Galilean Systems in AdS/CFT

with an eye to holographic superconductors

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Abstract

In this thesis, a brief introduction is given to the AdS/CFT correspondence and its uses in condensed matter physics. To treat Galilean physics in a relativistic setting, the introduction of a Galilean spacetime is motivated and discussed. We discuss the Clifford algebra and its geometrical significance, and introduce the Pin and Spin groups and the concept of spinors. Then we proceed to find the Killing spinors with Galilean symmetry and the corresponding Killing vectors.

Acknowledgments

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Contents

Chapter 1

Introduction

One fairly recent development in theoretical physics, is the so called AdS/CFT correspondence, discussed in this thesis. The correspondence relates a difficult quantum theory and a simpler gravitational theory — in its original formulation, Maldacena (1999), a very specific quantum theory, called super Yang-Mills, in a certain limiting case. Naturally, people have been trying to extend the correspondence to other quantum theories — including condensed matter systems such as high-temperature superconductors. Many such systems (including the superconductors) are expected to have Galilean symmetry, and it is those systems that we presently examine here.

In chapter two we treat the AdS/CFT correspondence, focusing on giving a feeling for the concepts involved, without going into the technicalities too much.

In chapter three we treat Galilean spacetime — Galilean physics is usually described with space and time being separate, but if we hope to treat a Galilean system using the AdS/CFT correspondence (or any other relativistic method for that matter), we need a unified spacetime. Doing this is not entirely trivial.

In chapter four we treat Clifford algebra and spinors. The goal is to understand what the $\mathrm{Spin}(9,1)$ group is, which is expressed in terms of the Clifford algebra. We detail the geometric aspects of Clifford algebra, and motivate the introduction of spinors.

In chapter five we take those generators of $\mathrm{Spin}(9,1)$ that correspond to the Galilean transformations (as discussed in the third chapter), and find a space of spinors that transform into each other under Galilean transformations. We use gauge symmetry to bring them on a simple form, and declare them to be Killing spinors. This enables us to write down the corresponding Killing vectors, and thus gives us some restrictions on the geometry that must be satisfied in order to have Galilean invariance.

In chapter six we talk of the natural next steps: the Killing spinor equation and field strengths.

Chapter 2

The AdS/CFT correspondence

The AdS/CFT correspondence is the basis for this work. I will not attempt to go into all the details of the AdS/CFT correspondence here, but I will try to give you a rough idea of the concepts involved. AdS stands for anti-de Sitter, and CFT for conformal field theory.

2.1 Anti-de Sitter space

Anti-de Sitter space is a maximally symmetric spacetime with negative curvature, denoted AdS_n for n spacetime dimensions.

To get some intuition for what this means, let us first consider the simpler case of a two-dimensional space. If the space is to be maximally symmetric, we have three alternatives: it can be a sphere, a plane, or a hyperbolic plane. The sphere has positive curvature, and is frequently depicted embedded in a three-dimensional Euclidean space. A hyperbolic space has constant negative curvature, and cannot be embedded in three-dimensional Euclidean space.[1] We can still, however, draw flat maps of a hyperbolic space, just as we draw flat maps of the surface of a sphere, but we have to distort areas and distances when we do it. One such map, called the Poincaré disk, is illustrated in figure 2.1. The hyperbolic plane is infinitely big, so to make room for all of it on a finite map, things very far from the reference point in the middle are drawn very small. Infinity is mapped to the circumference of the Poincaré disk. The Poincaré disk is a conformal map, which means that all the angles are preserved by the mapping. Due to the negative curvature of the space, a straight line in the hyperbolic plane looks like a circular arc in the map, and the sum of angles in a triangle is always less than 180°.

[1] This is Hilbert's theorem from 1901, which is treated in e.g. Do Carmo (1976).

