mills equation

Our next step will be generalise the HYM equation by considering other moment maps on $\mathcal{A}$. We will consider functions $f : \Omega^2(M, \mathrm{ad}(P)) \to \Omega^{2n}(M, \mathrm{ad}(P))$ and define deformed Hermitian Yang-Mills equations by

$$f(F_A) = 0. \tag{3.37}$$

To allow f to be a moment the symplectic form will be modified to be

$$\omega_{\mathcal{A}}(\alpha, \beta) = \int_M \mathrm{tr}(\alpha \wedge \beta^* \wedge f'(F_A)). \tag{3.38}$$

Note that the symplectic form previously introduced on $\mathcal{A}$ is recovered in the HYM case where $f(F_A) = F_A \wedge \omega^{n-1}$ since $f'(F_A) = \omega^{n-1}$. With this symplectic form the corresponding moment map to f is

$$m : \mathcal{A} \to \Omega^{2n}(M, \mathrm{ad}(P)) : A \mapsto f(F_A). \tag{3.39}$$

A corresponding modified stability condition suggested by Thomas in [36] comes from modifying the slope to

$$\mu(E)=\frac{\int_M \operatorname{tr} f(F_A)}{\operatorname{rank}(E)}. \tag{3.40}$$

The above condition should be seen as more motivational than definite, currently only a few special cases has been studied and the connection to stability in the GIT sense is far from clear. In fact the cases we will review suggest more complex stability conditions. In particular we will devote chapter 6 to the study of the MMMS [16] case and connections to Bridgeland stability.

Remark 3.12. *Relating back to mirror symmetry, on the symplectic side Thomas in [36] similarly proposes the moment map*

$$\operatorname{Im}\Omega|_L=0 \tag{3.41}$$

and the slope

$$\mu(L)=\frac{\int_L \operatorname{Im}\Omega}{\int_L \operatorname{Re}\Omega}=\frac{1}{\operatorname{vol}(L)}\int_L \operatorname{Im}\Omega. \tag{3.42}$$

As before we stress that this picture is motivational and for more details we refer the reader to Thomas original article.

Chapter 4

Stability of vector bundles

In this section we will introduce (semi)stable vector bundles and some of their important properties. We then proceed to the study of Hermitian-Einstein bundles and give a proof of the first half of the Donaldson-Uhlenbeck-Yau theorem, which is due to Lübke in [2]. The proof is also covered with additional additional details by Kobayashi in [28].

4.1 Introduction to stability of vector bundles

We begin by introducing (slope)stability of holomorphic vectors bundles. To do this we first need to define the slope of a holomorphic vector bundle and more generally over coherent sheafs.

Definition 4.1. *Let $\mathscr{E}$ be a coherent sheaf over a compact Kähler manifold (M, ω). The degree of $\mathscr{E}$ is then defined as*

$$\deg(\mathscr{E}) = \int_M c_1(\mathscr{E}) \wedge \omega^{n-1}, \tag{4.1}$$

where $c_1(\mathscr{E})$ is the first Chern class of $\mathscr{E}$. The slope of $\mathscr{E}$ is then defined as

$$\mu(\mathscr{E}) = \frac{\deg(\mathscr{E})}{\mathrm{rk}(\mathscr{E})}. \tag{4.2}$$

Definition 4.2. *A holomorphic vector bundle $\mathscr{E}$ over M is semistable if for each proper coherent subsheaf $\mathscr{S} \subset \mathscr{E}$*

$$\mu(\mathscr{S}) \leq \mu(\mathscr{E}). \tag{4.3}$$

If the inequality is strict we say that $\mathscr{E}$ is stable. If $\mathscr{E}$ is not semistable we say that $\mathscr{E}$ is unstable.

Remark 4.3. *We have defined stability for a vector bundle since this will be our main interest. However the definition is equally valid for coherent sheafs which we will use in the section on the Harder-Narasimhan filtration.*

This definition of stability of a vector bundle is commonly referred to as slope stability or μ-stability. However we will simply write stable and semistable up until chapter 6 when we introduce other forms of stability. The definition comes from geometric invariant theory and corresponds to if the bundle is a (semi)stable point in the moduli spaces of holomorphic vector bundles. In particular the moduli space of semistable bundles over a projective curve form a (quasi)projective variety.

Remark 4.4. *Every line bundle is stable since it has no proper coherent subsheafs.*

Remark 4.5. *In the case of a bundle over a projective algebraic curve X it is sufficient to consider subbundles since by proposition 2.14 every subsheaf is a subbundle in this case.*

To proceed with the analysis of (semi)stable bundles we will need some properties of the slope μ.

Proposition 4.6. *Let $\mathscr{E}$ and $\mathscr{F}$ be torsion-free coherent sheafs over M. Then*
(i) $\mu(\mathscr{E}\otimes\mathscr{F}) = \mu(\mathscr{E}) + \mu(\mathscr{F})$,
(ii) $\mu(\det\mathscr{E}) = \operatorname{rk}(\mathscr{E})\mu(\mathscr{E})$,
(iii) $\mu(\mathscr{E}^*) = -\mu(\mathscr{E})$.

Proof. We use the properties of c_1 and the rank to get
(i)

$$\frac{c_1(\mathscr{E}\otimes\mathscr{F})}{\operatorname{rk}(\mathscr{E}\otimes\mathscr{F})} = \frac{\operatorname{rk}(\mathscr{F})c_1(\mathscr{E}) + \operatorname{rk}(\mathscr{E})c_1(\mathscr{F})}{\operatorname{rk}(\mathscr{E})\operatorname{rk}(\mathscr{F})} = \frac{c_1(\mathscr{E})}{\operatorname{rk}(\mathscr{E})} + \frac{c_1(\mathscr{F})}{\operatorname{rk}(\mathscr{F})}. \tag{4.4}$$

(ii) Since $c_1(\mathscr{E}) = c_1(\det\mathscr{E})$ and $\operatorname{rk}(\det\mathscr{E}) = 1$ we have

$$\frac{c_1(\det\mathscr{E})}{\operatorname{rk}(\det\mathscr{E})} = \operatorname{rk}(\mathscr{E})\frac{c_1(\mathscr{E})}{\operatorname{rk}(\mathscr{E})}. \tag{4.5}$$

(iii) Finally $\operatorname{rk}(\mathscr{E}^*) = \operatorname{rk}(\mathscr{E})$ and

$$c_1(\mathscr{E}^*) = c_1(\det\mathscr{E}^*) = c_1((\det\mathscr{E})^*) = -c_1(\det\mathscr{E}) = -c_1(\mathscr{E}), \tag{4.6}$$

gives

$$\frac{c_1(\mathscr{E}^*)}{\operatorname{rk}(\mathscr{E}^*)} = -\frac{c_1(\mathscr{E})}{\operatorname{rk}(\mathscr{E})}. \tag{4.7}$$

The desired equalities for the slope then follows by integrating. □

Remark 4.7. *Note that that $\mathscr{E}$ being torsion-free is needed for (iii) in that it guarantees that there is a canonical isomorphism* $\det\mathscr{E}^* = (\det\mathscr{E})^*$.

We will also need to know how the slope μ behaves across short exact sequences.

Proposition 4.8. *If the sequence*

$$0 \to \mathscr{F} \to \mathscr{E} \to \mathscr{G} \to 0 \tag{4.8}$$

is a short exact sequence over coherent sheaves of positive rank then either $\mu(\mathscr{F}) > \mu(\mathscr{E}) > \mu(\mathscr{G})$ or $\mu(\mathscr{F}) < \mu(\mathscr{E}) < \mu(\mathscr{G})$ or the slopes are all equal. In particular if two of the sheaves have the same slope then so does the third.

Proof. We have additivity of c_1 and rank under short exact sequences

$$c_1(\mathscr{E}) = c_1(\mathscr{F}) + c_1(\mathscr{G}), \tag{4.9}$$

$$\operatorname{rk}(\mathscr{E}) = \operatorname{rk}(\mathscr{F}) + \operatorname{rk}(\mathscr{G}). \tag{4.10}$$

Assume that $\mu(\mathscr{E}) < \mu(\mathscr{F})$. We start from the equality

$$\frac{c_1(\mathscr{G})}{\operatorname{rk}(\mathscr{G})} = \frac{c_1(\mathscr{E}) - c_1(\mathscr{F})}{\operatorname{rk}(\mathscr{E}) - \operatorname{rk}(\mathscr{F})} = \frac{c_1(\mathscr{E})(1 - \frac{\operatorname{rk}\mathscr{F}}{\operatorname{rk}\mathscr{E}}) + c_1(\mathscr{E})\frac{\operatorname{rk}(\mathscr{F})}{\operatorname{rk}(\mathscr{E})} - c_1(\mathscr{F})}{\operatorname{rk}(\mathscr{E}) - \operatorname{rk}(\mathscr{F})} = \frac{c_1(\mathscr{E})}{\operatorname{rk}(\mathscr{E})} + \operatorname{rk}(\mathscr{F})\frac{\frac{c_1(\mathscr{E})}{\operatorname{rk}(\mathscr{E})} - \frac{c_1(\mathscr{F})}{\operatorname{rk}(\mathscr{F})}}{\operatorname{rk}(\mathscr{E}) - \operatorname{rk}(\mathscr{F})}. \tag{4.11}$$

Our assumption that $\mu(\mathscr{E}) < \mu(\mathscr{F})$ can be written

$$\frac{c_1(\mathscr{E})}{\operatorname{rk}(\mathscr{E})} < \frac{c_1(\mathscr{F})}{\operatorname{rk}(\mathscr{F})} \tag{4.12}$$

which together with the above equality implies that

$$\frac{c_1(\mathscr{G})}{\operatorname{rk}(\mathscr{G})} - \frac{c_1(\mathscr{E})}{\operatorname{rk}(\mathscr{E})} = \operatorname{rk}(\mathscr{F})\frac{\frac{c_1(\mathscr{E})}{\operatorname{rk}(\mathscr{E})} - \frac{c_1(\mathscr{F})}{\operatorname{rk}(\mathscr{F})}}{\operatorname{rk}(\mathscr{E}) - \operatorname{rk}(\mathscr{F})} < 0, \tag{4.13}$$

since $\operatorname{rk}(\mathscr{E}) > \operatorname{rk}(\mathscr{F})$. Hence it follows that $\mu(\mathscr{G}) < \mu(\mathscr{E}) < \mu(\mathscr{F})$.

We note that if we instead had assumed $\mu(\mathscr{F}) < \mu(\mathscr{E})$ all the inequalities reverse, so this would result in that $\mu(\mathscr{E}) < \mu(\mathscr{G})$. In addition the calculation instead starting with the inequalities $\mu(\mathscr{E}) > \mu(\mathscr{G})$ and $\mu(\mathscr{E}) < \mu(\mathscr{G})$ are completely analogous, which completes the proof. ☐

Proposition 4.9 (Proposition 5.7.9 in [28])**.** *Let $\mathscr{F}$ and $\mathscr{G}$ be torsion-free coherent sheafs over the Kähler manifold (M,ω). Then $\mathscr{F} \oplus \mathscr{G}$ is semistable if and only if $\mathscr{F}$ and $\mathscr{G}$ are semistable with $\mu(\mathscr{F}) = \mu(\mathscr{G})$.*

Proof. Assume that $\mathscr{F}$ and $\mathscr{G}$ are semi-stable with slope $\mu(\mathscr{F}) = \mu(\mathscr{G})$. By proposition 4.8 the exact sequence

$$0 \to \mathscr{F} \to \mathscr{F} \oplus \mathscr{G} \to \mathscr{G} \to 0, \tag{4.14}$$

implies that

$$\mu = \mu(\mathscr{F}) = \mu(\mathscr{G}) = \mu(\mathscr{F} \oplus \mathscr{G}). \tag{4.15}$$

Let $\mathscr{S}$ be a coherent subsheaf of $\mathscr{F} \oplus \mathscr{G}$ and denote the projections on $\mathscr{F}$ and $\mathscr{G}$ by $\mathscr{S}_{\mathscr{F}}$ and $\mathscr{S}_{\mathscr{G}}$ respectively. Then the additivity of the Chern class c_1 and rank under short exact sequences gives

$$\mu(\mathscr{S}) = \frac{\deg(\mathscr{S}_{\mathscr{F}}) + \deg(\mathscr{S}_{\mathscr{G}})}{\operatorname{rk}(\mathscr{S}_{\mathscr{F}}) + \operatorname{rk}(\mathscr{S}_{\mathscr{G}})} \leq \frac{\mu \operatorname{rk}(\mathscr{S}_{\mathscr{F}}) + \mu \operatorname{rk}(\mathscr{S}_{\mathscr{G}})}{\operatorname{rk}(\mathscr{S}_{\mathscr{F}}) + \operatorname{rk}(\mathscr{S}_{\mathscr{G}})} = \mu, \tag{4.16}$$

where we also used the semistable conditions that

$$\deg(\mathscr{S}_{\mathscr{F}}) \leq \mu \operatorname{rk}(\mathscr{S}_{\mathscr{F}}) \quad \text{and} \quad \deg(\mathscr{S}_{\mathscr{G}}) \leq \mu \operatorname{rk}(\mathscr{S}_{\mathscr{G}}). \tag{4.17}$$

Hence $\mathscr{F} \oplus \mathscr{G}$ is semistable.

For the other direction we first assume that $\mathscr{F}$ and $\mathscr{G}$ have different slopes. Without loss of generality let $\mu(\mathscr{F}) < \mu(\mathscr{G})$. Then by proposition 4.8 we have

$$\mu(\mathscr{F}) < \mu(\mathscr{F} \oplus \mathscr{G}) < \mu(\mathscr{G}). \tag{4.18}$$

Thus $\mathscr{F} \oplus \mathscr{G}$ is unstable since $\mathscr{G}$ is a subsheaf of $\mathscr{F} \oplus \mathscr{G}$.

Lastly assume that $\mu(\mathscr{F}) = \mu(\mathscr{G})$ and let one of them be unstable, say without loss of generality $\mathscr{F}$ is unstable. Then there is a coherent subsheaf $\mathscr{S}$ of $\mathscr{F}$ with $\mu(\mathscr{S}) > \mu(\mathscr{F}) = \mu(\mathscr{F} \oplus \mathscr{G})$ making $\mathscr{F} \oplus \mathscr{G}$ unstable. ☐

Remark 4.10. *Note that a bundle $\mathscr{E} = \mathscr{F} \oplus \mathscr{G}$ which is a direct sum of two stable bundles with the same slope is semistable but not stable since $\mathscr{F}$ and $\mathscr{G}$ are subbundles of $\mathscr{E}$.*

4.1.1 The Harder-Narasimhan filtration

We have so far dealt with (semi)stable bundles and noted that in the case of algebraic curves they form a projective variety. It remains the study and classification of unstable bundles. The filtration known as the Harder-Narasimhan filtration provides us with a way to also classify unstable bundles by viewing them as extensions of semistable ones.

Theorem 4.11 (Theorem 5.7.15 in [28]). *Let $\mathscr{E}$ be a torsion-free coherent sheaf over a compact Kähler manifold (M,ω). Then there is a unique filtration by coherent subsheaves*

$$0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_n = \mathscr{E} \tag{4.19}$$

such that $\mathscr{F}_i = \mathscr{E}_i/\mathscr{E}_{i-1}$ are semistable and have decreasing slopes,

$$\mu(\mathscr{F}_1) > \mu(\mathscr{F}_2) > \cdots > \mu(\mathscr{F}_{n-1}) > \mu(\mathscr{F}_n). \tag{4.20}$$

We call this filtration a Harder-Narasimhan filtration of $\mathscr{E}$.

To show the theorem we will rely on

Proposition 4.12 (Lemma 5.7.17 in [28]). *Let $\mathscr{E}$ be a torsion-free coherent sheaf over (M,ω). Then there is a unique coherent subsheaf $\mathscr{E}_1 \subset \mathscr{E}$ with the quotient $\mathscr{E}/\mathscr{E}_1$ being torsion-free and*

$$\begin{aligned} &\mu(\mathscr{F}) \leq \mu(\mathscr{E}_1), \\ &\mathrm{rk}(\mathscr{F}) < \mathrm{rk}(\mathscr{E}_1) \textit{ if } \mu(\mathscr{F}) = \mu(\mathscr{E}), \end{aligned} \tag{4.21}$$

for every coherent subsheaf $\mathscr{F} \subset \mathscr{E}$. Then $\mathscr{E}_1$ is semistable. We call $\mathscr{E}_1$ the maximal semistable/destabilizing subsheaf of $\mathscr{E}$.

The uniqueness in the proposition will follow from

Proposition 4.13 (Proposition 5.7.11 in [28]). *Let $\mathscr{E}$ and $\mathscr{E}'$ be semistable sheafs over a Kähler manifold (M,ω) with $\mu(\mathscr{E}) = \mu(\mathscr{E}')$. Then any non-zero morphism $f : \mathscr{E} \to \mathscr{E}'$ is an isomorphism.*

Proof. The sequence

$$0 \to \mathscr{E} \xrightarrow{f} f(\mathscr{E}) \xrightarrow{i} \mathscr{E}' \to 0 \tag{4.22}$$

is exact since f is surjective onto $f(\mathscr{E})$ and the inclusion map i is injective. Thus $\mathrm{rk}(\mathscr{E}) = \mathrm{rk}(f(\mathscr{E})) = \mathrm{rk}(\mathscr{E}')$ which makes $f : \mathscr{E} \to \mathscr{E}'$ an isomorphism. ☐

And the existence follows from

Lemma 4.14 (Lemma 5.7.16 in [28]). *Let $\mathscr{E}$ be a torsion-free coherent sheaf over (M,ω). Then there exists an integer m such that*

$$\mu(\mathscr{F}) \leq m, \tag{4.23}$$

for all coherent subsheafs $\mathscr{F} \subset \mathscr{E}$.

For a proof of this lemma we refer the reader to [28].

Using proposition 4.12 we can show the existence of a Harder-Narasimhan filtration by letting $\mathscr{F}_i = \mathscr{E}_i/\mathscr{E}_{i-1}$ be the maximal semistable subsheafs of $\mathscr{E}/\mathscr{E}_{i-1}$. First if $\mathscr{E}$ is semistable the filtration is

$$0 = \mathscr{E}_0 \subset \mathscr{E}_1 = \mathscr{E}. \tag{4.24}$$

If $\mathscr{E}$ is unstable we take $\mathscr{E}_1$ to be the maximal semistable subsheaf of $\mathscr{E}$. If $\mathscr{E}/\mathscr{E}_1$ is semistable we are done and have the filtration

$$0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \mathscr{E}_2 = \mathscr{E}. \tag{4.25}$$

Otherwise take $\mathscr{E}_2/\mathscr{E}_1$ to be the maximally semistable subsheaf of $\mathscr{E}/\mathscr{E}_1$.

Remark 4.15. *Note that any subsheaf of a quotient $\mathscr{E}/\mathscr{E}_1$ is of the form $\mathscr{E}_2/\mathscr{E}_1$ for some subsheaf $\mathscr{E}_2 \subset \mathscr{E}$.*

We then proceed letting $\mathscr{F}_i = \mathscr{E}_i/\mathscr{E}_{i-1}$ be the maximal semistable subsheaf of $\mathscr{E}/\mathscr{E}_{i-1}$ until we reach an integer $i = n$ where $\mathscr{E}/\mathscr{E}_{n-1}$ is semistable thus giving us the filtration

$$0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_{n-1} \subset \mathscr{E}_n = \mathscr{E}. \tag{4.26}$$

Note that the sequence terminates since $\mathrm{rk}(\mathscr{E}_i) < \mathrm{rk}(\mathscr{E}_{i+1})$. We shall now show that $\mathscr{F}_i$ satisfies the descending slope condition. By the third isomorphism theorem we have that

$$\frac{(\mathscr{E}_{i+1}/\mathscr{E}_{i-1})}{(\mathscr{E}_i/\mathscr{E}_{i-1})} \cong \frac{\mathscr{E}_{i+1}}{\mathscr{E}_i}, \tag{4.27}$$

which tells us that the sequence

$$0 \to \mathscr{F}_i = \frac{\mathscr{E}_i}{\mathscr{E}_{i-1}} \to \frac{\mathscr{E}_{i+1}}{\mathscr{E}_{i-1}} \to \mathscr{F}_{i+1} = \frac{\mathscr{E}_{i+1}}{\mathscr{E}_i} \to 0, \tag{4.28}$$

is exact. Then since $\mathscr{E}_{i+1}/\mathscr{E}_{i-1}$ is subsheaf of $\mathscr{E}/\mathscr{E}_{i-1}$ and $\mathscr{F}_i$ the maximal subsheaf we have that

$$\mu(\mathscr{F}_i) > \mu(\mathscr{E}_{i+1}/\mathscr{E}_{i-1}) > \mathscr{F}_{i+1}. \tag{4.29}$$

Thus the sequence of $\mathscr{F}_i$ is decreasing in slope verifying that we have constructed a Harder-Narasimhan filtration.

It remains to show uniqueness. Let

$$0 = \mathscr{E}'_0 \to \mathscr{E}'_1 \to \cdots \to \mathscr{E}'_{n-1} \to \mathscr{E}'_n = \mathscr{E} \tag{4.30}$$

also be a filtration of $\mathscr{E}$ with quotients $\mathscr{F}'_i = \mathscr{E}'_i/\mathscr{E}'_{i-1}$. We have $\mu(\mathscr{E}'_1) \leq \mu(\mathscr{E}_1)$. Let i be the least integer such that $\mathscr{E}_1 \subset \mathscr{E}'_i$. We have a non-zero morphism

$$\mathscr{E}_1 \to \mathscr{E}'_i \to \frac{\mathscr{E}'_i}{\mathscr{E}'_{i-1}} = \mathscr{F}'_i, \tag{4.31}$$

meaning $\mathscr{E}_1 \cong \mathscr{F}'_i$. Thus $\mathscr{F}'_i$ is maximal and since the slopes decrease with i we have that $i = 1$. Hence $\mathscr{E}_1 \cong \mathscr{E}'_1$. The result now follows by induction, $\mathscr{F}_2$ and $\mathscr{F}'_2$ are semistable subsheaves of $\mathscr{E}/\mathscr{E}_1$ so the same argument show that $\mathscr{F}_2 \cong \mathscr{F}'_2$ etc.

4.2 Stability of Hermitian-Einstein vector bundles

In section 3.2 we introduced the concept of a Hermitian-Einstein connection. We will now see that the bundles that admit such connections can be related to those that are stable. That is we will show the first half of the Donaldson-Uhlenbeck theorem, namely

Theorem 4.16 (Proposition 4 in [2] or Theorem 5.8.3 in [28])**.** *Let E be an Hermitian-Einstein bundle over a compact Kähler manifold (M, ω). Then E is semistable and a direct sum of stable Hermitian-Einstein subbundles with the same slope as that of E.*

Let us first note that in the case of line bundles we already have both directions of the Donaldson-Uhlenbeck theorem as in theorem 1.1.

Proposition 4.17. *Every Hermitian line bundle is stable and admits a Hermitian-Einstein metric.*

Proof. We know that every line bundle is stable since it has no proper subsheafs. Also the bundle is Hermitian-Einstein since tracing over the $U(1)$-bundle does nothing and the curvature form can be written $F = \bar{\partial} A = \varphi I$ for a suitable choice of φ. □

Proposition 4.18 (Corollary 4.1.6 in [28])**.** *Let $E \to M$ be a Hermitian-Einstein bundle with factor λ. Then*

$$\mu(\wedge^s E) = s\mu(E), \tag{4.32}$$

for $s \leq \operatorname{rk}(E)$.

Proof. By proposition 4.6 we know that the slope over tensor products is additive. Since E is Hermitian-Einstein this implies that $\wedge^s E$ is Hermitian-Einstein with factor $s\lambda$. Since the Hermitian-Einstein factor is proportional to the slope the proposition follows. To be more explicit the Einstein factors expressed in terms of the slopes are

$$\lambda = \frac{\pi n}{\operatorname{vol} M} \mu(E), \tag{4.33}$$

and

$$s\lambda = \frac{\pi n}{\operatorname{vol} M} \mu(\wedge^s E). \tag{4.34}$$

□

We have seen in proposition 4.9 that the direct sum of two semistable bundles is semistable if and only if they have share the same slope. With the correspondence between Hermitian-Einstein bundles and semistable bundles, by the Donaldson-Uhlenbeck-Yau theorem as stated in theorem 1.1, there should then be a corresponding statement for the direct sum of Hermitian-Einstein bundles.

Proposition 4.19 (Proposition 4.1.4 in [28])**.** *The direct sum $E = E_1 \oplus E_2$ is Hermitian-Einstein with constant λ if and only if E_1 and E_2 is Hermitian-Einstein with constant λ.*

Proof. Let F_1 and F_2 be the curvatures on E_1 and E_2 respectively. By the definition of the trace operator we have that

$$\Lambda(F_1 \oplus F_2)\omega^n = (F_1 \oplus F_2) \wedge \omega^{n-1} = F_1 \wedge \omega^{n-1} \oplus F_2 \wedge \omega^{n-1} = (\Lambda F_1)\omega^n \oplus (\Lambda F_2)\omega^n, \tag{4.35}$$

that is the trace on the curvature commutes with the direct sum

$$\Lambda(F_1 \oplus F_2) = \Lambda F_1 \oplus \Lambda F_2. \tag{4.36}$$

The requirement for $E_1 \oplus E_2$ to be Hermitian-Einstein is that

$$\Lambda(F_1 \oplus F_2) = \lambda I_{E_1 \oplus E_2} = \lambda I_{E_1} \oplus I_{E_2}, \tag{4.37}$$

for some constant λ. This is the case if and only if E_1 and E_2 are Hermitian-Einstein with constant λ. □

4.2.1 Over Riemann surfaces

We will now proceed with the proof of theorem 4.16. Since the case with M being a Riemann surface is simpler to show we begin working up to that case. However since the results will be also needed for the general case, we will work over an arbitrary Kähler manifold M on our way there.

First we will need to know how the curvature of a holomorphic vector bundle (E,h) is related to that of subbundle (S,h_S). Knowing this we will also know how the first Chern class c_1 are related. Denote the quotient bundle $Q = E/S$. Then we have the exact sequence

$$0 \to S \to E \to Q \to 0. \tag{4.38}$$

and we can write $E = S \oplus S^\perp$. The bundle $S^\perp$ is isomorphic to (Q,h_Q) as a complex vector bundle, however they need not be biholomorphic so they may not be isomorphic viewed as holomorphic bundles. We will denote the curvatures by F,F_S,F_Q and the corresponding connection forms D,D_S,D_Q.

It will be useful of for us to work in a coordinate system. Thus set $r = \text{rank}(E)$ and $s = \text{rank}(S)$ and let $e_1, \dots e_r$ be an local unitary frame for E such that $e_1, \dots ,e_s$ is a local frame for S. Unitary frame here means that it is orthonormal with respect to the Hermitian metric h.

Acting with the connection D on sections of respective bundle we have

$$\begin{aligned} D\xi &= D_S\xi + A\xi, \quad \xi \in \Omega^0(S) \\ D\eta &= D_Q\eta + B\eta, \quad \eta \in \Omega^0(S^\perp), \end{aligned} \tag{4.39}$$

with $A \in \Omega^{1,0}(\text{Hom}(S,S^\perp)$ or equivalently by identifying $S^\perp$ with Q we have $A \in \Omega^{1,0}(\text{Hom}(S,Q))$. A is known as the second fundamental form of the submanifold (S,h_s). When $A = 0$ we say that the connection D_S is projectively flat. For a proof that this decomposition is valid see [28].

In particular we have for

$$\begin{aligned} De_S &= D_Se_S + Ae_Q, \\ De_Q &= D_Qe_Q + Be_S, \end{aligned} \tag{4.40}$$

for basis elements e_S and e_Q in the previous introduced frames. We can now conclude that $B = -A^*$ since

$$0 = Dh(e_S,e_Q) = h(De_S,e_Q) + h(e_S,De_Q) = h(D_Se_S + Ae_Q,e_Q) + h(e_S,D_Qe_Q + Be_S) = A + B^*, \tag{4.41}$$

which gives $B \in \Omega^{0,1}(\text{Hom}(Q,S))$. In matrix notation eq. (4.39) can be stated as

$$D = \begin{bmatrix} D_S & -A^* \\ A & D_Q \end{bmatrix}, \tag{4.42}$$

and the curvatures are then related by

$$F = D^2 = \begin{bmatrix} D_S^2 - A^* \wedge A & -D_SA^* \\ D_QA & D_Q^2 - A \wedge A^* \end{bmatrix} = \begin{bmatrix} F_S - A^* \wedge A & -\partial_SA^* \\ \bar\partial A & F_Q - A \wedge A^* \end{bmatrix}, \tag{4.43}$$

where we kept only (1,1)-forms since we know that F is a (1,1)-form. Alternatively expressed in terms of $B = -A^*$,

$$F = \begin{bmatrix} F_S - B \wedge B^* & \partial_SB \\ -\bar\partial B^* & F_Q - B^* \wedge B \end{bmatrix}. \tag{4.44}$$

Proposition 4.20 (Proposition 1.4.18 in [28])**.** *Let E be a holomorphic bundle with Hermitian connection D and S a C^∞ subbundle of E. If S is invariant under D then S and $S^\perp$ are both holomorphic subbundles of E. In particular if S is projectively flat, $S^\perp$ is a holomorphic subbundle of E.*

Proof. Let s be a local holomorphic section and $s = s' + s''$ the decomposition into $S \oplus S^\perp$. Invariance under E means that $Ds' = D_S s'$, which is the case when $A = 0$, and thus as a consequence $Ds'' = D_{S^\perp} s''$. Hence Ds' and Ds'' are both (1,0)-forms which is the requirement for S and $S^\perp$ to be holomorphic respectively. $\square$

Proposition 4.21 (Proposition 1 in [2] or Proposition 5.2.3 in [28])**.** *Let (E,h) be a Hermitian-Einstein vector bundle over a compact Kähler manifold, and let*

$$0 \to E' \to E \to E'' \to 0, \tag{4.45}$$

be an exact sequence over Holomorphic vector bundles. Then

$$\mu(E') \le \mu(E). \tag{4.46}$$

Additionally if equality holds the sequence splits.

Proof. Set $r = \operatorname{rk} E$, $s = \operatorname{rk} E'$, $p = r - s$ and let $A = (A_{ab})$ be the (s,p)-dimensional matrix of the second fundamental form of E' with respect to the previously introduced unitary local frame. The adjoint matrix is then the (p,s)-dimensional matrix $A^* = (A^*_{ab}) = (\bar{A}_{ba})$. The curvature forms of (E,h) and (E',h') are then related in component form by

$$F'_{ab} = F_{ab} + \sum_{\lambda=1}^{p} A^*_{a\lambda} \wedge A_{\lambda b} = F_{ab} + \sum_{\lambda=1}^{p} \bar{A}_{\lambda a} \wedge A_{\lambda b} = F_{ab} - \sum_{\lambda=1}^{p} A_{\lambda b} \wedge \bar{A}_{\lambda a}, \qquad a,b \le s. \tag{4.47}$$

The Einstein condition gives that in our unitary frame $F_{aa} = \alpha$ for some (1,1)-form α. The first Chern classes are then

$$c_1(E) = \frac{i}{2\pi} \sum_{\lambda=1}^{r} F_{\lambda\lambda} = \frac{i}{2\pi} r\alpha \tag{4.48}$$

and

$$c_1(E') = \frac{i}{2\pi} \sum_{a=1}^{s} F'_{aa} = \frac{i}{2\pi} \sum_{a=1}^{s} F_{aa} - \frac{i}{2\pi} \sum_{a=1}^{s} \sum_{\lambda=1}^{p} A_{\lambda a} \wedge \bar{A}_{\lambda a} = \frac{i}{2\pi} s\alpha - \frac{i}{2\pi} \sum_{a=1}^{s} \sum_{\lambda=1}^{p} A_{\lambda a} \wedge \bar{A}_{\lambda a}. \tag{4.49}$$

Dividing by the ranks r and s respectively and integrating we have

$$\mu(E) = \frac{i}{2\pi} \int_M \alpha \wedge \omega^{n-1} \tag{4.50}$$

and

$$\mu(E') = \frac{i}{2\pi} \int_M \alpha \wedge \omega^{n-1} - \frac{i}{2\pi p} \int_M \sum_{a=1}^{1} \sum_{\lambda=1}^{p} A_{\lambda a} \wedge \bar{A}_{\lambda a} \wedge \omega^{n-1}. \tag{4.51}$$

Since $iA_{\lambda a} \wedge \bar{A}_{\lambda a}$ is positive and ω positive definite we conclude that

$$\mu(E') \le \mu(E), \tag{4.52}$$

with equality only when $A = 0$ i.e. when the connection is projectively flat. In this case $E'^\perp \cong E''$ by proposition 4.20 making the sequence split $E = E' \oplus E''$. . $\square$

Remark 4.22. *If we drop the requirement of compactness, we still have the inequality*

$$\frac{c_1(E') \wedge \omega^{n-1}}{\mathrm{rk}(E')} \leq \frac{c_1(E) \wedge \omega^{n-1}}{\mathrm{rk}(E)}, \tag{4.53}$$

with equality only when $A = 0$. The difference is thus only that the integration may not be finite, which could be dealt with by defining the degree integral on a maximal compact submanifold.

Corollary 4.23. *Let E be a Hermitian-Einstein bundle over a projective algebraic curve X, i.e. over a compact Riemann surface. Then E is semistable and a direct sum of stable Hermitian-Einstein subbundles with the same slope as E.*

Proof. Assume that E is not stable. Then there exists a proper subbundle E' of E with $\mu(E') \geq \mu(E)$. The short exact sequence

$$0 \to E' \to E \to E/E' \to 0, \tag{4.54}$$

implies by the previous proposition that $\mu(E') \leq \mu(E)$ and hence $\mu(E') = \mu(E)$ and that the sequence splits, $E = E' \oplus E/E'$. We also have by proposition 4.8 that E/E' has the same slope as E and E'.

The statement now follows by induction and proposition 4.17. That is assume the corollary is true for bundles of rank less than r, then the above shows it is true for a bundle $\mathscr{E}$ of rank r. □

4.2.2 Over Kähler manifolds

Having now shown that an irreducible Hermitian-Einstein bundle over a complex algebraic curve is stable we can proceed with the general case of Kähler manifolds in higher dimension. The difference from the case of a Riemann surface will be that the subsheafs are not necessarily locally free, i.e. vector bundles, so we will have to deal with the presence of singularity sets.

Our proof will rely on

Proposition 4.24 ([4] Proposition 3b)**.** *If a holomorphic bundle $\mathscr{E}$ over X admits a Hermitian-Einstein connection and $\mu(\mathscr{E}) < 0$ then $\mathscr{E}$ has no non-trivial global holomorphic sections. If $\mu(\mathscr{E}) = 0$, any holomorphic section is covariantly constant.*

Proof. The Laplacians are related by

$$\Delta_{\bar{\partial}} = \frac{1}{2}(\Delta - \lambda . I). \tag{4.55}$$

For $\mu(\mathscr{E}) < 0$ we have $\lambda < 0$ making $\Delta_{\bar{\partial}}$ into a strictly positive operator, that is $\Delta_{\bar{\partial}} s > 0$ for all non-trivial sections s of $\mathscr{E}$. Hence $\mathscr{E}$ does not admit any non-trivial global holomorphic sections since a holomorphic section is one with $\Delta_{\bar{\partial}} s = 0$. We also see that when $\mu(\mathscr{E}) = 0$ we have for any holomorphic section s that

$$\Delta s = \Delta_{\bar{\partial}} s = 0, \tag{4.56}$$

i.e. s is covariantly constant. □

We are now ready to proceed with the proof of that Hermitian-Einstein bundles are semistable.

Proof of theorem 4.16

We begin by showing that $\mathscr{E}$ is semistable. Let $\mathscr{F} \subset \mathscr{E}$ be a reflexive subsheaf of rank s such that $0 < s < r = \mathrm{rk}(\mathscr{E})$. The inclusion map

$$j : \mathscr{F} \to \mathscr{E} \tag{4.57}$$

induces by proposition 2.15 a monomorphism

$$\det(j) : \det \mathscr{F} = (\wedge^s \mathscr{F})^{**} \to (\wedge^s \mathscr{E})^{**} = \wedge^s \mathscr{E}, \tag{4.58}$$

where we used that $\mathscr{F}$ and $\mathscr{E}$ are reflexive.

Reminding ourselves that $\det \mathscr{F} \otimes (\det \mathscr{F})^* \cong \mathcal{O}_M$, since the determinant bundles are line bundles over M, we take the tensor product of the above morphism with $(\det \mathscr{F})^*$. We then obtain a non-trivial global holomorphic section

$$f : \mathcal{O}_M \to \wedge^s \mathscr{E} \otimes (\det \mathscr{F})^*, \tag{4.59}$$

on the $\mathscr{E} \otimes (\det \mathscr{F})^*$ bundle.

We have by proposition 4.6 that

$$\mu(\wedge^s \mathscr{E} \otimes (\det \mathscr{F})^*)) = \mu(\wedge^s \mathscr{E}) - \mu(\det \mathscr{F}) = s(\mu(\mathscr{E}) - \mu(\mathscr{F})). \tag{4.60}$$

Since we have a section on $\wedge^s \mathscr{E} \otimes (\det \mathscr{F})^*$ we have by proposition 4.24 that

$$\mu(\wedge^s \mathscr{E} \otimes (\det \mathscr{F})^*) \geq 0 \Rightarrow \mu(\mathscr{F}) \leq \mu(\mathscr{E}), \tag{4.61}$$

i.e. $\mathscr{E}$ is semistable.

For the second part of the theorem, stating that $\mathscr{E}$ decomposes into a direct sum of stable bundles, assume that $\mu(\mathscr{F}) = \mu(\mathscr{E})$. Set $\mathscr{G} = \mathscr{E}/\mathscr{F}$. We shall show that the exact sequence

$$0 \to \mathscr{F} \xrightarrow{i} \mathscr{E} \xrightarrow{j} \mathscr{G} \to 0 \tag{4.62}$$

splits.