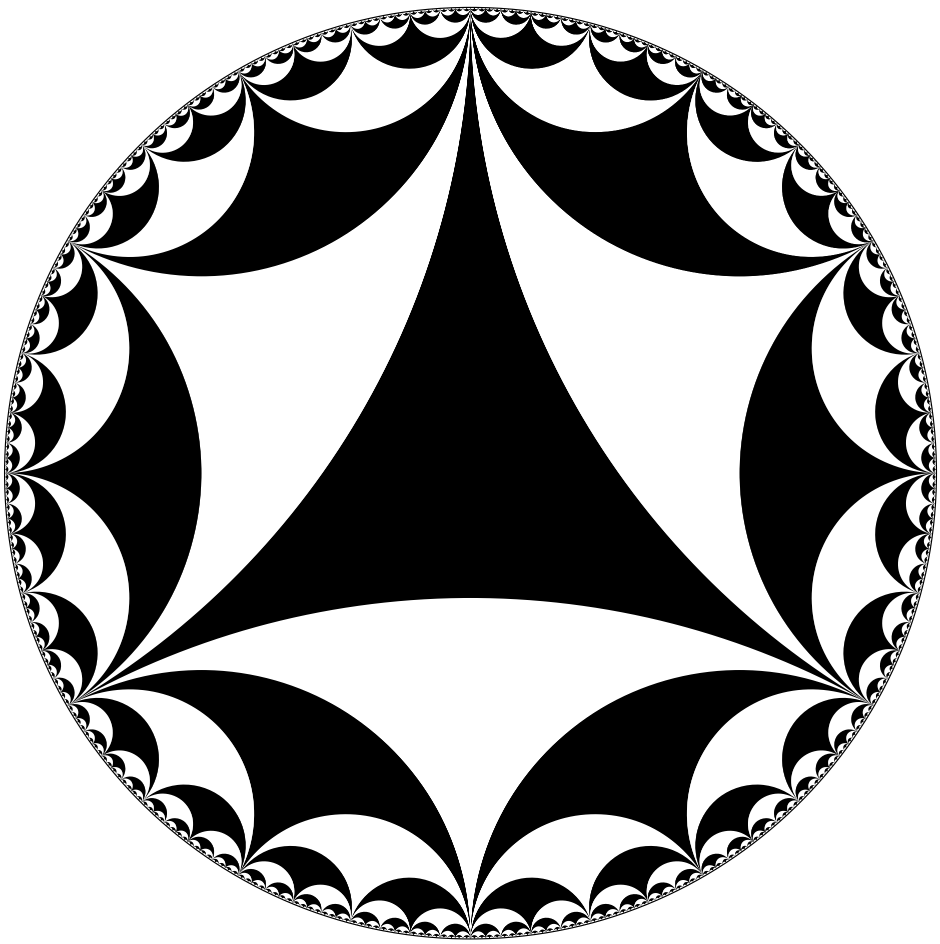

Figure 2.1: *The Poincaré disk, a map of the hyperbolic plane. The boundary is located infinitely far away from any given point in the interior. The black shapes drawn in the Poincaré disk are all ideal triangles in the hyperbolic plane, and the same thing holds for the areas left white. The triangles are congruent, which means that they can be transformed into each other by simple translations and rotations. They all have the same finite area, and infinite perimeter.*
(Image generated by Wikipedia user Saric and released into the public domain.)

This is all illustrated in figure 2.1 by using ideal triangles, i.e. triply asymptotic triangles. The sides of such triangles approach each other asymptotically — in effect, all three vertices of an ideal triangle lie on the circle at infinity.

Anti-de Sitter space is the Lorentzian analogue of hyperbolic space, i.e. a hyperbolic plane with a time dimension and possibly many spatial dimensions. It is a solution to the Einstein equations with a negative cosmological constant.

Another way of visualising anti-de Sitter space is to embed AdS in a pseudo-Euclidean space with *two* time dimensions. (Pseudo-Euclidean space with one time dimension is simply the familiar Minkowski space.) Embedded in this pseudo-Euclidean space, anti-de Sitter space takes the form of a hyperboloid of one sheet, with time in the circular direction. This is illustrated in figure 2.2. The embedding space comes with the metric

$$\mathrm{d}s^2 = -\mathrm{d}u^2 - \mathrm{d}v^2 + \mathrm{d}\vec{\boldsymbol{x}}^2 \tag{2.1}$$

where u and v are time coordinates and $\vec{\boldsymbol{x}}$ is a vector with all the spatial coordinates. Confining these coordinates to the hyperboloid