Denote the singularity set of $\mathscr{F}$ by $S(\mathscr{F})$ and set $\tilde{M} = M - S(\mathscr{F})$. Consider the exact sequence restricted to $\tilde{M}$

$$0 \to \mathscr{F}|_{\tilde{M}} \xrightarrow{i} \mathscr{E}|_{\tilde{M}} \xrightarrow{j} \mathscr{G}|_{\tilde{M}} \to 0, \tag{4.63}$$

making $\mathscr{F}|_{\tilde{M}}$ and $\mathscr{G}_{\tilde{M}}$ into vector bundles. Since this is a sequence over vector bundles with the same slope we have that the sequence splits by proposition 4.21 i.e.

$$\mathscr{E}|_{\tilde{M}} = \mathscr{F}|_{\tilde{M}} \oplus \mathscr{G}|_{\tilde{M}}. \tag{4.64}$$

By the splitting lemma there then exists a morphism $\phi : \mathscr{E}_{\tilde{M}} \to \mathscr{F}_{\tilde{M}}$ such that $\phi \circ i = \mathrm{d}_{\mathscr{F}|_{\tilde{M}}}$. Since $\tilde{M}$ is analytic and $\operatorname{codim} S \geq 3$ we have that the restriction map

$$\mathscr{F} \to \mathscr{F}|_{\tilde{M}} \tag{4.65}$$

is an isomorphism. Hence the splitting morphism extends uniquely to a splitting morphism

$$\phi' : \mathscr{F} \to \mathscr{E}, \tag{4.66}$$

with $\phi' \circ i = \mathrm{d}_{\mathscr{F}}$. Then again by the splitting lemma the sequence splits and

$$\mathscr{E} = \mathscr{F} \oplus \mathscr{G}. \tag{4.67}$$

The decomposition of $\mathscr{E}$ into a finite direct sum of stable bundles now follows by induction and proposition 4.17 the same way as in the proof of corollary 4.23.

Chapter 5

The Donaldson-Uhlenbeck-Yau theorem

In chapter 4 we showed that that the existence of a irreducible Hermitian-Einstein connection implied stability. This section will be devoted to the converse statement, that a stable bundle admits an Hermitian-Einstein connection. This will give us the full Donaldson-Uhlenbeck-Yau theorem as stated in the introduction which we repeat here for the convenience of the reader.

Theorem 5.1 (The Donaldson-Uhlenbeck-Yau theorem)**.** *A holomorphic vector bundle E over a compact Kähler manifold M is stable if and only if E admits an irreducible Hermitian-Einstein connection. This connection is then unique.*

The implication is that there is a one-to-one correspondence between semistable bundles and the space of Hermitian-Einstein connections since a semistable bundle is a direct sum of stable bundles by corollary 4.23. A proof of theorem 1.1 is provided in [7] by Uhlenbeck and Yau and a detailed discussion can also be found in [37]. Donaldson had previously shown the case when M is an algebraic surface in [4]. The case when M is an algebraic curve, i.e. a Riemannian surface, is known as the Narasimhan–Seshadri theorem and is discussed extensively in [3]. There is also an alternative proof by Donaldson in [38] that is also covered in [39] which more closely resembles his approach for algebraic surfaces. For the interested we would also like to direct the reader to the survey article [40] by Jun Li.

We will now turn our attention to the case when M is an algebraic surface which is the version we will prove in this section. To do this we will follow Donaldson's heat equation approach to the proof which is also covered by Kobayashi in [28] and by Siu in [39]. The approach to the proof is to use the evolution equation

$$\frac{\partial H_t}{\partial t} = -H_t(\Lambda F_t - \lambda I), \tag{5.1}$$

which we introduced in section 3.2. The idea will be to start at some Hermitian metric H_0 and show convergence to an Hermitian-Einstein metric.

Our eventual goal is to show

Proposition 5.2 (Proposition 21 in [4])**.** *Let $\mathscr{E}$ be a semi-stable bundle over an projective algebraic surface X. Then there is a holomorphic bundle $\mathscr{E}'$ with the same rank and degree as $\mathscr{E}$ such that*

(i) $\mathscr{E}'$ admits a Hermitian-Einstein connection
(ii) There is a non-zero holomorphic map $\alpha : \mathscr{E} \to \mathscr{E}'$.

This implies the theorem 5.1 in the case of an algebraic surface since by proposition 4.13 a morphism between two semistable bundles is an isomorphism and by theorem 4.16 $\mathscr{E}'$ is semi-stable. Thus $\mathscr{E}$ and $\mathscr{E}'$ are isomorphic which implies that $\mathscr{E}$ is Hermitian-Einstein.

5.1 Existence of a solution to the evolution equation

We will in section show that the evolution equation has a unique smooth solution H_t for all time. To do this we will follow Donaldson's heat equation approach, which will allow us to make use of the theory of parabolic PDEs. Equivalent to the evolution equation

$$\frac{\partial H_t}{\partial t} = -H_t(\Lambda F_t - \lambda I), \tag{5.2}$$

we have the Yang-Mills flow which we introduced in chapter 3 as

$$\frac{\partial}{\partial t} A = -\,\mathrm{d}_A^* F_A. \tag{5.3}$$

We then showed that the curvature F_A for a path down Yang-Mills flow satisfies the heat equation,

$$\frac{\partial}{\partial t} F_A = -\Delta_A F_A. \tag{5.4}$$

We remind ourselves that the curvature of two Hermitian metrics H and K are related by

$$F_K = F_H + \bar{\partial}_H (h^{-1}\partial_H h) = F_H + h^{-1}\left(\bar{\partial}_H \partial_H h - (\bar{\partial}_H h) h^{-1} (\partial_H h)\right), \tag{5.5}$$

setting $h = H^{-1}K$. Then using the above relation of the curvatures we can express the evolution equation equivalently in terms of initial Hermitian metric H_0 with $h = H_0^{-1} H_t$ as

$$\begin{cases} \frac{\partial h}{\partial t} = -\left(\Delta_0 h + h F_0\right), \\ h(0) = 1. \end{cases} \tag{5.6}$$

Equivalently

$$\frac{\partial h_t}{\partial t} = -h\left(\hat{F}_0 - \lambda + \Lambda \bar{\partial}_0 (h_t^{-1})(\partial_0 h_t)\right). \tag{5.7}$$

Since the evolution equation can equivalently be expressed as a heat equation it is known that we have short time existence.

Proposition 5.3. *For sufficient small $\epsilon > 0$ the evolution equation*

$$\frac{\partial H_t}{\partial t} = -H_t(\hat{F}_{H_t} - \lambda I), \tag{5.8}$$

have a smooth solution H_t for time t in $[0,\epsilon)$.

For a proof that the heat equation on Riemannian manifolds have a smooth short time solution we refer the reader to §11 in [41] by Hamilton.

To introduce a notion of distance between two metrics define

$$\begin{aligned} \tau(H,K) &= \operatorname{tr}(H^{-1}K), \\ \sigma(H,K) &= \tau(H,K) + \tau(K,H) - 2\operatorname{rk} E. \end{aligned} \tag{5.9}$$

We have that $\sigma(H,K) \geq 0$ with equality only when $H = K$ since H and K are positive definite and Hermitian implies

$$\operatorname{tr}((H^{-1}K)^{-1} + H^{-1}K) \geq 2\operatorname{tr}(I) = 2\operatorname{rk} E, \tag{5.10}$$

similar to how for any number $\lambda > 0$, we have $\lambda + \lambda^{-1} \geq 2$.

Proposition 5.4 (Proposition 13 in [4])**.** *Let H_t and K_t be two be solutions to the evolution equation. Then for $\sigma = \sigma(H_t,K_t)$ we have the inequality*

$$\left(\frac{\partial}{\partial t} + \Delta\right)\sigma \leq 0. \tag{5.11}$$

Proof. We are done if we show that τ satisfy the same inequality, that is

$$\left(\frac{\partial}{\partial t} + \Delta\right)\tau \leq 0. \tag{5.12}$$

We have

$$\frac{\partial \tau}{\partial t} = \operatorname{tr}\left(\frac{\partial H^{-1}}{\partial t}K + H^{-1}\frac{\partial K}{\partial t}\right) = \left(-H^{-1}\frac{\partial H}{\partial t}H^{-1}K + H^{-1}\frac{\partial K}{\partial t}\right). \tag{5.13}$$

Inserting the time evolution of H and K by eq. (5.1) gives

$$\frac{\partial \tau}{\partial t} = \operatorname{tr}\left(H^{-1}H(\hat{F}_H - \lambda I)H^{-1}K - H^{-1}K(\hat{F}_K - \lambda I)\right) = \operatorname{tr}\left(h(\hat{F}_H - \hat{F}_K\right), \tag{5.14}$$

setting $h = H^{-1}K$ and where we used the cyclicity of the trace. The curvatures are related by

$$F_K = F_H + h^{-1}\left(\bar{\partial}_H\partial_H h - (\bar{\partial}_H h)h^{-1}(\partial_H h)\right), \tag{5.15}$$

which gives

$$\frac{\partial \tau}{\partial t} = \operatorname{tr}\left(h\Lambda\left(-h^{-1}\left(\bar{\partial}_H\partial_H h + (\bar{\partial}_H h)h^{-1}(\partial_H h)\right)\right)\right) = -\operatorname{tr}(\bar{\partial}_H\partial_H h) + \Lambda\operatorname{tr}\left((\bar{\partial}_H h)h^{-1}(\partial_H h)\right). \tag{5.16}$$

Then since $\Delta h = \operatorname{tr}(\bar{\partial}_H\partial_H h)$,

$$\frac{\partial \tau}{\partial t} + \Delta\tau = \Lambda\operatorname{tr}\left((\bar{\partial}_H h)h^{-1}(\partial_H h)\right) \leq 0. \tag{5.17}$$

□

We can now apply the maximum principle of the heat equation to show uniqueness and short time convergence. We will use the version that is stated and proved in [28],

Theorem 5.5 (Maximum principle for the parabolic PDE)**.** *Let M be a compact Riemannian manifold and $f : M \times [0,T)$ a smooth function satisfying*

$$\left(\frac{\partial}{\partial t} + c\Delta\right) f \leq 0. \tag{5.18}$$

Then $\sup_{x \in M} f(x,t)$ is decreasing.

Applying the maximum principle to proposition 5.4 we have that $\sup_M \sigma(H,K)$ is decreasing in time. However we also know that $\sigma(H,K)$ is positive and that if $H_0 = K_0$ then $\sigma(H_0,K_0) = 0$. Thus starting at two metrics equal at $t = 0$ we have that $\sigma(H_t,K_t) = 0$ which gives

Corollary 5.6 (Corollary 14 in [4])**.** *Let H_t and K_t be smooth solutions to the evolution equation defined for $0 \leq t < T$ with the same initial conditions $H_0 = K_0$. Then H_t and K_t agree for all time t in $[0,T)$.*

We can also show C^0 convergence for any finite time T.

Proposition 5.7 (Corollary 15 in [4])**.** *Suppose H_t is a smooth solution to the evolution equation for $0 \leq t < T$. Then H_t converges in C^0 to some continuous metric H_T as $t \to T$.*

Proof. We show that H_t is uniformly Cauchy, that is for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\sup_M \sigma(H_t,H_{t'}) < \epsilon \text{ for all } t,t' > T - \delta. \tag{5.19}$$

By continuity at $t = 0$ there exists a $\delta > 0$ such that

$$\sup_M \sigma(H_t,H_{t'}) < \epsilon \text{ for all } t,t' < \delta. \tag{5.20}$$

But since we have shown that $\sigma(H_t,H_{t+\tau})$ is a decreasing function of t we know that

$$\sup_M \sigma(H_t,H_{t'}) < \epsilon, \tag{5.21}$$

for the interval $(T - \delta,T)$. C^0 convergence follows since we can identity the space of Hermitian metrics with $GL(k,\mathbb{C})/U(k)$ which is complete. □

Our next task is to show the existence of a unique smooth solution to the evolution equation for any time. For convenience set

$$e = |F_A|^2, \quad \hat{e} = |\Lambda F_A|^2, \quad e_k = |\nabla_A^k F_A|^2, \tag{5.22}$$

for $k \geq 0$ where ∇_A^k is the k times iterated covariant derivative. We will show that these functions are bounded under Yang-Mills flow using the maximum principle for parabolic PDE. To do this we first need to find inequalities for these functions under the heat operator

$$\frac{\partial}{\partial t} + \Delta. \tag{5.23}$$

We have so far worked with the Hodge Laplacian Δ_A. One can also form the Bochner Laplacian $\nabla_A^* \nabla_A$ from the connection. Note that ∇_A^* here is the formal adjoint of ∇_A, that is with respect to the L^2 inner product and not the inner product on $\mathfrak{g}$. When acting on 0-forms, $\Omega^0(\mathfrak{g})$ the Bochner Laplacian coincide with the Hodge Laplacian whereas when acting on $\alpha \in \Omega^2(\mathfrak{g})$ the Laplacians are related by the Weitzenböck identity

$$\Delta_A \alpha = \nabla_A^* \nabla_A \alpha + \{F_A,\alpha\} + \{R,\alpha\}. \tag{5.24}$$

R here denotes the Riemann curvature of M and $\{F_A,\alpha\}$ and $\{R,\alpha\}$ are some multilinear expressions. The exact expression of these terms is not important to us since we are only interested in finding a bound in terms of $|\alpha|$. We begin by showing

Proposition 5.8 (Proposition 16 in [4]). *Let $A = A_t$ be gauge-equivalent to a path satisfying Yang-Mills flow. Then*

$$\begin{aligned} (i)\ &\left(\frac{\partial}{\partial t} + \Delta\right) \operatorname{tr}(F_A) = 0, \\ (ii)\ &\left(\frac{\partial}{\partial t} + \Delta\right) e \leq c\left(e^{3/2} + e\right), \\ (iii)\ &\left(\frac{\partial}{\partial t} + \Delta\right) \hat{e} \leq 0, \\ (iv)\ &\left(\frac{\partial}{\partial t} + \Delta\right) e_k \leq c_k e_k^{1/2} \sum_{i+j=k} e_i^{1/2}(e_j^{1/2} + 1), \end{aligned} \tag{5.25}$$

for some constants c and c_k. Note that (ii) is a special case of (iv) with $k = 0$.

Proof. Taking the trace of eq. (5.4) we get the first identity. For the other inequalities we use that for $\alpha \in \Omega^*(\mathfrak{g})$ the Laplacian acts as

$$\Delta|\alpha|^2 = 2\langle \nabla_A^* \nabla_A \alpha,\alpha\rangle - 2|\nabla_A \alpha|^2. \tag{5.26}$$

In particular

$$-2\langle \nabla_A^* \nabla_A \alpha,\alpha\rangle \leq \Delta|\alpha|^2. \tag{5.27}$$

By the evolution equation and he Weitzenböck identity then have

$$\begin{aligned} \frac{\partial e}{\partial t} = 2\langle \frac{\partial F_A}{\partial t}, F_A\rangle = -2\langle \Delta_A F_A,F_A\rangle = -2\langle \nabla_A^* \nabla_A F_A,F_A\rangle - 2\langle \{F_A,F_A\},F_A\rangle - 2\langle \{R,F_A\},F_A\rangle = \\ - 2\langle \Delta_A F_A,F_A\rangle + \{F_A,F_A,F_A\} + \{R,F_A,F_A\} \leq -\Delta|F_A|^2 + \{F_A,F_A,F_A\} + \{R,F_A,F_A\}, \end{aligned} \tag{5.28}$$

where again $\{F_A,F_A,F_A\}$ and $\{R,F_A,F_A\}$ are some multilinear expressions. Since they are multilinear they are bounded by $ce^{3/2} = c|F_A|^3$ and $ce = c|F_A|^2$ respectively for some sufficient large constant c. Thus we have shown (ii).

For (*iii*) we use that the corresponding Yang-Mills flow for $\hat{F}_A$ is given by

$$\frac{\partial}{\partial t}\hat{F}_A = -\Delta_A \hat{F}_A = -\nabla_A^* \nabla_A \hat{F}_A, \tag{5.29}$$

where the last identity follows from that $\hat{F}_A \in \Omega^0(\mathfrak{g})$. Thus

$$\frac{\partial}{\partial t}\hat{e} = 2\left(\frac{\partial \hat{F}}{\partial t},\hat{F}\right) = -2(\nabla_A^* \nabla_A \hat{F}_A, F_A) = -\Delta|\hat{F}_A|^2 - 2|\nabla_A \hat{F}_A| \leq -\Delta\hat{e}. \tag{5.30}$$

For (iv) we first need to know how the time derivative act on the covariant derivative ∇_A,

$$\frac{\partial}{\partial t}\nabla_A = \nabla_A \frac{\partial}{\partial t} + \left\{ \frac{\partial A}{\partial t} \right\} = \nabla_A \frac{\partial}{\partial t} + \{\mathrm{d}_A^* F_A\} = \nabla_A \frac{\partial}{\partial t} + \{\nabla_A F_A\}\,, \tag{5.31}$$

and thus

$$\frac{\partial}{\partial t}\nabla_A^k = \nabla_A^k \frac{\partial}{\partial t} + \sum_{i+j=k-1} \nabla_A^i \left\{ \nabla_A F_A, \nabla_A^j F_A \right\}. \tag{5.32}$$

Acting on F_A and using the Weitzenböck identity then gives

$$\begin{aligned}\frac{\partial}{\partial t}\left(\nabla_A^k F_A\right) = -\nabla_A^k\left(\nabla_A^*\nabla_A F_A + \{F_A, F_A\} + \{R, F_A\}\right) + \sum_{i+j=k-1} \nabla_A^i \{\nabla_A F_A, \nabla_A^j F_A\} = \\ - \nabla_A^k(\nabla_A^* \nabla_A F_A) + \sum_{i+j=k} \left(\{\nabla^i F, \nabla^j F\} + \{\nabla_i R, \nabla_j F\}\right).\end{aligned} \tag{5.33}$$

To get a ∇_A^k term we commute derivatives

$$\nabla_A^k(\nabla_A^*\nabla_A F_A) - \nabla_A^*\nabla_A(\nabla_A^k F_A) = \sum_{i+j=k} \{\nabla_A^i F_A, \nabla_A^j F_A\}, \tag{5.34}$$

which gives

$$\left(\frac{\partial}{\partial t} + \nabla_A^* \nabla_A\right)(\nabla^k F) = \sum_{i+j=k} \left(\{\nabla^i F, \nabla^j F\} + \{\nabla_i R, \nabla_j F\} \right). \tag{5.35}$$

Thus

$$\begin{aligned}\frac{\partial}{\partial t} e_k = 2\left(\frac{\partial}{\partial t}(\nabla_A^k F_a), \nabla_A^k F_A\right) = \\ - 2\left(\nabla_A^*\nabla_A(\nabla_A^k F_A), \nabla_A^k F_A\right) + \sum_{i+j=k} \left(\{\nabla_A^i F_A, \nabla_A^j F_A, \nabla_A^k F_A\} + \{\nabla_A^i R, \nabla_A^j F_A, \nabla_A^k F_A\} \right) \leq \\ - \Delta e_k + c_k e_k^{1/2} \sum_{i+j=k} e_j^{1/2}(e_i^{1/2} + 1),\end{aligned} \tag{5.36}$$

for sufficient large choice of c_k. □

We can now use the maximum principle to bound $\operatorname{tr} F_A$, e and e_k. Since $|\operatorname{tr} F_A|$ and $\hat{e}$ are positive it then follows by the maximum principle that

Corollary 5.9 (Corollary 17(i) in [4]). *Let H_t be a smooth solution to the evolution equation. Then $\sup_M |\operatorname{tr} F_A|$ and $\sup_M \hat{e}$ are decreasing and bounded for all t.*

The inequality in (iv) is harder to apply since it is non-linear. We know that the linear equation

$$\left(\frac{\partial}{\partial t} + \Delta\right) f = C(f+1), \quad f(0)e_k(0), \tag{5.37}$$

has a smooth solutions for all time $t > 0$. To conclude that the evolution equation for e_k has a smooth solution we need an at most linear inequality,

$$\left(\frac{\partial}{\partial t} + \Delta\right) e_k \leq C(e_k + 1). \tag{5.38}$$

To be able to show this we will need another assumption on F_A, that of uniform boundedness.

Corollary 5.10 (Corollary 17(ii) in [4]). *Assume that F_A is uniformly bounded for $0 \leq t < T$. Then $\sup e_k$ is uniformly bounded for $0 \leq t < T$.*

Proof. By assumption $|F_{A_t}| < B$ for all $x \in M$ and $t \in [0,T)$. Then by (ii) we certainly have

$$\left(\frac{\partial}{\partial t} + \Delta\right) e \leq C(e + 1). \tag{5.39}$$

Suppose that $|\nabla^i_A F_A| < B_i$ for all $i < k$. It then follows by (iv) that

$$\left(\frac{\partial}{\partial t} + \Delta\right) e_k \leq C(e_k + 1), \tag{5.40}$$

and thus induction on k the inequality is true for all k provided that the inequality implies that e_k is bounded. To show this is the case we apply the maximum principle to $(e_k - f)e^{-Ct}$,

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \Delta\right) \left((e_k - f)e^{-Ct}\right) &= e^{-Ct}\left(\frac{\partial}{\partial t} + \Delta\right)(e_k - f) - Ce^{-Ct}(e_k - f) = \\ &\underbrace{e^{-Ct}\left(-\left(\frac{\partial}{\partial t} + \Delta\right) f + Cf\right)}_{=-Ce^{-Ct}} + \underbrace{e^{-Ct}\left(\left(\frac{\partial}{\partial t} + \Delta\right) f - Ce_k\right)}_{\leq Ce^{-Ct}} \leq 0.\end{aligned} \tag{5.41}$$

We can then conclude by the maximum principle that $e^{-Ct}(e_k - f)$ is decreasing and hence that $e_k \leq f$ and in particular that e_k is bounded. ☐

We can relax the uniformly bounded requirement of F_A to that of being bounded in L^p. To do this we will make use of the fundamental solution to the heat equation, the heat kernel which we can locally write as

$$H_t(x,y) = \frac{1}{(4\pi t)^n} e^{-\frac{r^2}{4t}}, \tag{5.42}$$

where n is the complex dimension of M and $r = d(x,y)$ the distance function between two points x and y. When we take the L^p norm of the heat kernel it will be convenient to do a change of variables, $u = \sqrt{\frac{p}{4t}}r$. Then the integrand is

$$H_t(x,y)^p r^{2n-1}\,\mathrm{d}r = (4\pi t)^{-np} e^{-u^2} \left(\frac{4t}{p}\right)^n u^{2n-1}\,\mathrm{d}u. \tag{5.43}$$

Lemma 5.11 (Lemma 18 in [4]). *Let $A = A_t$ gauge-equivalent to a path satisfying Yang-Mills flow. Suppose that F_t is bounded in L^q for $q > 3n$. Then F_t is uniformly bounded for $t < T$.*

Proof. We have that that the L^p norm of the heat kernel for each fixed $x \in M$ satisfies

$$||H_t(x,-)||_{L^p(M)} \leq Ct^{\frac{-n(p-1)}{p}} \left(\underbrace{\int_0^\infty u^{2n-1} e^{-u^2} \,\mathrm{d}u}_{=\frac{(n-1)!}{2}} \right)^{1/p} \leq C' t^{\frac{-n(p-1)}{p}}, \tag{5.44}$$

for some constants C and C'. When the exponent is larger than -1 the integral

$$\int_0^T ||H_t(x,-)||_{L^p(M)} \,\mathrm{d}t, \tag{5.45}$$

converges. This is true for all p when $n = 1$, while for $n > 1$ the requirement is

$$\frac{-n(p-1)}{p} > -1 \Leftrightarrow p < \frac{n}{n-1}. \tag{5.46}$$

When the exponent is greater than -1, Thus for $p < \frac{n}{n-1}$, we have

$$\int_0^T ||H_t(x,-)||_{L^p(M)} \,\mathrm{d}t \leq c_p(T), \tag{5.47}$$

for some constant $c_p(T)$. We can now apply the inequality (ii),

$$\left(\frac{\partial}{\partial t} + \Delta\right) e \leq c\left(e^{3/2} + e\right), \tag{5.48}$$

to get

$$e_t \leq H_t e_0 + c \int_0^t H_{t-\tau}(e_\tau^{3/2} + e_\tau) \,\mathrm{d}\tau. \tag{5.49}$$

We know that $H_t e_0$ is bounded by $\sup_M e_0$ while the second term is less

$$c \int_0^t ||H_{t-\tau}||_{L^p} ||e^{3/2} + e||_{L^{p'}} \,\mathrm{d}\tau, \tag{5.50}$$

by Hölder's inequality for conjugated indices p and p',

$$\frac{1}{p} + \frac{1}{p'} = 1. \tag{5.51}$$

We have that the integral is bounded for $p < \frac{n}{n-1}$ and if $e^{3/2}$ is uniformly bounded for $p' > n$. Equivalently that $|F_A|$ bounded in L^q for $q > 3n$. □

To apply the lemma we need to show that F_t is bounded in L^p for all p. To do this we need part of the Sobolev embedding theorem,

Theorem 5.12 (Theorem 6.5.4 in [28])**.** *Let M be a compact Riemannian manifold of real dimension m. Then for integers $\gamma, k,$ and p satisfying the inequality*

$$0 \leq \gamma < k - \frac{m+2}{p}, \tag{5.52}$$

the operator

$$L^p_k(M \times [0,T]) \to C^\gamma(M \times [0,T]), \tag{5.53}$$

is compact.

We can now show

Lemma 5.13 (Lemma 19 in [4] or Lemma 6.5.22 in [28])**.** *Let H_t for $0 \leq t < T$ be any family of Hermitian metrics on a holomorphic bundle E such that*
(i) H_t converges in C^0 to some continuous metric H_T as $t \to T$
(ii) $\sup_M |\hat{F}_{H_t}|$ is uniformly bounded for $t < T$.
Then H_t is bounded in C^1 and F_{H_t} is bounded in L^p for any finite p, uniformly in $t < T$.

Proof. Since there is no natural C^1 norm on the space of metrics we specify one by choosing a local holomorphic trivialization of the vector bundle and take the supremum of the matrix representation of the metrics H_t. That is our C^1 norm of H_t is given by

$$\sup_M |\nabla H_t|. \tag{5.54}$$

Suppose in contradiction to the hypothesis that the H_t metrics are not bounded in C^1. That is there exists a sequence of $t_i \to T$ such that

$$m_i = \sup_M |\nabla H_i| \to \infty. \tag{5.55}$$

Let x_i be the points where the supremum is attained. By taking a subsequence converges to some point $x \in M$. Thus we have a local problem and can choose local coordinates $z = (z_1, \ldots, z_n)$ in the polydisc

$$D_1 = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n | \ |z_i| < 1 \text{ for all } i\}. \tag{5.56}$$

By translating the coordinates we may suppose that supremum

$$m_i = \sup_{D_1} |\nabla H_i|, \tag{5.57}$$

is achieved at $z = \mathbf{0}$.

Next set

$$\tilde{H}_i(z) = H_i\left(\frac{z}{m_i}\right), \tag{5.58}$$

which gives by the chain rule that the supremum

$$\sup_{D_1} |\nabla \tilde{H}_i| = 1, \tag{5.59}$$

is attained at $z = 0$.

However by assumption (ii) and expressing $\Lambda F_{\tilde{H}_i}$ in terms of $\tilde{H}_i$,

$$|\Lambda F_{\tilde{H}_i}| = |\tilde{H}_i^{-1}(\Delta \tilde{H}_i - i\Lambda \partial \tilde{H}_i \tilde{H}_i^{-1} \bar{\partial} \tilde{H}_i)| \tag{5.60}$$

is bounded in D^1. Then since by (i) $\tilde{H}_i$ and $\nabla \tilde{H}_i$ are bounded on D_1 it follows that $|\Delta \tilde{H}_i|$ is bounded independently of i in D_1.

By estimates of the operator Δ on L^p spaces it follows that $\tilde{H}_i$ is bounded on a slightly smaller polydisc $D_{1-\delta}$. For the details we refer to the proof in [28] and the general theory for elliptic operators in [41].

By theorem 5.12 for $p > 2n+2$ the operator $L_2^p \to C^1$ is compact. Thus choosing such a p there is a subsequence of $\tilde{H}_i$ that converge in C^1 to some $\tilde{H}_\infty$. By (i) $\tilde{H}_\infty$ must have constant components making $\nabla \tilde{H}_\infty = 0$ which contradicts

$$|\nabla \tilde{H}(\mathbf{0})_\infty| = \lim_{i\to\infty} |\nabla \tilde{H}_i(\mathbf{0})| = 1. \tag{5.61}$$

Thus we can conclude that H_t is bounded in C^1. It then follows by the same L^p estimate of the Laplacian and that $\hat{F}_{H_t}$ being uniformly bounded that H_t is bounded in L_2^p for all p and that F_{H_t} is bounded in L^p. $\square$

We can now show existence of a solution to the evolution equation for all time.

Proposition 5.14 (Proposition 20 in [4]). *Let E be a holomorphic vector bundle over M with initial metric h_0. Then the evolution equation*

$$\frac{\partial H_t}{\partial t} = -H_t\left(\Lambda F_H - \lambda I\right), \tag{5.62}$$

has a unique smooth solution for $0 \le t < \infty$.

Proof. We know that we have a solution for short time by proposition 5.3 and that a solution to the evolution equation is unique by corollary 5.6.

Suppose that H_t is a solution to the evolution equation for finite time $0 \le t < T$. By proposition 5.7 and corollary 5.9 the conditions in lemma 5.13 are satisfied and thus F_H is bounded in L^p for any $p < \infty$. Then by lemma 5.11 F_H is bounded uniformly and by corollary 5.10 the k times iterated derivative of the curvature is uniformly bounded for all t and by lemma 5.13 H_t is then bounded in C^k for all k. Thus we have convergence in C^∞ as $t \to T$ which allows us to extend the solution by short time existence to $[0, T+\epsilon)$. Proceeding this way we can extend the solution to any time. $\square$

5.2 The Donaldson functional

To show convergence the Hermitian metrics H_t to an Hermitian metric H_∞, Donaldson introduced a functional $L(h)$ whose gradient flow correspond to the evolution equation. We will devote this section to the introduction this functional.

We begin by introducing two second characteristic classes which we enable us to give an explicit expression for the functional. Set

$$R_1(H,K) = \log(\det(K^{-1}H)), \tag{5.63}$$

and

$$R_2(H,K) = i\int_0^t \operatorname{tr}(H_t^{-1}\dot{H}_t F_t)\,\mathrm{d}t. \tag{5.64}$$

Consider a family of metrics H_t while the metric K is fixed. Reminding ourselves of the differentiation rule for the determinant $\partial_t \det(A) = \det(A)\operatorname{tr}(A^{-1}\partial_t A)$ for an invertible matrix A we get

$$\partial_t R_1(H_t,K) = \partial_t \log(\det(K^{-1}H_t)) = \partial_t \log(\det(H_t)) = \frac{\partial_t \det(H_t)}{\det(H_t)} = \operatorname{tr}(H_t^{-1}\dot{H}_t). \tag{5.65}$$

We also have that

$$\partial_t R_2(H_t,K) = i\operatorname{tr}(H_t^{-1}\dot{H}_t F_t). \tag{5.66}$$

With $v_t = H_t^{-1}\partial_t H_t$ we summarise this as

$$\partial_t R_1(H_t,K) = \operatorname{tr}(v_t), \tag{5.67}$$

$$\partial_t R_2(H_t,K) = i\operatorname{tr}(v_t F_t). \tag{5.68}$$

The characteristic classes are associated to $\operatorname{tr} F_H$ and $-\operatorname{tr} F_H^2$ meaning that

$$\begin{aligned} i\bar{\partial}\partial R_1(H_t,K) &= \operatorname{tr}(F_H) - \operatorname{tr}(F_K), \\ i\bar{\partial}\partial R_2(H_t,K) &= -\operatorname{tr}(F_H)^2 + \operatorname{tr}(F_K)^2. \end{aligned} \tag{5.69}$$

This follows from that

$$\operatorname{tr}(F_H) = i\bar{\partial}\partial \log\det H, \tag{5.70}$$

i.e.

$$\operatorname{tr}(F_H) - \operatorname{tr}(F_K) = i\bar{\partial}\partial \log\det(K^{-1}H). \tag{5.71}$$

One can also show by direction computation that $\partial_t F_{H_t} = \bar{\partial}\partial v_t$ which gives us the equation for R_2. In particular

$$i\bar{\partial}\partial\left(R_2 - \frac{\lambda}{n}R_1\right) = -\left(\Psi(F_H) - \Psi(F_K)\right), \tag{5.72}$$

where Ψ is defined by

$$\Psi(F_H) = \operatorname{tr}(F_H^2) + \frac{\lambda}{n}\operatorname{tr}(F_H)\wedge\omega. \tag{5.73}$$

We now define the Donaldson functional as

$$L_M(H,K) = \int_M (R_2 - \frac{\lambda}{n}R_1\omega)\wedge\frac{\omega^{n-1}}{(n-1)!}, \tag{5.74}$$

where λ is the Hermitian-Einstein constant. In particular over a projective algebraic curve C the functional becomes

$$L_C(H,K) = \int_C (R_2 - \lambda R_1\omega), \tag{5.75}$$

and over an algebraic surface X

$$L_X(H,K) = \int_X (R_2 - \frac{\lambda}{2}R_1\omega)\wedge\omega. \tag{5.76}$$

Fixing the metric K gives a functional $L(H)$ which shall be shown to have the desired property.

Again considering a family of Hermitian metrics H_t and considering $L(H_t) = L(H_t,K)$ for a fixed Hermitian metric K we have that

$$\partial_t(R_2 - \frac{\lambda}{n}R_1\omega) = \left(\operatorname{tr}(v_tF_t) - \frac{\lambda}{n}\operatorname{tr}(v_t)\omega\right) = \operatorname{tr}(v_t(F_t - \frac{\lambda}{n}\omega)), \tag{5.77}$$

and thus

$$\frac{\mathrm{d}L(H_t)}{\mathrm{d}t} = \frac{1}{(n-1)!}\int_M \operatorname{tr}(v_t(F_t\wedge\omega^{n-1} - \frac{\lambda}{n}\omega^n)) = \int_M \operatorname{tr}(v_t(\Lambda F_t - \lambda I))\frac{\omega^n}{n!} = \langle \Lambda F_t - \lambda I, v_t\rangle_{L^2}. \tag{5.78}$$

The variation is then zero only when

$$\Lambda F_t = \lambda I, \tag{5.79}$$

i.e. when H_t is Hermitian-Einstein. We have shown

Proposition 5.15. *A Hermitian metric H_0 is a critical point of $L(H)$ if and only if H_0 is Hermitian-Einstein.*

As in the case of the Yang-Mills functional we will consider the associated gradient flow of the functional. The L^2 gradient flow of the Donaldson functional is

$$\frac{\mathrm{d}}{\mathrm{d}t}L(H_t) = -||\Lambda F_t - \lambda I||^2_{L^2}. \tag{5.80}$$

Our goal is to show existence and convergence of this flow, thus allowing us to arrive at a Hermitian-Einstein metric.

Taking the second derivative of the functional gives

$$\frac{\mathrm{d}^2L(H_t)}{\mathrm{d}t^2} = -\frac{\mathrm{d}}{\mathrm{d}t}||\Lambda F_t - \lambda I||^2_{L^2} = -2\langle\frac{\partial}{\partial t}\Lambda F_t, \Lambda F_t - \lambda I\rangle_{L^2} = 2\langle\Delta\Lambda F_A, \Lambda F_t - \lambda I\rangle_{L^2} = 2||\nabla\Lambda F_A||^2_{L^2} \geq 0. \tag{5.81}$$

That is the functional is convex and we have shown

Proposition 5.16 (Proposition 8 in [4])**.** *If H is a critical point of $L(H)$ then H is an absolute minimum of the functional $L(H)$.*

And as a consequence

Corollary 5.17. *If H_0 and H_1 are two critical points of $L(H_t)$, then they define the same connection.*

Next we will need to know how R_1 and R_2 behave over an exact sequence of holomorphic bundles

$$0 \to E' \to E \to E'' \to 0. \tag{5.82}$$

A Hermitian metric H on E induces metrics H' and H'' on the bundles E' and E'' respectively.

Proposition 5.18 (Proposition 7 in [4] or Proposition 6.10.2 in [28])**.** *For any pair of Hermitian metrics H and K on E we have*
(i) $R_1(H,K) = R_1(H',K') + R_1(H'',K'')$,
(ii) $R_2(H,K) = R_2(H',K') + R_2(H'',K'') - i\,(\operatorname{tr}(B_H \wedge B^*_H) - \operatorname{tr}(B_K \wedge B^*_K))$
modulo $\partial\Omega^{0,1} + \bar{\partial}\Omega^{1,0}$.

Proof. By the definition of R_1 as the second characteristic class associated to $c_1(E)$ we know that that R_1 is additive over short exact sequences. For (ii) we have splitting C^∞-bundle homomorphisms π and λ,

$$\begin{tikzcd} 0 \arrow[r] & E' \arrow[r, shift left, "\alpha"] & E \arrow[l, shift left, "\pi"] \arrow[r, shift left, "\beta"] & E'' \arrow[l, shift left, "\lambda"] \arrow[r] & 0. \end{tikzcd} \tag{5.83}$$

That is

$$\begin{aligned} &\pi \circ \alpha = I_{E'}, \\ &\beta \circ \lambda = I_{E'}, \\ &\alpha \circ \pi + \lambda \circ \beta = I_E. \end{aligned} \tag{5.84}$$

Two different splitting morphisms are related by an element $S \in \mathrm{Hom}(E'',E')$ as

$$\begin{cases} \pi_2 = \pi_1 - S\beta \\ \lambda_2 = \lambda_1 + \alpha S. \end{cases} \tag{5.85}$$

Next we join the metrics H and K by a 1-parameter family H_t such that $H_0 = K$ and $H_1 = H$, which induces morphisms $\pi = \pi_t$ and $\lambda = \lambda_t$.