$$-u^2 - v^2 + \vec{\boldsymbol{x}}^2 = -b^2 \tag{2.2}$$

for some constant b gives us anti-de Sitter space. The constant b is sometimes called the "radius" of the anti-de Sitter spacetime (cf. Maldacena, 1999). A change of coordinates given by

$$\begin{cases} u = \sqrt{b^2 + r^2}\cos\dfrac{t}{b}, \\ v = \sqrt{b^2 + r^2}\sin\dfrac{t}{b}, \\ \vec{\boldsymbol{x}} = r\vec{\boldsymbol{n}} \qquad\qquad \text{where } |\vec{\boldsymbol{n}}| = 1, \end{cases} \tag{2.3}$$

gives us the intrinsic metric

$$\mathrm{d}s^2 = -\frac{b^2 + r^2}{b^2}\,\mathrm{d}t^2 + \frac{b^2}{b^2 + r^2}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2, \tag{2.4}$$

where Ω represents an angle or a solid angle of the appropriate dimension.

Note that the embedding of anti-de Sitter space as a one-sheet hyperboloid can only express a limited period of time — as it stands, (2.3) gives us closed timelike curves and a periodic time coordinate t. Equation (2.4) does make sense even if t is not periodic, however, so we take the view that the embedding only captures a limited time period. We want to avoid closed

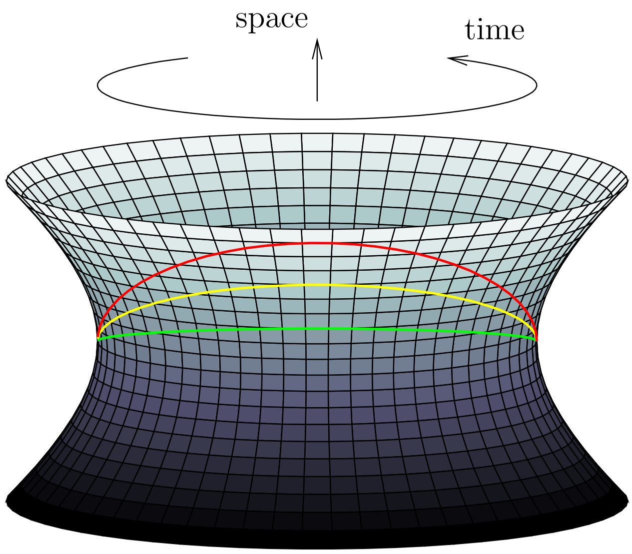

Figure 2.2: *This hyperboloid can be used to represent* AdS, *if we interpret the embedding space right. The hyperboloid is not embedded in Euclidean space, but a Lorentzian spacetime with two time directions (forming the horizontal plane in the picture) and one or more spatial directions (forming the vertical direction). The coloured lines are time-like geodesics given by equation* (2.9), *and represent the paths through space-time that an object that you throw would travel. The harder you throw it, the further away it goes in the spatial direction, but no matter how hard you throw it, it always comes back after the same fixed time.*

time-like curves, since we hope to apply the theory to real physical systems (such as condensed matter) where closed time-like curves are not possible.

Now, if we take a spatial slice of constant t in (2.4), we get a hyperbolic space, just as the one illustrated in figure 2.1. The hyperbolic space captures many interesting properties of anti-de Sitter space, but when we look at it as a spacetime, as in figure 2.2, we will uncover an even more remarkable result — that if you stand in an anti-de Sitter space and throw a stone in some arbitrary direction away from you, it will always return to you in the same finite time, no matter how hard you throw it.

To see this, we compute the time-like geodesics of anti-de Sitter space. If there are no forces acting on an object, it will follow such a geodesic. Computing the geodesics is perhaps easiest if we do it in the embedding space, with variables $X^\mu \mathrel{\hat{=}} (u, v, \vec{\boldsymbol{x}})$, rather than using the intrinsic metric.[2] The usual procedure for finding geodesics involves solving the Euler–Lagrange equations corresponding to the Lagrangian $L = \frac{1}{2}\dot{X}^2$, where the dot denotes derivatives with respect to proper time. $\dot{X}^2$ is taken to mean $\dot{X}_\mu \dot{X}^\mu$. In this particular case, however, the Lagrangian $L = \frac{1}{2}\dot{X}^2$ would give us the geodesics of the embedding space, rather than the anti-de Sitter space. We need to add the constraint that we move on anti-de Sitter space, $X^2 = -b^2$, and such constraints can be added using a Lagrange multiplier Λ. Thus we take the Lagrangian to be

$$L(X, \dot{X}) \coloneqq \tfrac{1}{2}\dot{X}^2 + \Lambda(X^2 + b^2). \tag{2.5}$$

The canonical momenta are

$$\frac{\partial L}{\partial \dot{X}^\mu} = \frac{\partial}{\partial \dot{X}^\mu}\left(\tfrac{1}{2}\eta_{\nu\lambda}\dot{X}^\nu \dot{X}^\lambda\right) = \dot{X}_\mu.$$

This Lagrangian has a manifest symmetry under rotations and boosts of the embedding space, which means that the corresponding generator is conserved:

$$k_{\mu\nu} = X_\mu \dot{X}_\nu - X_\nu \dot{X}_\mu. \tag{2.6}$$

Taking the derivative of $X^2 = -b^2$ gives us $X^\mu \dot{X}_\mu = 0$, which we use to calculate $k_{\mu\nu}k^{\mu\nu} = -2b^2\dot{X}^2$. Thus, we find that $\dot{X}^2$ is a constant of motion, because $k_{\mu\nu}$ is a constant of motion.

The Euler-Lagrange equations corresponding to (2.5) give us

$$\begin{cases} \ddot{X}_\mu = 2\Lambda X_\mu, \\ X^2 = -b^2. \end{cases}$$

[2] We are roughly following Bengtsson (1998).

The first step is always to solve for the Lagrangian multiplier Λ, which can be done by multiplying the first equation by X^μ:

$$X^\mu \ddot{X}_\mu = 2\Lambda X^2,$$

$$X^\mu \ddot{X}_\mu = -2\Lambda b^2.$$

Taking the derivative of $X^\mu \dot{X}_\mu = 0$ gives us $X^\mu \ddot{X}_\mu = -\dot{X}^2$, so

$$\Lambda = \frac{\dot{X}^2}{2b^2}.$$

Thus we get the differential equation for the geodesics:

$$\ddot{X}_\mu = \underbrace{\frac{\dot{X}^2}{b^2}}_{\text{const.}} X_\mu. \tag{2.7}$$

If $\dot{X}^2 > 0$ it describes a space-like geodesic. If $\dot{X}^2 = 0$ it is light-like. For time-like geodesics, $\dot{X}^2 < 0$, and the solution to the above equation (2.7) is of the form

$$X^\mu(\tau) = m^\mu \cos\left(\frac{\sqrt{-\dot{X}^2}}{b}\tau\right) + n^\mu \sin\left(\frac{\sqrt{-\dot{X}^2}}{b}\tau\right), \tag{2.8}$$

where $m^2 = n^2 = -b^2$ and $m_\mu n^\mu = 0$. If τ is to be the proper time, we also need $\dot{X}^2 = -1$, which can be seen directly from the metric equation $\mathrm{d}\tau^2 = -\eta_{\mu\nu}\mathrm{d}X^\mu \mathrm{d}X^\nu$ (divide by $\mathrm{d}\tau^2$). Thus the angular frequency is $\omega = 1/b$.

Let us study what happens when you throw something away from you in anti-de Sitter space. Without loss of generality (because the space is maximally symmetric), assume that we start at $X^\mu \mathrel{\hat{=}} (b, 0, \vec{\mathbf{0}})$. Inserting this into (2.8) gives us $m^\mu \mathrel{\hat{=}} (b, 0, \vec{\mathbf{0}})$, and the conditions $n^2 = -b^2$ and $m_\mu n^\mu = 0$ give us $n^\mu \mathrel{\hat{=}} (0, \pm\sqrt{b^2 + \vec{\boldsymbol{n}}^2}, \vec{\boldsymbol{n}})$. Inserting these m^μ and n^ν into (2.8), this class of solutions (related to any general solution by translation) becomes:

$$\begin{cases} u(\tau) = b\cos\omega\tau \\ v(\tau) = \sqrt{b^2 + \vec{\boldsymbol{n}}^2}\sin\omega\tau \\ \vec{\boldsymbol{x}}(\tau) = \vec{\boldsymbol{n}}\sin\omega\tau \end{cases} \tag{2.9}$$