We also know from eq. (4.44) that the curvatures connected by an element $B \in \Omega^{0,1}(\mathrm{Hom}(E'',E'))$ as

$$F_t = \begin{bmatrix} F_t' - B \wedge B^* & \partial B \\ -\bar{\partial} B^* & F_t'' - B^* \wedge B \end{bmatrix}. \tag{5.86}$$

Note here that we denote the exterior covariant derivative by $\partial = \partial_{H_t}$ while we will use ∂_t to the denote the time derivative $\frac{\mathrm{d}}{\mathrm{d}t}$. Note also that B is given by

$$B = \pi \circ \bar{\partial} \circ \lambda, \tag{5.87}$$

and depends on the time t and the adjoint B^* is taken with respect to the Hermitian metric H_t. We have that the time derivative of B is given by

$$\partial_t B = (\partial_t \pi)\bar{\partial}\lambda + \pi\bar{\partial}(\partial_t \lambda) = -\dot{S}\beta\bar{\partial}\lambda + \pi\bar{\partial}\partial_t(\alpha S) = -(\partial_t S_t)\bar{\partial} + \bar{\partial}(\partial_t S_t) = \bar{\partial}(\dot{S}). \tag{5.88}$$

We also need to know how $v_t = H_t^{-1}\partial_t H_t$ is related to $v_t' = H_t'^{-1}\partial_t H_t'$ and $v_t'' = H_t''^{-1}\partial_t H_t''$. To do this we pick a local basis. Let $e_1, \ldots, e_p$ and $e_{p+1}, \ldots, e_r$ be local unitary frames for the bundles E' and E'' respectively. Then we have a local unitary frame for E given by $(\alpha(e_1), \ldots, \alpha(e_p),\lambda(e_{p+1}), \ldots \lambda(e_r))$. Using this basis we have

$$(\partial_t H)(\alpha(e_i),\lambda(e_j)) = -H(\alpha(e_i),\partial_t\lambda(e_J)) = -H(\alpha(e_i),\alpha(\dot{S})) = -H'(e_i,(\dot{S})e_j). \tag{5.89}$$

Thus in matrix form $v_t =$ are related to v_t' and v_t'' by

$$v_t = \begin{bmatrix} v_t' & -\dot{S} \\ -(\dot{S})^* & v_t'' \end{bmatrix}. \tag{5.90}$$

Calculating the trace of $v_t F_t$ we then have

$$\mathrm{tr}(v_t R_t) - \mathrm{tr}(v_t' R_t') - \mathrm{tr}(v_t'' R_t'') = -\mathrm{tr}(v_t' B \wedge B^*) - \mathrm{tr}(v_t'' B^* \wedge B) + \mathrm{tr}(\dot{S}\bar{\partial}B^*) - \mathrm{tr}((\dot{S})^*\partial B). \tag{5.91}$$

We also have

$$\begin{aligned}\operatorname{tr}(\dot{S}\bar{\partial}B^*) &= \bar{\partial}(\dot{S}\bar{\partial}B^*) - \operatorname{tr}(\dot{B}\wedge B^*),\\ \operatorname{tr}((\dot{S})^*\bar{\partial}B) &= \partial((\dot{S})^*\partial B) + \operatorname{tr}(B\wedge(\dot{B})^*).\end{aligned} \tag{5.92}$$

Then modulo $\partial\Omega^{0,1} + \bar{\partial}\Omega^{0,1}$,

$$\begin{aligned}\operatorname{tr}(\dot{S}\bar{\partial}B^*) &= -\operatorname{tr}(\dot{B}\wedge B^*),\\ \operatorname{tr}((\dot{S})^*\bar{\partial}B) &= \operatorname{tr}(B\wedge(\dot{B})^*).\end{aligned} \tag{5.93}$$

Thus we can write eq. (5.91) as

$$\begin{aligned}\operatorname{tr}(v_tR_t) - \operatorname{tr}(v_t'R_t') - \operatorname{tr}(v_t''R_t'') = -\operatorname{tr}(v_t'B\wedge B^*) + \operatorname{tr}(B\wedge v_t''B^*) - \operatorname{tr}(\dot{B}\wedge B^*) - \operatorname{tr}(B\wedge(\dot{B})^*) =\\ -\operatorname{tr}(\dot{B}\wedge B^*) - \operatorname{tr}\left(B\wedge(B^*v_t' + (\dot{B})^* - v_t''B^*)\right).\end{aligned} \tag{5.94}$$

Next we show that

$$B^*v_t' + (\dot{B})^* - v_t''B^* = \partial_t(B^*). \tag{5.95}$$

By the identity $H'(\xi,B\eta) = H''(B^*\xi,\eta)$ we have

$$(\partial_tH')(\xi,B\eta) + H'(\xi,\partial_tB\eta) = (\partial_tH'')(B^*\xi,\eta) + H''(\partial_t(B^*)\xi,\eta). \tag{5.96}$$

Rewriting this using

$$\begin{aligned}(\partial_tH')(\xi,B\eta) = H'(v_t'\xi,B\eta) &= H''(B^*v_t'\xi,\eta),\\ H'(\xi,\partial_tB\eta) &= H''((\partial_tB)^*,\eta),\\ (\partial_tH'')(B^*\eta,\xi) &= H''(v_t''B^*\eta,\xi),\end{aligned} \tag{5.97}$$

gives

$$H''(B^*v_t'\xi,\eta) + H''((\partial_tB)^*,\eta) = H''(v_t''B^*\xi,\eta) + H''(\partial_t(B^*)\xi,\eta). \tag{5.98}$$

Since ξ and η are arbitrary, eq. (5.95) follows and thus

$$\operatorname{tr}(v_tR_t) - \operatorname{tr}(v_t'R_t') - \operatorname{tr}(v_t''R_t'') = -\operatorname{tr}(\partial_tB\wedge B^*) - \operatorname{tr}(B\wedge\partial_t(B^*)) = -\partial_t\operatorname{tr}(B\wedge B^*). \tag{5.99}$$

From this (ii) follows by integrating from $t = 0$ to $t = 1$. □

We can now establish two essential properties of the Donaldson functional, that the functional is bounded from below and that we can relate the functional over manifold M to that of the functional over a hypersurface M' of M. Note that the following proposition assumes the Harder-Narasimhan theorem so it will be only useful for showing the Donaldson-Uhlenbeck-Yau theorem for manifolds with complex dimension $n \geq 2$.

Proposition 5.19 (Corollary 9 in [4])**.** *Let E be a semi-stable vector bundle over an algebraic curve C. Then the functional $L_C(-,K)$ is bounded from below.*

Proof. We know this is true if E is stable since by the Harder-Narasimhan theorem the bundle then admits an Hermitian-Einstein connection giving a minimum of the functional. Thus assume that E is semi-stable but not stable. We can then find a subbundle $E' \subset E$ that is stable with $\mu(E) = \mu(E)$. We also have that $E'' = E/E'$ is semi-stable with $\mu(E'') = \mu(E)$ from the exact sequence

$$0 \to E' \to E \to E'' \to 0. \tag{5.100}$$

Since λR_1 is proportional the slope μ which is additive over exact sequences and we then only have to consider R_2. Then integrating proposition 5.18 we have

$$L(H,K) = L(H',K') + L(H'',K'') + ||B_H||^2 - ||B_K||^2, \tag{5.101}$$

since

$$-i\,\mathrm{tr}(B \wedge B^*) \wedge \omega^{n-1} = |B|^2 \frac{\omega^n}{n}. \tag{5.102}$$

We know $||B_H|| \geq ||B_K||^2$ and the hypothesis then follows by induction on the rank of the bundle, that is $L(H',K')$ and $L(H'',K'')$ are bounded below by constants depending only on K' and K'' respectively. $\square$

We will next consider a hypersurface M' in the linear system $|\mathcal{O}_M(m)|$ of divisors on M. M' is then defined by $s = 0$ for a section of the line bundle $H^{\otimes m}$. We will use the Poincaré-Lelong formula for line bundles, which states that

$$\omega = \frac{|M'|}{m} - i\bar{\partial}\partial f. \tag{5.103}$$

for

$$f = \frac{1}{\pi m} \log |s|, \tag{5.104}$$

and with $|M'|$ the current of integration of the divisor defined by s. We refer to [42] the background and proof of the Poincaré-Lelong formula and [28] on how it applies to holomorphic line bundles.

Proposition 5.20 (Proposition 10 in [4])**.** *Let M and M' and f be as above. Then the respective functionals $L_M(H,K)$ and $L_{M'}(H,K)$ are related by*

$$L_M(H,K) = \frac{1}{m(n-1)} L_{M'}(H,K) - \int_M f\,(\Psi(F_H) - \Psi(F_K)) \wedge \frac{\omega^{n-2}}{(n-1)!}. \tag{5.105}$$

Proof. Using eq. (5.103) gives

$$\begin{aligned} L_M = \int_M (R_2 - \frac{\lambda}{n} R_1 \omega) \wedge \frac{\omega^{n-1}}{(n-1)!} = \frac{1}{n-1} \int_M (R_2 - \frac{\lambda}{n} R_1 \omega) \wedge \frac{\omega^{n-2}}{(n-2)!} \wedge \left(\frac{|M'|}{m} - i\bar{\partial}\partial f \right) = \\ \frac{1}{m(n-1)} \int_{M'} (R_2 - \frac{\lambda}{n} R_1 \omega) \wedge \frac{\omega^{n-2}}{(n-2)!} - \frac{1}{n-1} \int_M (R_2 - \frac{\lambda}{n} R_1 \omega) \wedge \frac{\omega^{n-2}}{(n-2)!} \wedge (i\bar{\partial}\partial f) = \\ \frac{1}{m(n-1)} L_{M'} - \int_M (R_2 - \frac{\lambda}{n} R_1 \omega) \wedge \frac{\omega^{n-2}}{(n-1)!} \wedge (i\bar{\partial}\partial f). \end{aligned} \tag{5.106}$$

Note here the constant λ/n is the correct constant for M' as well since

$$\frac{\lambda_M}{n} = \frac{\lambda_{M'}}{n-1} \tag{5.107}$$

and as a consequence of that

$$\lambda = \frac{2\pi n}{\mathrm{vol}(M)} \mu(E), \tag{5.108}$$

is proportional to n while the volume and slope do not change.

Integrating the last term in eq. (5.106) by parts gives

$$\int_M if\bar{\partial}\partial\left(R_2-\frac{\lambda}{n}\omega\right)\wedge\frac{\omega^{n-2}}{(n-1)!}=-\int_M f\left(\Psi(F_H)-\Psi(F_K)\right)\wedge\frac{\omega^{n-2}}{(n-1)!}. \tag{5.109}$$

Thus

$$L_M=\frac{1}{m(n-1)}L_{M'}-\int_M f\left(\Psi(F_H)-\Psi(F_K)\right)\wedge\frac{\omega^{n-2}}{(n-1)!}. \tag{5.110}$$

□

5.3 The choice of a good gauge

In the previous two sections we have studied the Yang-Mills flow with the goal of showing convergence towards an Hermitian-Yang-Mills connection. For this to be possible the vector bundle needs to be semistable so to finish the proof we will need to connect our previous work to the assumption of stability. Let us also note that so far we have worked in all generality with M as a Kähler manifold. This will now change where we will let the base manifold be an projective algebraic surface X. The reason for this is the usage of Uhlenbeck's results from gauge theory which restrict the proof strategy to the surface case.

We will use that it is possible to restrict the bundle over an algebraic surface X to a curve C in such a way as preserve stability. This is by the Metha-Ramanathan theorem in [43], which they later extended to arbitrary dimension in [44]. For surfaces we only need the version presented in [4], which is

Theorem 5.21 (Theorem 6.10.14 in [28] or Theorem 6.1 in [43].)**.** *Let E be a semistable bundle over a projective algebraic variety X with an ample line bundle H. Then for some integer m there exists a curve C in $|\mathcal{O}_X(m)|$ such that the restriction $E|_C$ is semistable.*

Using the theorem to restrict to a curve and preserve stability we can show that the Donaldson functional over X is bounded below.

Proposition 5.22 (Lemma 24 in [4])**.** *Let E be a semi-stable holomorphic vector bundle over a projective algebraic surface X. Then the functional $L_X(H_t,H_0)$ is bounded below.*

Proof. By proposition 5.20 we have

$$L_X=L_C-\int_X f(\Psi(F_H)-\Psi(F_K)). \tag{5.111}$$

By proposition 5.19 L_C is bounded below since $E|C$ is semi-stable. By corollary 5.9 $\operatorname{tr}(F_H)$ is bounded for all time t. Decomposition in self-dual and anti-self dual parts we have

$$\operatorname{tr}F_H^2=(|F_H^-|^2-|\hat{F}_H|^2)\,\mathrm{dvol}\,. \tag{5.112}$$

Since $\hat{F}_H$ is also bounded by corollary 5.9 and since $f\in L^1(X)$ it follows that $-\int_X f\Psi(F_H)$ is bounded below. □

Corollary 5.23 (Corollary 25 in [4])**.** *Let E be a semi-stable holomorphic vector bundle over a projective algebraic surface X. For a path H_t down Yang-Mills flow $\hat{F}_{H_t}$ converges to λI in C^0 as $t\to\infty$.*

Proof. From the gradient flow of the functional

$$\frac{\mathrm{d}}{\mathrm{d}t} L_X(H_t,H_0) = -||\Lambda F_t - \lambda I||^2_{L^2}, \tag{5.113}$$

we know that $L_X(H_t,H_0)$ is a decreasing function of t. We have also shown that $L_X(H_t,H_0)$ is convex and bounded below which implies that $\partial_t L_X$ goes to 0 in L^2 as $t \to \infty$. Thus we have convergence $\Lambda F_t \to \lambda I$ in L^2 norm.

By corollary 5.9 $\sup_X \Lambda F$ is bounded and decreasing so it converges in C^0 by the monotone convergence theorem. Thus the C^0 convergence to λI follows. □

To complete the proof of proposition 5.2 we will also need to know that the evolution equation converges to a connection A_∞. For this we will use Uhlenbeck's result on the choice of a good gauge. A good gauge for our context means that the connection A locally satisfy

$$\mathrm{d}^* A = 0. \tag{5.114}$$

This is known as the Coulomb gauge since in three dimensions it correspond to $\nabla \cdot A = 0$ in vector notation which is commonly used in the study of electromagnetism.

The importance of this gauge becomes clear when we consider the condition for a connection to be ASD, that its the self-dual part is zero

$$F_A^+ = (\mathrm{d}A)^+ + (A \wedge A)^+ = 0. \tag{5.115}$$

We also remind ourselves that we can equivalently express the ASD condition as $\Lambda F_A = 0$. Combining the ASD condition with the gauge condition $\mathrm{d}^* A = 0$ yields an elliptic system, allowing us to use existing theory in this area. In addition to the articles by Uhlenbeck [5] and [6] we would highly encourage the interested reader to see the section on Uhlenbeck's theorem in [35] for the details on this.

Now let $\tilde{A}$ be a $SU(r)$ connection on a trivial bundle over the open unit sphere B^4. Then Uhlenbeck choice of Good gauge states

Theorem 5.24 (Theorem 1.3 in [5], Proposition 22 in [4], or Theorem 2.3.7 in [35])**.** *There exists an absolute constant $K > 0$ and constants C_p such that if*

$$||F_{\tilde{A}}||_{L^2} \leq K, \tag{5.116}$$

then the connection $\tilde{A}$ is a gauge-equivalent to a connection A satisfying

$$\mathrm{d}^* A = 0, \tag{5.117}$$

$$||A||_{L^p_1} \leq C_p ||F_A||_{L^p} = C_p ||F_{\tilde{A}}||_{L^p} \quad \textit{for} \quad p \geq 2. \tag{5.118}$$

For the proof of the theorem we recommend the more accessible alternative proof in [35]. The theorem will allow us to choose a gauge where the connection in terms of the curvature and satisfying $\mathrm{d}^* A = 0$.

Using results from the theory of elliptic operators one can show

Corollary 5.25 (Corollary 23 in [4])**.** *Let $\tilde{A}_i$ be a sequence of connections as in theorem 5.24. Suppose that their self-dual parts $F^+_{\tilde{A}_t} \to 0$ as $i \to \infty$ in C^0. Then there exists a constant $K_2 \leq K$ such that if*

$$||F_{\tilde{A}_i}||_{L^2} \leq K_2, \tag{5.119}$$

then there exists a subsequence of the sequence A_i, as given by the proposition, that converges weakly in L_1^p, for all p on $\frac{1}{2}B^4$ to an ASD connection A_∞.

For the proof we refer to [4] and [35].

To extend the convergence result from B^4 to X we can cover X by of a collection of open balls. The local convergence result of corollary 5.25 can then be extended to the complement of a finite set $\{x_1, \ldots, x_l\}$. Thus we have a connection A_∞ over $M - \{x_1, \ldots, x_n\}$. To extend the connection to the entirety of M, we can use Uhlenbeck's theorem on the removability of singularities for Yang-Mills connections.

Theorem 5.26 (Theorem 4.1 in [6])**.** *Let A be a Yang-Mills connection in a bundle E over $B^4 - \{0\}$. If the L^2 norm of the curvature F_A is finite, then there exists a gauge such that the bundle E extends to a smooth bundle E' over B^4 and the connection A extends to a smooth Yang-Mills connection A' on E'.*

That is we have a Hermitian-Yang-Mills connection A_∞ on E'. And by the integrability requirement $F_\infty^{0,2} = 0$ E' is a holomorphic bundle. We can also note that $\mu(E) = \mu(E')$ since we have convergence of $\operatorname{tr} F_t$ by corollary 5.9 making $c_1(E) = c_1(E')$. Lastly to complete the proof of proposition 5.2 we need to show the existence of a non-zero holomorphic map $E \to E'$ for which we refer to [4] or [39]. As a consequence we have the Donaldson-Uhlenbeck-Yau theorem in the case of bundles over an algebraic surface X.

Chapter 6

The deformed Hermitian-Yang-Mills equation & Bridgeland stability

We are now ready to connect our earlier work on the Donaldson-Uhlenbeck-Yau theorem to mirror symmetry. To do this we will begin by introducing derived categories and triangulated categories in section 6.1. In section 6.2 we introduce Bridgeland stability on triangulated categories as a generalisation of μ-stability. Finally in section 6.3 we study properties of the deformed Hermitian-Yang-Mills equation and how it connects to Bridgeland stability on the bounded derived category of coherent sheafs $\mathcal{D}^b(\mathrm{Coh}(M))$ over a Calabi-Yau manifold M.

6.1 Derived and triangulated categories

Our aim in this section is to understand the bounded category of coherent sheafs $\mathcal{D}^b\,\mathrm{Coh}(M)$. From a physics perspective the motivation for $\mathcal{D}^b\,\mathrm{Coh}(M)$ is they are the B-branes in the B-model. To describe the complex geometry of the B-branes an initial approach would be view them as complex vector bundles and later extend that to the category of coherent sheafs $\mathrm{Coh}(M)$. However not every B-brane could be described by the objects in $\mathrm{Coh}(M)$, a "larger" category was needed. This would be the category of bounded coherent sheafs $\mathcal{D}^b\,\mathrm{Coh}(M)$ whose objects are cochain complexes over $\mathrm{Coh}(M)$. For our purposes we will choose to accept $\mathcal{D}^b\,\mathrm{Coh}(M)$ as the correct category, however the reader interested in their connection to physics and B-branes we refer to [45], [46], [47] and the overview in [9].

We will now proceed with introducing derived categories. We will keep our introduction motivational, aiming to make the construction plausible without getting bogged down in details of proofs. Our coverage is heavily based on [48] and [49] to which we also refer the reader for a more in dept coverage of the topic.

6.1.1 The category of chain complexes

Instead of working with an abelian category $\mathcal{A}$ we will work with cochain complexes on $\mathcal{A}$ which we will simply refer to as complexes. We remind the reader that a complex is a sequence of objects in an abelian category connected by morphisms

$$\dots \xrightarrow{\mathrm{d}^{i-1}} A^{i-1} \xrightarrow{\mathrm{d}^{i}} A^{i} \xrightarrow{\mathrm{d}^{i+1}} A^{i+1} \xrightarrow{\mathrm{d}^{i+2}} \dots \tag{6.1}$$

such that $\mathrm{d} \circ \mathrm{d}^{i-1} = 0$. Given an abelian category $\mathcal{A}$ we can form a new abelian category, the category of complexes $\mathrm{Kom}(\mathcal{A})$. The zero object in this category consists of the zero complex

$$\dots \to 0 \to 0 \to 0 \to \dots, \tag{6.2}$$

and a morphism between complexes $f : A^* \to B^*$ is represented by the commutative diagram

$$\begin{tikzcd}
\dots \arrow[r, "\mathrm{d}_A^{i-2}"] & A^{i-1} \arrow[r, "\mathrm{d}_A^{i-1}"] \arrow[d, "f^{i-1}"] & A^{i} \arrow[r, "\mathrm{d}_A^{i}"] \arrow[d, "f^{i}"] & A^{i+1} \arrow[r, "\mathrm{d}_A^{i+1}"] \arrow[d, "f^{i+1}"] & \dots \\
\dots \arrow[r, "\mathrm{d}_B^{i-2}"] & B^{i-1} \arrow[r, "\mathrm{d}_B^{i-1}"] & B^{i} \arrow[r, "\mathrm{d}_B^{i}"] & B^{i+1} \arrow[r, "\mathrm{d}_B^{i+1}"] & \dots
\end{tikzcd} \tag{6.3}$$

Note in particular that since the diagram commutes

$$f^{i+1} \circ \mathrm{d}_A^{i} = \mathrm{d}_B^{i} \circ f^{i}, \tag{6.4}$$

resulting in that f induces maps

$$f^{i+1} : \mathrm{Im}(\mathrm{d}_A^{i}) \to \mathrm{Im}(\mathrm{d}_B^{i}) \tag{6.5}$$

and

$$f^{i} : \ker(\mathrm{d}_A^{i}) \to \ker(\mathrm{d}_B^{i}). \tag{6.6}$$

The existence of kernels, direct sums and quotients in $\mathrm{Kom}\,\mathcal{A}$ follow from their existence in $\mathcal{A}$. The kernel $\ker f$ is the complex of kernels

$$\dots \to \ker f^{i-1} \to \ker f^{i} \to \ker f^{i+1} \to \dots \tag{6.7}$$

and we have direct sums $A^* \oplus B^*$ and quotients A^*/B^* given by the complexes of

$$\dots \to A^{i-1} \oplus B^{i-1} \to A^{i} \oplus B^{i} \to A^{i+1} \oplus B^{i+1} \to \dots \tag{6.8}$$

and

$$\dots \to A^{i-1}/B^{i-1} \to A^{i}/B^{i} \to A^{i+1}/B^{i+1} \to \dots \tag{6.9}$$

respectively.

With the category $\mathrm{Kom}(\mathcal{A})$ we have the shift functor $T : \mathrm{Kom}(\mathcal{A}) \to \mathrm{Kom}(\mathcal{A})$ defined by $TA^i = A^{i+1}$ and $d^i_{TA^*} = -d^{i+1}_{A^*}$. That is the sequence is shifted as

$$\begin{tikzcd}
\dots \arrow[r, "\mathrm{d}^{i-2}"] & A^{i-1} \arrow[r, "\mathrm{d}^{i-1}"] \arrow[d, "T"] & A^{i} \arrow[r, "\mathrm{d}^{i}"] \arrow[d, "T"] & A^{i+1} \arrow[r, "\mathrm{d}^{i+1}"] \arrow[d, "T"] & \dots \\
\dots \arrow[r, "-\mathrm{d}^{i-1}"] & A^{i} \arrow[r, "-\mathrm{d}^{i}"] & A^{i+1} \arrow[r, "-\mathrm{d}^{i+1}"] & A^{i+2} \arrow[r, "-\mathrm{d}^{i+2}"] & \dots
\end{tikzcd} \tag{6.10}$$

where the sign is added by convention which simplifies the notation when we later work with cones and triangles. We usually write the shift as $A^*[1] = TA^*$ or more generally for an integer n we have the shift $A^*[n] = T^n A^*$.

Next we will need the concept of quasi-isomorphisms (qis) between complexes. Denote the i:th cohomology group by

$$H^i(A^*) = \frac{\ker d^i}{\operatorname{Im} d^{i-1}}. \tag{6.11}$$

Definition 6.1. *We say that a morphism* $f : A^* \to B^*$ *is a quasi-isomorphism if the induced maps* $H^i(f) : H^i(A^*) \to H^i(B^*)$ *is are isomorphisms.*

Definition 6.2. *We say that the complexes* A^* *and* B^* *are quasi-isomorphic if there exists a third complex* C^* *and morphisms*

$$\begin{tikzcd} & C^* \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A^* & & B^* \end{tikzcd} \tag{6.12}$$

such that $H^*(f)$ *and* $H^*(g)$ *are isomorphisms, that is* f *and* g *are quasi-isomorphism.*

Remark 6.3. *We note that quasi-isomorphic complexes must have isomorphic cohomology groups* $H^i(A^*) \cong H^i(B^*)$*. However the converse is not true in general.*

The derived category $\mathcal{D}(\mathcal{A})$ is the category where the quasi-isomorphisms in $\operatorname{Kom}(\mathcal{A})$ are isomorphisms. The process of making a collection of morphisms into isomorphisms is known as localization of the category. Thus stated in these terms $\mathcal{D}(\mathcal{A})$ is the category formed by the localization of the quasi-isomorphisms in $\operatorname{Kom}(\mathcal{A})$.

A motivation for wanting to consider quasi-isomorphic objects to be equal comes when considering resolution of sheafs. Given a coherent sheaf $\mathscr{F}$ over a projective variety X there is exists an exact sequence

$$0 \to \mathscr{F} \xrightarrow{f} \mathscr{E}^0 \xrightarrow{\mathrm{d}^0} \mathscr{E}^1 \xrightarrow{\mathrm{d}^1} \dots \xrightarrow{\mathrm{d}^{n-1}} \mathscr{E}^n \to 0, \tag{6.13}$$

where $\mathscr{E}^i$ are locally free sheafs i.e. vector bundles. This is known as a resolution of $\mathscr{F}$. We can then study $\mathscr{F}$ in terms of its resolution of locally free sheafs and wish to not make a distinction between the complexes

$$0 \to \mathscr{F} \to 0, \tag{6.14}$$

and

$$0 \to \mathscr{E}^0 \to \mathscr{E}^1 \to \mathscr{E}^2 \to \dots \to \mathscr{E}^n \to 0. \tag{6.15}$$

There is a quasi-isomorphism between these complexes given by the map f. To see this, let us write up the diagram for the morphisms

$$\begin{tikzcd} \dots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathscr{F} \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \\ \dots \arrow[r] & 0 \arrow[r] & \mathscr{E}^0 \arrow[r, "\mathrm{d}^0"] & \mathscr{E}^1 \arrow[r, "\mathrm{d}^1"] & \mathscr{E}^2 \arrow[r, "\mathrm{d}^2"] & \dots \arrow[r, "\mathrm{d}^{n-1}"] & \mathscr{E}^n \arrow[r] & 0 \arrow[r] & \dots \end{tikzcd} \tag{6.16}$$

Let us first note that since the sequence was exact we know that $H^p(\mathscr{E}) = 0$ for $p \geq 1$ so the zero maps induces isomorphisms on the cohomology groups. We then only have to check the morphisms f. We have that the cohomology groups are

$$H^0(\mathscr{F}) = \mathscr{F} \tag{6.17}$$

and

$$H^0(\mathscr{E}) = \ker \mathrm{d}^0 = \operatorname{Im} f. \tag{6.18}$$

Since

$$0 \to \mathscr{F} \xrightarrow{f} \mathscr{E}^0 \xrightarrow{\mathrm{d}^0} \mathscr{E}^1 \tag{6.19}$$

is exact it follows that f is one-to-one, or equivalently $\ker f = 0$. Then by the first isomorphism theorem

$$\operatorname{Im} f \cong \frac{\mathscr{F}}{\ker f} \cong \mathscr{F}. \tag{6.20}$$

Thus the map $H^0(f) : H^0(\mathscr{F}) \to H^0(\mathscr{E})$ is an isomorphism and f a quasi-isomorphism.

6.1.2 Chain homotopy

Instead of directly going to the derived category we first introduce the homotopy category of chain complex $\mathcal{K}(\mathcal{A})$ and then construct $\mathcal{D}(\mathcal{A})$ by localization of $\mathcal{K}(\mathcal{A})$. There are two reasons for this, first it makes it easier to see how the morphisms in $\mathcal{D}(\mathcal{A})$ looks like and secondly $\mathcal{K}(\mathcal{A})$ is a more intuitive category to both understand and work with in practice. Ultimately we will introduce functors between the categories such that the diagram

$$\begin{tikzcd}
\mathrm{Kom}(\mathcal{A}) \arrow[rr] \arrow[dr] & & \mathcal{D}(\mathcal{A}) \\
& \mathcal{K}(\mathcal{A}) \arrow[ur] &
\end{tikzcd} \tag{6.21}$$

commutes. Let us first note that the objects in both $\mathcal{K}(\mathcal{A})$ and $\mathcal{D}(\mathcal{A})$ as well as in $\mathrm{Kom}(\mathcal{A})$ are chain complexes and the difference between them will be in the morphisms between the objects. While $\mathrm{Kom}(\mathcal{A})$ is an abelian category $\mathcal{K}(\mathcal{A})$ and $\mathcal{D}(\mathcal{A})$ are not. However we will show that $\mathcal{K}(\mathcal{A})$ and $\mathcal{D}(\mathcal{A})$ have unlike $\mathrm{Kom}(\mathcal{A})$ another important property, that of having a triangulated structure making them triangulated categories. We now proceed with introducing $\mathcal{K}(\mathcal{A})$.

Definition 6.4. *Let A^* and B^* be chain complexes and $f : A^* \to B^*$ and $g : A^* \to B^*$ chain morphisms. A chain homotopy between f and g is then a collection of maps $h^i : A^i \to B^{i-1}$ such that*

$$f^i - g^i = h^{i+1} \circ \mathrm{d}_A^i + \mathrm{d}_B^{i-1} \circ h^i. \tag{6.22}$$

In diagram form chain homotopy then takes the shape

$$\begin{tikzcd}
\cdots \arrow[r] & A^{i-1} \arrow[r, "\mathrm{d}_A^{i-1}"] \arrow[dl] \arrow[d, shift right, "f^{i-1}"'] \arrow[d, shift left, "g^{i-1}"] & A^i \arrow[r, "\mathrm{d}_A^i"] \arrow[dl, "h^i"'] \arrow[d, shift right, "f^i"'] \arrow[d, shift left, "g^i"] & A^{i+1} \arrow[r] \arrow[dl, "h^{i+1}"'] \arrow[d, shift right, "f^{i+1}"'] \arrow[d, shift left, "g^{i+1}"] & \cdots \arrow[dl] \\
\cdots \arrow[r] & B^{i-1} \arrow[r, "\mathrm{d}_B^{i-1}"] & B^i \arrow[r, "\mathrm{d}_B^i"] & B^{i+1} \arrow[r] & \cdots
\end{tikzcd} \tag{6.23}$$

Let $f \sim g$ denote the homotopy equivalence of f and g. The morphisms in $\mathcal{K}(\mathcal{A})$ are then defined as

$$\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(A,B) = \mathrm{Hom}(A,B)/\sim, \tag{6.24}$$

that is quotienting out chain homotopy.

A homotopy equivalence is a morphism $f : A^* \to B^*$ that is an isomorphism in $\mathcal{K}(\mathcal{A})$, that is there exists a morphism $g : B^* \to A^*$ such that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$. Note that f and g are quasi-isomorphisms, thus all homotopy equivalences are quasi-isomorphisms. As a consequence homotopy equivalent complexes are quasi-isomorphic by the maps

$$\begin{tikzcd} A^* \arrow[r, shift left, "f"] & B^* . \arrow[l, shift left, "g"] \end{tikzcd} \tag{6.25}$$

6.1.3 The triangulated structure

Let us begin by the definition of a triangle.

Definition 6.5. *A triangle consists of three objects* (A,B,C) *in some category with translation with morphisms* $A \to B \to C \to A[1]$*. We represent this by the diagram*

$$\begin{tikzcd} A \arrow[rr] & & B \arrow[dl] \\ & C \arrow[ul, dashed] & \end{tikzcd} \tag{6.26}$$

Next consider a morphism $f \in \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(A^*,B^*)$. We introduce the mapping cone $C(f)$ as the complex

$$C(f) = A^*[1] \oplus B^* = \dots \to A^i \oplus B^{i-1} \xrightarrow{\mathrm{d}_C^i} A^{i+1} \oplus B^i \xrightarrow{\mathrm{d}_C^{i+1}} A^{i+2} \oplus B^{i+1} \to \dots \tag{6.27}$$

with

$$\mathrm{d}_C^i = \begin{bmatrix} \mathrm{d}_{A[1]}^i & 0 \\ f^i[1] & \mathrm{d}_B^i \end{bmatrix} = \begin{bmatrix} -\mathrm{d}_A^{i+1} & 0 \\ f^{i+1} & \mathrm{d}_B^i \end{bmatrix}. \tag{6.28}$$

Note that d_C^i differ from the natural morphisms for direct sums. We have that d_C^i is a boundary operator since

$$\mathrm{d}_C^{i+1} \mathrm{d}_C^i = \begin{bmatrix} \mathrm{d}_A^{i+2} \mathrm{d}_A^{i+1} & 0 \\ -f^{i+2} \mathrm{d}_A^{i+1} + \mathrm{d}_B^{i+1} f^{i+1} & \mathrm{d}_B^{i+1} \mathrm{d}_B^i \end{bmatrix} = 0, \tag{6.29}$$

where

$$-f^{i+2} \mathrm{d}_A^{i+1} + \mathrm{d}_B^{i+1} f^{i+1} : A^{i+1} \to B^{i+2}, \tag{6.30}$$

is the zero morphism by the commutative diagram of a morphism between complexes as seen in eq. (6.3). Thus $A^*[1] \oplus B^*$ is a complex. For any morphisms We can now form the triangle

$$A^* \xrightarrow{f} B^* \xrightarrow{\iota} C(f) \xrightarrow{\pi} A^*[1], \tag{6.31}$$

where the morphism ι is the inclusion map $b^i \mapsto (0,b^i)$ and π the surjective morphisms $(a^{i+1},b^i) \mapsto a^{i+1}$.

These triangles formed from a morphisms $f : A^* \to B^*$ and the mapping cone and any triangle

$$A^* \to B^* \to C^* \to A^*[1] \tag{6.32}$$

isomorphic to such a triangle are known as distinguished. The class of distinguished triangles makes $\mathcal{K}(\mathcal{A})$ and $\mathcal{D}(\mathcal{A})$ into triangulated categories. Triangulated categories are categories with translator function T and with a class of distinguished triangles that satisfy four axioms which we will get to shortly.

In particular we have that the mapping cone $C(\mathrm{id})$ of the identity map $\mathrm{id} : A^* \to A^*$ is homotopy equivalent to the zero complex by the map

$$h = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \tag{6.33}$$

since

$$h\,\mathrm{d}_C^i + \mathrm{d}_C^{i-1} h = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\mathrm{d}_A^{i+1} & 0 \\ 1 & \mathrm{d}_A^i \end{bmatrix} + \begin{bmatrix} -\mathrm{d}_A^i & 0 \\ 1 & \mathrm{d}_A^{i-1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & \mathrm{d}_A^i \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\mathrm{d}_A^i \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{6.34}$$

So $C(\mathrm{id})$ is isomorphic to the zero complex in $\mathcal{K}(\mathcal{A})$. It follows that the triangle

$$A^* \xrightarrow{\mathrm{id}} A^* \to 0 \to A^*[1] \tag{6.35}$$

is distinguished. In fact what we have shown is that $\mathcal{K}(\mathcal{A})$ and $\mathcal{D}(\mathcal{A})$ satisfy the first axiom of triangulated categories, TR1.

Axiom 6.6 (TR1)**.** *i) Any triangle*

$$A \xrightarrow{id} A \to 0 \to A[1] \tag{6.36}$$

is distinguished.
ii) Any triangle that is isomorphic to a distinguished triangle is distinguished.
iii) Any morphisms $f : A \to B$ can be completed to form a distinguished triangle

$$A \xrightarrow{f} B \to C \to A[1]. \tag{6.37}$$

Another property of triangulated categories that we will need is shift invariance, meaning we stay in the class of distinguishing triangles shifting the triangle.

Axiom 6.7 (TR2)**.** *The triangle*

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1] \tag{6.38}$$

is distinguished if and only if the triangle

$$B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{-f[1]} B[1] \tag{6.39}$$

is distinguished.

Before showing that our distinguished triangles satisfy this property, let us list one more axiom.

Axiom 6.8 (TR3)**.** *Given a commutative diagram of two distinguished triangles*

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d] & B_1 \arrow[r] \arrow[d] & C_1 \arrow[r] & A_1[1] \arrow[d] \\
A_2 \arrow[r] & B_2 \arrow[r] & C_2 \arrow[r] & A_2[1]
\end{tikzcd} \tag{6.40}$$

there exists a morphisms $C_1 \to C_2$ *such that the diagram*

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d] & B_1 \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d] & A_1[1] \arrow[d] \\
A_2 \arrow[r] & B_2 \arrow[r] & C_2 \arrow[r] & A_2[1]
\end{tikzcd} \tag{6.41}$$

commutes.

Note that by using the shift property of TR2 on the property of TR3 we get having any two of the three morphisms says that a third must exist. There is also a quite complicated forth axiom TR4 that we will not need but the interested reader can find stated in [50] along with a proof that derived categories satisfy this axiom.

Let us now show that the triangles formed by the mapping cone in $\mathcal{K}(\mathcal{A})$ satisfy the axioms.

Proposition 6.9. *The distinguished triangles in the derived category* $\mathcal{D}(\mathcal{A})$ *satisfy the axioms TR1, TR2, and TR3 of a triangulated category, making* $\mathcal{D}(\mathcal{A})$ *a triangulated category.*

Proof. We have already shown that TR1 is satisfied. Next we continue by showing TR3 where the relevant diagram is

$$\begin{tikzcd}
A_1^* \arrow[r, "f_1"] \arrow[d] & B_1^* \arrow[r] \arrow[d] & C(f_1) \arrow[r] \arrow[d] & A_1^*[1] \arrow[d] \\
A_2^* \arrow[r, "f_2"] & B_2^* \arrow[r] & C(f_2) \arrow[r] & A_2^*[1]
\end{tikzcd} \tag{6.42}$$

We take the morphism $A_1[1] \oplus B_1 = C(f_1) \to C(f_2) = C(f_2) = A_2[1]$ to be the one induced by the maps $A_1^{i+1} \to A_2^{i+1}$ and $B_1^i \to B_2^i$, which makes the diagram commute. Note that the map $C(f_1) \to C(f_2)$ is indeed a chain morphism since the boundary operator of the cones is defined identically.