A stationary observer would have $\vec{\boldsymbol{n}} = \vec{\mathbf{0}}$, and any object said observer throws into space, would have a nonzero $\vec{\boldsymbol{n}}$, determining how far into space (and in what direction) the object reaches before returning. Some trajectories of the form (2.9) are illustrated in figure 2.2 on page 6. Note that all objects return to $\vec{\boldsymbol{x}} = \vec{\boldsymbol{0}}$ when $\omega\tau = \pi$: everything converges to $X^\mu = (-b, 0, \vec{\boldsymbol{0}})$. No matter how hard an object is thrown, it always returns after the same finite time, both as measured by the observer and as measured by the proper time of the object thrown into space.

For small τ, $u(\tau) \simeq b$ while $v(\tau) \propto \tau$, meaning that v is the relevant time coordinate when calculating the initial speed:

$$\frac{|\vec{\boldsymbol{x}}|}{v} = \frac{|\vec{\boldsymbol{n}}|}{\sqrt{b^2 + \vec{\boldsymbol{n}}^2}}.$$

This is bounded from above by the speed of light, $c = 1$. In that limit $\vec{\boldsymbol{n}} \to \infty$. Thus, in the limit where the initial velocity goes to the speed of light, the object travels to infinity and back, returning in the *same finite time* as any other object thrown. This is not merely a coordinate effect — the boundary really is infinitely far away as measured along any spacelike geodesic, and yet light can travel from any point to infinity and back in finite time.

The points infinitely far away form the boundary of anti-de Sitter space. To make this statement mathematically precise, consider points $X^\mu \mathrel{\hat{=}} (u, v, \vec{\boldsymbol{x}})$ in the embedding space very far away from the origin, while still on the anti-de Sitter subspace. We define new coordinates $\tilde{X}^\mu = RX^\mu$, i.e. $\tilde{u} = Ru, \tilde{v} = Rv, \tilde{\vec{\boldsymbol{x}}} = R\vec{\boldsymbol{x}}$, and consider the limit $R \to \infty$. Equation (2.2) becomes

$$-\tilde{u}^2 - \tilde{v}^2 + \tilde{\vec{\boldsymbol{x}}}^2 = -\frac{b^2}{R^2} \to 0.$$

In this calculation tR works just as well as R for any $t \in \mathbb{R}$ (except zero), so we define the boundary as an equivalence class:

$$\begin{cases} -u^2 - v^2 + \vec{\boldsymbol{x}}^2 = 0, \\ (u, v, \vec{\boldsymbol{x}}) \sim (tu, tv, t\vec{\boldsymbol{x}}). \end{cases} \tag{2.10}$$

We can also find coordinates on the boundary that do not involve an explicit equivalence relation; for more details, see e.g. Petersen (1999).

This description allows us to define functions on the boundary, and do e.g. conformal field theory on the boundary. And though the boundary is infiniely far away from any point in AdS, it is connected to all the points (in the sense that you can send signals back and forth between them).

Often, when using the AdS/CFT correspondence, we want something more than simple anti-de Sitter space. We want something more general, that is still anti-de Sitter space asymptotically (so that we get a boundary as above). Typically, these more general geometries involve black holes in an anti-de Sitter background.

2.2 Correspondence to conformal field theory

Field theory in this context means quantum field theory. Conformal field theory means a quantum field theory that is invariant under the conformal transformations: boosts, rotations, translations, dilatations (scaling transformations), and the so called special conformal transformations. This is essentially an extension of the Poincaré group to include the dilatations and the special conformal transformations. The latter can be described as inversion, followed by translation, followed by inversion again. The special conformal transformations can map lines to circles and vice versa, always preserving angles.

Many, but not all, scale-invariant theories are also conformal field theories, while all conformal field theories are scale-invariant. Sometimes, when we ask if a theory is conformally invariant, the property we are interested in is really the scale invariance rather than e.g. the special conformal transformations. Scale invariance means that there is no characteristic length scale in the theory. When studying superconductors using AdS/CFT this means that we have to look for a phase transition or critical point, because the defining property of criticality is precisely that the correlation length becomes infinite — the quantum fluctuations occur at all length scales.

There is a connection between anti-de Sitter space and conformal symmetry. More specifically, if we study how the isometry group of AdS acts on the boundary, we find the conformal group.

Conversely, starting with a certain conformal field theory, Maldacena (1999) found, in the Hilbert space, the states of type IIB supergravity on $\mathrm{AdS}_5 \times S^5$, leading him to the Maldacena conjecture: "type IIB string theory on $(\mathrm{AdS}_5 \times S^5)_N$ plus some appropriate boundary conditions (and possibly also some boundary degrees of freedom) is dual to $\mathcal{N} = 4$, $d = 3 + 1$, $\mathrm{U}(N)$ super-Yang–Mills." AdS_5 is five-dimensional anti-de Sitter space, S^5 is the five-dimensional sphere, and the subscript on $(\mathrm{AdS}_5 \times S^5)_N$ "indicates the fact that the 'radii' in Planck units are proportional to $N^{1/4}$." In the limit where quantum interactions are strong, string theory becomes classical supergravity. Thus, we get a classical theory equivalent to a theory with so strong quantum interactions that it becomes very difficult indeed to compute anything.

This means that we can hope to do calculations in the classical theory and apply the results to the otherwise intractable strongly interacting quantum theory — the AdS/CFT correspondence would provide a new tool for calculations in certain strongly interacting theories. Granted, the original

Maldacena conjecture deals with a very specific quantum field theory, but there are hopes for a more general gauge/gravity duality, with applications to high-temperature superconductors, for example.

2.3 Applications to condensed matter physics

Condensed matter physics aims to describe the condensed phases of matter — the most familiar ones being solids and liquids, but the term also includes more exotic condensed phases such as superconductors and Bose–Einstein condensates. Under certain conditions, typically near a so called critical point, some condensed matter theories can be described as conformal field theories. If the theory is also strongly coupled, it becomes interesting from the perspective of the AdS/CFT correpsondence, which, if applicable, would then provide a new calculational tool for treating the theory. Strongly coupled thoeries are difficult to handle, so such a tool could provide great additional value. As has been discussed by Herzog (2009), there are superconductors with high characteristic temperature that could be approximated by just such a theory: strongly coupled, conformal field theory.

There is one problem, that we turn our attention to next. Condensed matter physics is normally non-relativistic, Galilean physics. But the boundary of anti-de Sitter space is a relativistic spacetime. In order to use the AdS/CFT correspondence to study a strongly coupled theory, said theory must somehow live on the boundary. For a system with Lorentz symmetry that is easily done, because such systems are naturally described in terms of a spacetime, but for a system with Galilean symmetry space and time are normally considered separate entities. We need a Galilean spacetime.

Chapter 3

Galilean spacetime

3.1 On non-relativistic physics

Relativistic physics is used whenever the velocities involved in a system approaches the speed of light in vacuum, c. In other systems, we can usually get a significant simplification of the problem by treating it using non-relativistic physics. Superconductors operate at very low temperatures, where relativistic effects are normally negligible. At the same time we would like to be able to use some powerful theoretical tools from relativistic theories, such as the AdS/CFT correspondence from string theory, to study them. Since these tools have been developed in a relativistic setting, it seems difficult to apply them to non-relativistic physics. The aim of this section is to show how this can be done.

The difficulty is actually not quite as bad as the terminology above would suggest. Non-relativistic physics might more accurately be called *Galilean physics* (relativistic physics would be *Lorentzian physics*), because Galilean physics is actually relativistic, in the sense that it obeys the special principle of relativity:

> "If a system of co-ordinates K is chosen so that, in relation to it, physical laws hold good in their simplest form, the *same* laws hold good in relation to any other system of co-ordinates K' moving in uniform translation relatively to K. This postulate we call the 'special principle of relativity.'"
>
> — Einstein (1916): The Foundation of the General Theory of Relativity, §1.