Next we look at TR2. Given the distinguished triangle

$$A^* \xrightarrow{f} B^* \xrightarrow{\iota} C(f) \to A^*[1] \tag{6.43}$$

we consider the mapping cone

$$C(\iota) = B^*[1] \oplus C(f) = B^*[1] \oplus A^*[1] \oplus B^* \tag{6.44}$$

which has the boundary operator

$$\mathrm{d}^i_{C(g)} = \begin{bmatrix} -\,\mathrm{d}_B^{i+1} & 0 \\ \iota & \mathrm{d}^i_{C(f)} \end{bmatrix} = \begin{bmatrix} -\,\mathrm{d}_B^{i+1} & 0 & 0 \\ 0 & -\,\mathrm{d}_A^{i+1} & 0 \\ \mathrm{id} & f^{i+1} & \mathrm{d}^i_B \end{bmatrix}. \tag{6.45}$$

We form the distinguished triangle

$$B^* \xrightarrow{\iota} C(f) \xrightarrow{\tau} C(\iota) \xrightarrow{-f[1]} B^*[1]. \tag{6.46}$$

Our goal is to show that this triangle is isomorphic to the triangle

$$B^* \xrightarrow{\iota} C(f) \to A^*[1] \xrightarrow{-f[1]} B^*[1]. \tag{6.47}$$

That is we need to show that the diagram

$$\begin{tikzcd}
B^* \arrow[r, "\iota"] \arrow[d, "\mathrm{id}"] & C(f) \arrow[r, "\pi"] \arrow[d, "\mathrm{id}"] & A^*[1] \arrow[r, "-f[1]"] \arrow[d, shift right, "s"'] & B^*[1] \arrow[d, "\mathrm{id}"] \\
B^* \arrow[r, "\iota"] & C(f) \arrow[r, "\tau"] & C(\iota) \arrow[r, "\pi'"] \arrow[u, shift right, "t"'] & B^*[1]
\end{tikzcd} \tag{6.48}$$

commutes in $\mathcal{K}(\mathcal{A})$ with s is an isomorphisms in $\mathcal{K}(\mathcal{A})$ with inverse t. Let us first note that without the morphisms s and t the diagram commutes in $\mathrm{Kom}(\mathcal{A})$.

Define the morphism

$$s = (-f[1], \mathrm{id}, 0) \tag{6.49}$$

which makes the diagram

$$\begin{tikzcd}
A^*[1] \arrow[r, "-f[1]"] \arrow[d, "s"] & B^*[1] \arrow[d, "\mathrm{id}"] \\
C(\tau) \arrow[r, "\pi'"] & B^*[1]
\end{tikzcd} \tag{6.50}$$

commute in $\mathrm{Kom}\,\mathcal{A}$.

We define the inverse morphism t as the

$$t : (b^{i+1}, a^{i+1}, b^i) \mapsto a^{i+1}. \tag{6.51}$$

We have to check that s are an isomorphism with inverse t in $\mathcal{K}(\mathcal{A})$. Let us first note that

$$t \circ s = \mathrm{id}_{A^*[1]} \tag{6.52}$$

so t is a right inverse in $\mathrm{Kom}(\mathcal{A})$. For the composition in the reverse order we have

$$s \circ t : (b^{i+1}, a^{i+1}, b^i) \mapsto (-f^{i+1}a^{i+1}, a^{i+1}, 0) \tag{6.53}$$

which we need to show is homotopy equivalent to $\mathrm{id}_{C(\tau)}$. We first note that

$$\mathrm{id} - s \circ t = \begin{bmatrix} 1 & f[1] & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{6.54}$$

We see that with chain homotopy

$$h = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6.55}$$

we have

$$h\,\mathrm{d}^i_{C(\tau)} + \mathrm{d}^{i-1}_{C(\tau)} h = \begin{bmatrix} 1 & f[1] & \mathrm{d}^i_B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -\mathrm{d}^i_B \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & f[1] & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{6.56}$$

Thus $f \circ g$ is homotopy equivalent to the identity map.

Finally we need to show that

$$\begin{tikzcd}
C(f) \arrow[r, "\pi"] \arrow[d, "\mathrm{id}"] & A^*[1] \arrow[d, "s"] \\
C(f) \arrow[r, "\tau"] & C(\tau)
\end{tikzcd} \tag{6.57}$$

commutes in $\mathcal{K}(\mathcal{A})$. This follows from that

$$\begin{tikzcd}
C(f) \arrow[r, "\pi"] \arrow[d, "\mathrm{id}"] & A^*[1] \\
C(f) \arrow[r, "\tau"] & C(\tau) \arrow[u, "t"]
\end{tikzcd} \tag{6.58}$$

commutes in $\mathrm{Kom}(\mathcal{A})$ and since s is an isomorphism with inverse t in $\mathcal{K}(\mathcal{A})$. $\square$

One particular useful aspect of the triangles are that they induce long exact sequences of cohomology

$$\dots H^{i-1}(C(f)) \to H^i(A^*) \xrightarrow{H^i(f)} H^i(B^*) \to H^i(C(f)) \to H^{i+1}(A) \to \dots \tag{6.59}$$

Compare this to how short exact sequences in an abelian category generate long exact sequence of cohomology by the zig-zag/snake lemma. Thus distinguished triangles in triangulated categories have the same role as short exact sequences in abelian categories. In particular if $C(f)$ is exact we have

$$0 \to H^i(A^*) \xrightarrow{H^i(f)} H^i(B^*) \to 0 \tag{6.60}$$

making f a quasi-isomorphism.

6.1.4 The derived category

Our next step is to pass to the derived category $\mathcal{D}(\mathcal{A})$ by localizing the quasi-isomorphisms. As already stated the objects in $\mathrm{Kom}(\mathcal{A}), \mathcal{K}(\mathcal{A})$, and $\mathcal{D}(\mathcal{A})$ are the same, chain complexes. That is

$$\mathrm{Ob}(\mathrm{Kom}(\mathcal{A})) = \mathrm{Ob}(\mathcal{K}(\mathcal{A})) = \mathrm{Ob}(\mathcal{D}(\mathcal{A})). \tag{6.61}$$

Our purpose is then to explicitly describe the morphisms in $\mathcal{D}(\mathcal{A})$. Consider a quasi-isomorphism $C^* \to A^*$ and a morphism $C^* \to B^*$. Since the quasi-isomorphisms are isomorphisms in $\mathcal{D}(\mathcal{A})$, we have an induced map $A^* \to B^*$. In fact the map $C^* \to B^*$ is equal to the map $A^* \to B^*$ in $\mathcal{D}(\mathcal{A})$. The morphisms in $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(A^*, B^*)$ are then represented by roofs

$$\begin{tikzcd}
& C^* \arrow[dl, "qis"'] \arrow[dr] & \\
A^* & & B^*
\end{tikzcd} \tag{6.62}$$

Two different roofs may be equivalent in that they represent the same morphism. Thus we need to know when two such roofs are considered equal and also how to compose two morphisms in $\mathcal{D}(\mathcal{A})$.

We begin by defining when two roofs are equivalent. Two roofs

$$\begin{tikzcd} & C_1^* \arrow[dl, "qis"'] \arrow[dr] & \\ A^* & & B^* \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & C_2^* \arrow[dl, "qis"'] \arrow[dr] & \\ A^* & & B^* \end{tikzcd} \tag{6.63}$$

are equivalent if they are dominated by a third roof, i.e. there exists a commutative diagram in $\mathcal{K}(\mathcal{A})$,

$$\begin{tikzcd} & & C^* \arrow[ddll, bend right, "qis"'] \arrow[ddrr, bend left] \arrow[dl] \arrow[dr] & & \\ & C_1^* \arrow[dl] \arrow[drrr] & & C_2^* \arrow[dlll] \arrow[dr] & \\ A^* & & & & B^* \end{tikzcd} \tag{6.64}$$

where the composition map $C^* \to C_1^* \to A^*$ is a quasi-isomorphism. Note that this implies that the map $C^* \to C_1^*$ is a quasi-isomorphism. We can understand why this defines an equivalence of roofs by thinking of it as that there is a third roof

$$\begin{tikzcd} & C^* \arrow[dl, "qis"'] \arrow[dr] & \\ A^* & & B^* \end{tikzcd} \tag{6.65}$$

consistent with the two others.

We can now also begin to understand why we passed to the homotopy category of chain complexes $\mathcal{K}(\mathcal{A})$ as an intermediate step. Let us denote the composition maps $C^* \to C_1^* \to A^*$ and $C^* \to C_2^* \to A^*$ by f and g respectively. We can see that f and g are chain homotopic by the commutative diagram

$$\begin{tikzcd} & C^* \arrow[r] \arrow[dl] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & C_2^* \arrow[dl] \\ C_1^* \arrow[r] & A^* & \end{tikzcd} \tag{6.66}$$

and since f is a quasi-isomorphism so is g.

Next we consider the composition of the morphisms in eq. (6.63). The composition is represented by the commutative diagram

$$\begin{tikzcd} & & C_0^* \arrow[dl, "qis"'] \arrow[dr] & & \\ & C_1^* \arrow[dl, "qis"'] \arrow[dr] & & C_2^* \arrow[dl, "qis"'] \arrow[dr] & \\ A^* & & B^* & & C^* \end{tikzcd} \tag{6.67}$$

in $\mathcal{K}(\mathcal{A})$.

We have so far allowed us to have infinite chains in $\mathrm{Kom}(\mathcal{A})$. If we restrict ourselves to only consider bounded chains, meaning they begin and end in an infinite sequence of zeroes, we get the bounded category of chain complexes $\mathrm{Kom}^b(\mathcal{A})$. We then have the corresponding bounded categories $\mathcal{K}^b(\mathcal{A})$ and $\mathcal{D}^b(\mathcal{A})$. We now have a sufficient knowledge of derived categories to proceed with introducing stability conditions on them. Let us end by emphasising three main points of this section.

- Consider the category of chain complexes $\mathrm{Kom}(\mathcal{A})$ instead of $\mathcal{A}$.
- Quasi-isomorphisms in $\mathrm{Kom}(\mathcal{A})$ becomes isomorphisms in $\mathcal{D}(\mathcal{A})$.
- The derived category have distinguished triangles.

In particular we have seen that working with chain complexes of an abelian category is a powerful tool that among other things tell gives us the chain cohomology.

6.2 Bridgeland stability

We will now introduce Bridgeland stability on triangulated categories. In particular we will look at Bridgeland stability on the bounded derived category of coherent sheafs $\mathcal{D}^b(\mathrm{Coh}(M))$ over a Calabi-Yau manifold M. We will begin by reformulating μ-stability in a form that makes the transition to Bridgeland stability natural. Our introduction is based on lecture notes by Huybrecht in [51] and the original article by Bridgeland [22]. The reader may also find the lectures notes in [52] that also cover the material useful. [53]

6.2.1 Reformulation of μ-stability

We will now reformulate slope stability in slightly different terms leading up to Bridgeland stability. Consider a holomorphic vector bundle E or more generally a coherent sheaf over a compact Kähler manifold M.

Define the central charge of the bundle

$$Z(E) = i\,\mathrm{rk}(E) - \deg(E) = i\,\mathrm{rk}(E) - c_1(E), \tag{6.68}$$

and the phase by

$$\phi(E) = \frac{1}{\pi}\operatorname{Arg} Z(E) = \frac{1}{\pi}\operatorname{Im}\log Z(E) = -\frac{1}{\pi}\operatorname{arccot}\frac{\deg(E)}{\mathrm{rk}(E)}, \tag{6.69}$$

where the factor of π is added to make $\phi(E)$ take values in $(0,1)$. Note that here arccot is defined to take values in $(0,\pi)$ and is thus a strictly decreasing function. This allows the slope (semi)stability condition on E to be restated as the inequality

$$\phi(E') \leq \phi(E), \tag{6.70}$$

for each coherent subsheaf $E' \subset E$. We illustrate the stability condition in fig. 6.1. Note that the central charge can be written

$$Z(E) = m(E)e^{i\pi\phi(E)} = \left(\mathrm{rk}(E)^2 + \deg(E)^2\right)e^{i\pi\phi(E)}, \tag{6.71}$$

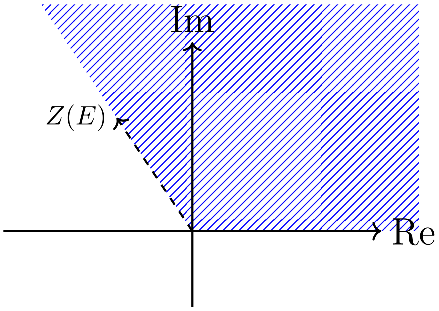

Figure 6.1: The bundle E is stable if and only if any coherent subsheaf E' have central charge $Z(E')$ in the dashed region.

making the mass $m(E) \in \mathbb{R}_{>0}$ positive and $Z(E)$ located in the upper half plane. We also have that the central charge is additive over short exact sequences, i.e.

$$0 \to F \to E \to G \to 0 \tag{6.72}$$

implies

$$Z(E) = Z(F) + Z(G). \tag{6.73}$$

Thus this relation on any short exact sequence generate the Grothendieck group $K(\mathrm{Coh}(M))$.

Consider the case when the base manifold is a projective algebraic curve C. Let us now see how slope stability can be stated in terms of the bounded derived category of coherent sheafs $\mathcal{D}^b(\mathrm{Coh}(C))$. We begin by looking at sheafs $E \in \mathrm{Coh}(C)$. In $\mathcal{D}^b(\mathrm{Coh}(C))$ such sheaf is represented by sequences with E at the zeroth degree

$$0 \to E \to 0. \tag{6.74}$$

As shown in chapter 4 we have a Harder-Narasimhan filtration of E

$$0 = E_0 \to E_1 \to E_2 \to \cdots \to E_{n-1} \to E_n = E, \tag{6.75}$$

with the quotients $\mathscr{F}_i = E_i/E_{i-1}$ satisfying

$$\phi(\mathscr{F}_1) > \phi(\mathscr{F}_2) > \cdots > \phi(\mathscr{F}_n). \tag{6.76}$$

If we instead view the filtration in terms of complexes the mapping cones of the morphisms in the filtration gives us the triangles

$$E_i^* \xrightarrow{\iota^i} E_{i+1}^* \xrightarrow{\tau^i} C(\iota^i) \xrightarrow{\pi^i} E_i^*[1]. \tag{6.77}$$

The Harder-Narasimhan property can then be formulated as the diagram

$$\begin{tikzcd}
0 = E_0 \arrow[rr, "\iota^0"] & & E_1 \arrow[rr, "\iota^1"] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[rr, "\iota^{n-1}"] & & E_n = E, \arrow[dl] \\
& C(f_0) \arrow[ul, dashed] & & C(f_1) \arrow[ul, dashed] & & & & C(f_{n-1}) \arrow[ul, dashed] &
\end{tikzcd} \tag{6.78}$$

with

$$\phi(C(f_0)) < \phi(C(f_1)) < \cdots < \phi(C(f_n)). \tag{6.79}$$

At this point the above relation tells us nothing since we have yet to define what the phase of mapping cones are. To rectify this we begin by explicitly writing up the triangles in sequence form

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & E_i \arrow[r] \arrow[d, "\iota"] & 0 \arrow[d] \\
 & 0 \arrow[r] \arrow[d] & E_{i+1} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & E_i \arrow[r, "\iota"] \arrow[d, "\mathrm{id}"] & E_{i+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E_i \arrow[r] & 0 &
\end{tikzcd} \tag{6.80}$$

We have not lost the quotient sheafs $\mathscr{F}_i$, in fact reminding ourselves of why we introduced quasi-isomorphisms in the first place we have that the mapping cone $C(\iota^i)$ is isomorphic to $\mathscr{F}_i$ since we have a quasi-isomorphism

$$\begin{tikzcd}
0 \arrow[r] & E_i \arrow[r] \arrow[d] & E_{i+1} \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & \mathscr{F}_i \arrow[r] & 0
\end{tikzcd} \tag{6.81}$$

Next let us look at an arbitrary $E^* \in \mathcal{D}^b(\mathrm{Coh}(C))$. The Harder-Narasimhan filtration on the individual sheafs then induce a Harder-Narasimhan filtration on the sequence. Explicitly the triangles formed by the mapping cone of the filtration is

$$\begin{tikzcd}
\dots \arrow[r] & E_i^{-1} \arrow[r] \arrow[d, "\iota^{-1}"] & E_i^0 \arrow[r] \arrow[d, "\iota^0"] & E_i^1 \arrow[r] \arrow[d, "\iota^1"] & \dots \\
\dots \arrow[r] & E_{i+1}^{-1} \arrow[r] \arrow[d, "\tau^{-1}"] & E_{i+1}^0 \arrow[r] \arrow[d, "\tau^0"] & E_{i+1}^1 \arrow[r] \arrow[d, "\tau^1"] & \dots \\
\dots \arrow[r] & E_i^0 \oplus E_{i+1}^{-1} \arrow[r] \arrow[d, "\pi^{-1}"] & E_i^1 \oplus E_{i+1}^0 \arrow[r] \arrow[d, "\pi^0"] & E_i^2 \oplus E_{i+1}^1 \arrow[r] \arrow[d, "\pi^1"] & \dots \\
\dots \arrow[r] & E_i^0 \arrow[r] & E_i^1 \arrow[r] & E_i^2 \arrow[r] & \dots
\end{tikzcd} \tag{6.82}$$

Note that these sequences begin and end with 0 since we work in the bounded category. We may now ask ourselves how the slope is defined for general sequences. To do this we will define concepts of hearts and slicings.

We define the heart $\mathcal{D}^{\heartsuit}(\mathrm{Coh}(M))$ as sequences with an element $E \in \mathrm{Coh}(M)$ at the zero degree of the sequence with all other degrees zero, i.e. sequences

$$0 \to E \to 0. \tag{6.83}$$

Thus we have the identification

$$\mathcal{D}^{\heartsuit} \cong \mathrm{Coh}(M). \tag{6.84}$$

This definition agrees with the heart of the standard choice t-structure given by

$$\mathcal{D}^{\geq 0} = \{E^* \mid H^i(E^*) = 0 \text{ for } i > 0\} \tag{6.85}$$

and

$$\mathcal{D}^{\leq 0} = \{E^* \mid H^i(E^*) = 0 \text{ for } i < 0\}. \tag{6.86}$$

That is the heart is

$$\mathcal{D}^{\heartsuit} = \mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}, \tag{6.87}$$

which agrees with our earlier definition of heart since any sequence with $H^i = 0$ for some degree i is quasi-isomorphic, and hence isomorphic in $\mathcal{D}^b(\mathrm{Coh}(C))$, to a sequence with zero in position i.

Next we define the slicing of the heart as

$$\mathcal{P}(\phi) = \{E^* \in \mathcal{D}^{\heartsuit}(\mathrm{Coh}(M)) \mid E^* \text{ is semistable of phase } \phi\}. \tag{6.88}$$

To deal with shifted sequences we set

$$\mathcal{P}(\phi)[1] = \mathcal{P}(\phi + 1) \tag{6.89}$$

and thus get the slicing defined for $\phi \in \mathbb{R}$. The heart can then be recovered as $\mathcal{D}^{\heartsuit}(\mathrm{Coh}(M)) = \mathcal{P}((0{,}1])$, where the choice of where to put the soft bracket is a matter of convention.

Let us also note that

Proposition 6.10. *If* $\phi_1 > \phi_2$ *and* $E \in \mathcal{P}(\phi_1)$ *and* $F \in \mathcal{P}(\phi_2)$ *then*

$$\mathrm{Hom}_{\mathcal{D}}(E{,}F) = 0. \tag{6.90}$$

This is a direct consequence of

Lemma 6.11. *If* $E, F \in \mathrm{Coh}(M)$ *are semistable with phases satisfying* $\phi(E) > \phi(F)$*, then*

$$\mathrm{Hom}(E, F) = 0. \tag{6.91}$$

In particular the lemma tells us that there is no isomorphism between objects in the slicing with different phases.

Bridgeland stability conditions are in general hard to construct. We have seen an example in terms of μ-stability on sheafs over a projective algebraic curve, however even going up to surfaces the theory quickly becomes very complicated. Some progress have been made in this area, specifically for K3 surfaces see [54] and [55]. The most interesting case to physicists are that of three dimensional Calabi-Yau manifolds, in which case no Bridgeland stability condition has yet been found.

6.2.2 Definition of Bridgeland stability

These definitions and reformulation of μ-stability may at this point seem arbitrary. However they are exactly the reformulation that agreeing with the definition of stability given by Bridgeland in [22].

Definition 6.12. *A Bridgeland stability condition* $(Z, \mathcal{P})$ *on a triangulated category* $\mathcal{D}$ *consists of a group homomorphism known as the central charge* $Z : K(\mathcal{D}) \to \mathbb{C}$ *and a full additive subcategory* $\mathcal{P}(\phi) \subset \mathcal{D}$ *for all* $\phi \in \mathbb{R}$*, satisfying*
(i) if $E \in \mathcal{P}(\phi)$ *then* $Z(E) = m(E)e^{i\pi\phi}$ *for some* $m(E) \in \mathbb{R}_{>0}$*,*
(ii) $\mathcal{P}(\phi)[1] = \mathcal{P}(\phi + 1)$*,*
(iii) if $\phi_1 > \phi_2$ *and* $A \in \mathcal{P}(\phi_1)$*,* $B \in \mathcal{P}(\phi_2)$ *then* $\mathrm{Hom}_{\mathcal{D}}(A,B) = 0$*,*
(iv) For every $E \in \mathcal{D}$ *there is an n-tuple*

$$\phi_1 > \phi_2 > \cdots > \phi_n, \tag{6.92}$$

and a collection of triangles

$$\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E, \arrow[dl] \\
& A_1 \arrow[ul, dashed] & & A_2 \arrow[ul, dashed] & & & & A_n \arrow[ul, dashed] &
\end{tikzcd} \tag{6.93}$$

with $E_i \in \mathcal{D}$ *and* $A_i \in \mathcal{P}(\phi_i)$*. We denote by* $\mathrm{Stab}(\mathcal{D})$ *the space of Bridgeland stability conditions on* $\mathcal{D}$*.*

Note in particular that (iv) is the Harder-Narasimhan property. The properties $(ii), (iii), (iv)$ is the definition of a slicing. Thus one can also formulate the definition as

Definition 6.13. *A Bridgeland stability condition* $(Z, \mathcal{P})$ *on a triangulated category* $\mathcal{D}$ *consists of a central charge* $Z : K(\mathcal{D}) \to \mathbb{C}$ *and slicing* $\mathcal{P}$ *on* $\mathcal{D}$ *such that if* $E \subset \mathcal{P}(\phi)$ *then*

$$Z(E) = m(E)e^{i\pi\phi}, \tag{6.94}$$

for some $m(E) \in \mathbb{R}_{>0}$*.*

Identically to before the (semi)stability of objects in $\mathcal{D}$ are formulated as an inequality of the phases.

Definition 6.14. *We say that* $E \in \mathcal{D}$ *is semistable in respect to* Z *if for every nonzero proper subset* $0 \neq A \subset E$

$$\phi(A) \leq \phi(E). \tag{6.95}$$

If the inequality is strict we say that E *is stable.*

6.3 The deformed-Hermitian-Yang-Mills equation

In chapter 3 we discussed how the Hermitian-Yang-Mills equation can be formulated as the zeroes of a moment map and how this led to deformed-Hermitian-Yang-Mills (dHYM) equations. Of particular interest from a physics perspective is the MMMS equation, named after Mariño. Minasian, Moore, and Strominger who derived the equation in [16]. By convention the MMMS equation is referred to as the deformed-Hermitian-Yang-Mills equation since it is the main such equation that is studied. We will follow this convention and henceforth refer of the MMMS equation as the deformed Hermitian-Yang-Mills (dHYM) equation.

Let (M, J, ω, Ω) be a Calabi-Yau threefold and let $X \subset M$ be a complex submanifold of complex dimension n. Note that this implies that (X, ω) is Kähler.

Definition 6.15. *The deformed Hermitian-Yang-Mills equation is defined as*

$$\operatorname{Im}\left(e^{-i\theta}(\omega+F)^n\right)=0, \tag{6.96}$$

where θ is a real number and F the curvature two-form.

As when we introduced ordinary Yang-Mills theory the curvature form will be defined in terms of a connection on a principal bundle, however with one significant difference. In addition to a $U(1)$ connection one-form $A \in \Omega^1(X, \mathfrak{u}(1))$ we will have a connection two-form $B \in \Omega^2(X, \mathfrak{u}(1))$. A connection is then considered to be the pair (A, B) with the curvature defined as

$$F = F_A - B = \mathrm{d}A + A \wedge A - B. \tag{6.97}$$

Note that we have written out the general form of the curvature, while in our case of an abelian gauge group the $A \wedge A$ term vanishes. We also have the integrability condition

$$F^{0,2}=0, \tag{6.98}$$

which implies that $A \in \Omega^{0,1}(X, \mathfrak{u}(1))$ and $B \in \Omega^{1,1}(X, \mathfrak{u}(1))$.

The gauge theory we have introduced where we admit higher differential forms in the connection have a natural description in terms of higher gauge theory, see [56] and [57]. However for our case where we have $U(1)$ connections the only change is the introduction of the two-form B.

In the context of string theory the B-field is known as the Kalb-Ramond field and comes as a generalisation of the electromagnetic potential when moving up in dimension from particles to higher dimensional objects such as strings. In this context the curvature is known as the modified field string and also include the constant α' known as the Regge slope or simply as the string constant,

$$\mathscr{F} = 2\pi\alpha'(F_A - B). \tag{6.99}$$

The importance for us is that the dHYM equation can be expanded in terms of α' and also that the curvature, which we shall continue to denote as F, may also depend on a B-field.

Remark 6.16. *Physically the deformed-Hermitian-Yang-Mills equation correspond to B-branes in the B-model $D(-1)$, $D1$, $D3$, and $D5$-branes which wrap manifolds C of real dimension $0, 2, 4, 6$ respectively. The supersymmetric cycles are then represented by line bundles $L \to C$ giving us $U(1)$ connections. See [9] for details.*

We can restate the dHYM equation as

$$\operatorname{Im}(\omega+F)^n = \tan\theta \operatorname{Re}(\omega+F)^n. \tag{6.100}$$

by using the relation

$$\operatorname{Im}\left(e^{-i\theta}(\omega+F)^n\right) = \cos\theta \operatorname{Im}(\omega+F)^n - \sin\theta \operatorname{Re}(\omega+F). \tag{6.101}$$

Expressed in the form in eq. (6.100) the existence of solution to the dHYM equation can be expressed as if there exists a imaginary valued (1,1)-form F such that

$$(\omega+F)^n \tag{6.102}$$

has constant argument at all points on the compact Kähler manifold (X, ω).

Since the Kähler form ω is real and the curvature F is imaginary the left side will involve only even odd terms of F while the right side even terms when expanding the parenthesis. In one dimension the dHYM equation becomes

$$F = \tan\theta\,\omega. \tag{6.103}$$

which is the Hermitian-Yang-Mills equation.

Lets remind ourselves of our aim, the stability conjecture conjecture 1.2 discussed in the introduction. For the convenience of the reader we restate it here.

Conjecture 6.17 (Conjecture 1.1 in [24])**.** *There is a Bridgeland stability condition on $\mathcal{D}^b\,\mathrm{Coh}(M)$ so that the holomorphic vector bundle E is stable if and only if it admits a deformed Hermitian-Yang-Mills metric.*

In section 6.2 we showed that slope stability can be formulated as a Bridgeland stability condition in the case of one dimensional manifolds. Thus the conjecture is shown in the case of dimension one by slope stability and the Donaldson-Uhlenbeck-Yau theorem. The two and three dimensional cases are much more complicated, in fact we noted in section 6.2 the existence of Bridgeland stability conditions in dimension are not yet known. Our approach and the approach taken by Collins and Yau in [24] is to begin by studying when dHYM equation admits a solution. Specifically since we wish to connect the dHYM equation to a Bridgeland stability condition we are after algebraic conditions for when the dHYM equation admit a solution as opposed to analytic conditions, since Bridgeland stability is formulated in the language of algebra.

Writing out the dHYM equation in two and three dimensions explicitly gives

$$2F \wedge \omega = \tan\theta(F^2 + \omega^2), \tag{6.104}$$

and

$$3F \wedge \omega^2 + F^3 = \tan\theta\left(3F^2 \wedge \omega + \omega^3\right), \tag{6.105}$$

respectively. We can truncate the dHYM equation to the lowest order F or equivalently in α' resulting in the equation

$$nF \wedge \omega^{n-1} = \tan\theta\,\omega^n, \tag{6.106}$$

which we recognise as the Hermitian Yang-Mills equation. That is in the limit of the small curvature the Hermitian Yang-Mills equation is recovered. Thus we already know that solutions to the dHYM equation should have a limit that solves the HYM equation and if conjecture 1.2 is true we would expect that the Bridgeland stability condition should reduce to slope stability in an appropriate limit.

We can now begin analysing when the dHYM equation admits a solution. Let (X, ω) be a Kähler manifold and consider a real cohomology class $[\alpha] \in H^{1,1}(X, \mathbb{R})$. As pointed out if the dHYM equation has a solution the argument is constant. Set

$$\hat{\theta} = \mathrm{Arg}\int_X (\omega + i\alpha)^n. \tag{6.107}$$

If particular if the dHYM equation has a solution we have that

$$\hat{\theta} = \mathrm{Arg}(\omega + i\alpha)^n. \tag{6.108}$$

In general we can write

$$(\omega + i\alpha)^n = r_\omega(\alpha) e^{i\Theta_\omega(\alpha)} \omega^n. \tag{6.109}$$

where $r_\omega(\alpha)$ and $\Theta_\omega(\alpha)$ are real valued smooth functions on X. The dHYM equation can then be expressed as

$$\Theta_\omega(\alpha) = \hat{\theta}, \qquad \mod 2\pi. \tag{6.110}$$

Introducing local holomorphic coordinates at a point $x \in X$

$$\omega = \frac{i}{2} \sum_{k=1}^{n} \mathrm{d}z_k \wedge \mathrm{d}\bar{z}_k, \qquad \alpha = \frac{i}{2} \sum_{k=1} \lambda_k \, \mathrm{d}z_k \wedge \mathrm{d}\bar{z}_k, \tag{6.111}$$

we get

$$\begin{gathered}(\omega + i\alpha)^n = \left(\frac{i}{2} \sum_{k=1}^{n} (1 + i\lambda_k) \, \mathrm{d}z_k \wedge \mathrm{d}\bar{z}_k \right)^n = \\ \left(\frac{i}{2} \right)^n \left(\prod_{k=1}^{n} (1 + i\lambda_k) \right) \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 \wedge \cdots \wedge \mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n = \left(\prod_{k=1}^{n} (1 + i\lambda_k) \right) \omega^n.\end{gathered} \tag{6.112}$$

Writing this in polar form in accordance with eq. (6.109) we get

$$r_\omega(\alpha) e^{i\Theta_\omega(\alpha)} = \prod_{k=1}^{n} (1 + i\lambda_k) = \prod_{k=1}^{n} \sqrt{1 + \lambda_k^2} e^{i \arctan \lambda_k} = \sqrt{\prod_{k=1}^{n} 1 + \lambda_k^2} e^{\left(i \sum_{k=1}^{n} \arctan \lambda_k\right)}, \tag{6.113}$$

which gives

$$r_\omega(\alpha) = \sqrt{\prod_{k=1}^{n} (1 + \lambda_k^2)} \quad \text{and} \quad \Theta_\omega(\alpha) = \sum_{k=1}^{n} \arctan(\lambda_k). \tag{6.114}$$

As a consequence if α is a solution to the dHYM equation the angle θ is constant and since $r_\omega(\alpha) > 0$ we have

$$\int_X (\omega + i\alpha)^n = \int_X r_\omega(\alpha) e^{i\theta} \omega^\theta \in \mathbb{R}_{>0} e^{i\theta}. \tag{6.115}$$

That is the integral is non-zero, which we formulate as

Proposition 6.18 (Lemma 2.1 in [23])**.** *If there exists solution α to the dHYM equation, then*

$$\int_X (\omega + i\alpha)^n \in \mathbb{C}^*. \tag{6.116}$$

In dimension $n = 3$ this give rise to a non-trivial cohomological obstruction on α. Explicitly we have that

$$\int_X (\omega^3 - 3\omega \wedge \alpha^2) + i \int_X (3\omega^2 \wedge \alpha - \alpha^3) \neq 0. \tag{6.117}$$

which tells us that either

$$(\omega^3 - 3\omega \wedge \alpha^2) \notin [0], \quad \text{or} \quad (3\omega^2 \wedge \alpha - \alpha^3) \notin [0]. \tag{6.118}$$

Let us look at how this can be connected to Bridgeland stability. Since Bridgeland stability is formulated in terms of algebra we want to find algebraic conditions to the solution to of the dHYM equation and formulate finding an algebraic definition of the lifted angle.

Let us assume that $[\alpha] = c_1(L)$ for some holomorphic line bundle. In this case we can write

$$\int_X (\omega + ic_1(L))^n = i^n \int_X (-i\omega + c_1(L))^n = n!i^n \int_X e^{-i\omega + c_1(L)} = n!i^n \int_X e^{-i\omega} \operatorname{ch}(L), \tag{6.119}$$

since only top forms contribute to the integration. Note also that we used $e^{\omega+\alpha} = e^\omega e^\alpha$ as a consequence of that ω and α commute, whereas otherwise we would have needed the Baker–Campbell–Hausdorff formula. This leads to the central charge on $\mathcal{D}^b(\operatorname{Coh}(L))$ considered in [23],

$$L \mapsto Z_\omega(L) = -\int_X e^{-i\omega} \operatorname{ch}(L). \tag{6.120}$$

We note that this agrees with the central charge found in the B-model of string theory. Since multiplying by i correspond to a $\pi/2$ rotation we have that

$$\operatorname{Arg} \int_X (\omega + ic_1(L))^n = \operatorname{Arg}(-i^n Z_\omega(L)) = \operatorname{Arg} Z_\omega(L) + \frac{(n-2)\pi}{2}. \tag{6.121}$$

Next consider a path $\gamma(t) : [1, \infty) \to \mathbb{C}$ defined by

$$\gamma(t) = Z_{t\omega}(L) = -\int_X e^{-it\omega} \operatorname{ch}(L). \tag{6.122}$$

Writing this out in dimension $n = 3$ we get

$$\begin{aligned}\gamma(t) = -\int_X e^{-it\omega} \operatorname{ch}(L) = -\int_X \left(1 - it\omega - \frac{t^2\omega^2}{2} + i\frac{t^3\omega^3}{6}\right)\left(1 + c_1(L) + \frac{c_1(L)^2}{2} + \frac{c_1(L)^3}{6}\right) = \\ \int_X \left(-\frac{c_1(L)^3}{6} + t^2\frac{c_1(L)\wedge\omega^2}{2}\right) + i\int_X \left(t\frac{c_1(L)\wedge\omega^2}{2} - t^3\frac{\omega^3}{6}\right).\end{aligned} \tag{6.123}$$

Similarly for $n = 1$ and $n = 2$ we have

$$\gamma(t) = -\int_X (1 - it\omega)(1 + c_1(L)) = -\int_X (c_1(L) - it\omega) = -\int_X c_1(L) + i\int_X t\omega \tag{6.124}$$

and

$$\gamma(t) = -\int_X \left(\frac{c_1(L)^2}{2} - it\omega c_1(L) - \frac{t^2\omega^2}{2}\right) = \int_X \left(-\frac{c_1(L)^2}{2} + t^2\frac{\omega^2}{2}\right) + i\int_X tc_1(L)\wedge\omega \tag{6.125}$$

respectively.

If $\gamma(t) \in C^*$, Collins-Xie-Yau defines the algebraic lifted angle θ as the winding angle of $\gamma(t)$ when t goes from ∞ to 1. Note that the requirement that $\gamma(t) \in \mathbb{C}^*$ ensures that the lifted angle is well defined, thus we need to check that this requirement is satisfied.

Let us start by analysing $n = 1$. In this case we have that $\gamma(t) \in C^*$. Next looking at $n = 2$ we still have $\gamma(t) \in C^*$. This follows from that if

$$\int_X c_1(L) \wedge \omega = 0, \tag{6.126}$$

the Hodge index theorem tells us that

$$\int_X t\omega^2 - c_1(L)^2 \leq 0. \tag{6.127}$$

For a statement and proof of this see [58].

In dimension $n = 3$, we are no longer guaranteed that $\gamma(t) \in \mathbb{C}^*$. However Collins-Xie-Yau showed that if we further assume the existence of a solution to the dHYM equation we can avoid this problem.

Proposition 6.19 (Proposition 3.3 in [23] or Proposition 8.2 in [24])**.** *Let $L \to X$ be a holomorphic line bundle over the Kähler threefold (X, ω). If L admits a solution to the dHYM equation with a lifted angle $\theta \in (\pi, \frac{3\pi}{2})$, then we have the Chern inequality*

$$\left(\int_X \omega^3\right)\left(\int_X \frac{c_1(L)^3}{6}\right) < 3\left(\int_X \frac{c_1(L)^2 \wedge \omega}{2}\right)\left(\int_X c_1(L) \wedge \omega^2\right), \tag{6.128}$$

which implies that $\gamma(t) \in \mathbb{C}^*$ for all $t \in [1,\infty)$. Furthermore the algebraic lifted angle equals the lifted angle.

We will end our discussion here, however we should note that the results towards Bridgeland stability we have reviewed here is further developed in by Collins-Yau in [24]. However even so finding a Bridgeland stability condition corresponding to the dHYM equation would still require a lot of further work. We have also only looked at dimensions $n \leq 3$ in our discussion. In higher dimensions even further complication arise as Collins-Yau notes in [24].

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