This was already explained in Galileo (1632, pages 165–166), though not quite as succinctly, by considering how experiments in a cabin between the decks of a large ship would be affected by the movements of the ship — "(so long as the motion is uniforme, and not fluctuating this way and that way) you shall not discern any the least alteration in all the forenamed effects".

Both Galilean physics and Lorentzian physics obey the special principle of relativity, which will make our work easier. The difference is that Galilean physics is invariant under Galilean transformation, whereas Lorentzian physics is invariant under Lorentz transformations.

3.2 On space and time

Space has three dimensions, commonly denoted x, y, z. We want to be a bit more general, though, so we say that space has d dimensions, and collect the spatial coordinates in a vector $\vec{\boldsymbol{x}}$. This way we can treat two-dimensional surfaces as well as higher-dimensional spaces using the same formalism. In Lorentzian physics space and time come together and form a $d+1$ dimensional *spacetime*, with $(d+1)$-vectors $x = x^\mu \partial_\mu \hat{=} (x^t, \vec{\boldsymbol{x}})$. In Galilean physics space and time are normally seen as being completely separate entities, rather than a unified spacetime. So how can you tell if you have a spacetime or just a space and a time?

One might look at the Lorentz transformation and argue that we have a spacetime since the transformation mixes space and time. Space and time can be "rotated" into each other. But that is not really true, since time remains a very special dimension, that cannot be fully rotated into space the way the x direction can be rotated into the y direction. The Galilean transformation,

$$\begin{cases} t \to t, \\ \vec{\boldsymbol{x}} \to \vec{\boldsymbol{x}} - \vec{\boldsymbol{V}} t, \end{cases} \tag{3.1}$$

also mixes space and time, albeit only in the transformation of the spatial coordinates, and time is special here too.

No, the real characteristic of a spacetime is the existence of an invariant scalar product.

Vectors $\vec{\boldsymbol{x}}$ in space have a scalar product, $\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{y}}$, that is invariant under rotations. This is the main reason why it is reasonable to look at space as one d dimensional entity, rather than d different and disconnected entities. A spacetime deserves to be called a spacetime if it manages to add time to this scalar product, and form a new scalar product invariant not only under rotations, but under boosts as well. If we have a Lorentzian spacetime and the boosts are Lorentz boosts, we can form the invariant scalar product

$$x \cdot y = -x^t y^t + \vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{y}}.$$

This scalar product reduces to the ordinary scalar product of space if $x^t = y^t = 0$, and it is invariant under Lorentz transformations; these are the two crucial properties that we are looking for in a scalar product for a spacetime. If the spacetime is flat we can think of a vector as the difference between two points — a vector from A to B. Then the requirement $x^t = 0$ means that x is an equal-time vector, with no time difference between A and B. (If the spacetime is not flat, think infinitesimally, $\mathrm{d}x^\mu$ instead of x^μ, throughout this discussion.) It is not unreasonable to identify an equal-time vector x with the purely spatial vector $\vec{\boldsymbol{x}}$.

Now, how would we go about to create a scalar product invariant under Galilean transformations? We would like it to reduce to the ordinary scalar product of space, $\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{y}}$, when there is no time difference, so we start by seeing how $\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{y}}$ transforms under (3.1).

$$\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{y}} \to (\vec{\boldsymbol{x}} - \vec{\boldsymbol{V}}t) \cdot (\vec{\boldsymbol{y}} - \vec{\boldsymbol{V}}t) = \underbrace{\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{y}}}_{\text{good}} \underbrace{- \vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{V}}t - \vec{\boldsymbol{y}} \cdot \vec{\boldsymbol{V}}t + \vec{\boldsymbol{V}}^2 t^2}_{\text{bad}}. \tag{3.2}$$

We got back the original product $\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{y}}$, which is good, but we got a lot of undesirable extra terms as well, so $\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{y}}$ is clearly not invariant under Galilean transformations as it stands. The task is now to change the scalar product using the time coordinate — perhaps multiply with a function of the time coordinates, or add a function of the time coordinates — in such a way as to remove the unwanted terms. But since the time coordinates transform as $t \to t$, no function of t could possibly cancel a $\vec{\boldsymbol{V}}$.

It looks like there is no such thing as a Galilean spacetime, since there is no invariant scalar product — not if we insist on the equal-time scalar product being the ordinary spatial scalar product, anyway. The time coordinates cannot save the spatial part from violating Galilean invariance, the way it was possible in the Lorentzian case.

Naturally, I would not be writing a section on Galilean spacetime if it were impossible to construct such a thing. But we need something to save the invariance of the scalar product here, and the time coordinate is not going to do the job. What we need is a new coordinate, an additional coordinate that transforms in such a way that it can enter into the scalar product and cancel all the bad terms in (3.2). While a Lorentzian spacetime (such as Minkowski space) is a $d+1$ dimensional spacetime, vectors in the Galilean spacetime are actually $d+2$ dimensional.

To see how to proceed, let us take the low-velocity limit of a Lorentzian theory.

3.3 The low-velocity limit

In the special theory of relativity, a particle is described by its coordinates x^μ and its momentum $p^\mu \mathrel{\hat{=}} (p^t, \vec{\boldsymbol{p}})$, where $E = cp^t$ is the energy of the particle. The mass m of the particle can be found from the four-momentum p^μ by $p_\mu p^\mu = -m^2c^2$. Inserting $p_\mu p^\mu = -(p^t)^2 + (\vec{\boldsymbol{p}})^2$, we get $E^2 = m^2c^4 + (\vec{\boldsymbol{p}})^2c^2$.

$$E = (m^2c^4 + (\vec{\boldsymbol{p}})^2c^2)^{\frac{1}{2}} = mc^2 \left(1 + \frac{(\vec{\boldsymbol{p}})^2}{m^2c^2}\right)^{\frac{1}{2}} =$$

$$= mc^2 \left(1 + \frac{1}{2} \cdot \frac{(\vec{\boldsymbol{p}})^2}{m^2c^2} + \mathcal{O}\left(\frac{(\vec{\boldsymbol{p}})^4}{c^4}\right)\right)$$

$$E = mc^2 + \frac{(\vec{\boldsymbol{p}})^2}{2m} + (\vec{\boldsymbol{p}})^2 \cdot \mathcal{O}\left(\frac{(\vec{\boldsymbol{p}})^2}{c^2}\right)$$

We take the low-velocity limit, $v \ll c$, by neglecting terms containing v/c or higher powers thereof. In this limit $\vec{\boldsymbol{p}} = m\vec{\boldsymbol{v}}$, and the surviving terms of E are simply

$$E = mc^2 + \frac{(\vec{\boldsymbol{p}})^2}{2m}. \tag{3.3}$$

Occasionally, people take the limit $c \to \infty$ instead of considering $v \ll c$. I have not done so, since it is convenient to be able to put $c = 1$ when working with spacetimes. Also, $c \to \infty$ messes up the rest energy term, which we would have to drop — and then the theory would not strictly be the limit of the Lorentzian theory.

So, now taking $c = 1$, our Galilean theory of a particle is characterised by $t, \vec{\boldsymbol{x}}, E$ and $\vec{\boldsymbol{p}}$. They transform under a Galilean boost as

$$\begin{cases} t \to t \\ \vec{\boldsymbol{x}} \to \vec{\boldsymbol{x}} - \vec{\boldsymbol{V}}t \\ \vec{\boldsymbol{p}} \to \vec{\boldsymbol{p}} - \vec{\boldsymbol{V}}m \\ E \to E - \vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{p}} + \frac{1}{2}m(\vec{\boldsymbol{V}})^2 \end{cases}$$

Now, in the Lorentzian theory x^μ and p^μ, indeed all vectors, transform linearly, as $x^\mu \to \Lambda^\mu{}_\nu x^\nu$ and $p^\mu \to \Lambda^\mu{}_\nu p^\nu$. But the above is not of that form. It is not a linear transformation, because the mass m appears in the transformation. The mass m can be found from E and $\vec{\boldsymbol{p}}$ and (3.3), but it is not a linear function of E and $\vec{\boldsymbol{p}}$.

The solution, following Pinski (1968), is to simply add the mass to the momentum vector p^μ. Define the momentum $(d+2)$-vector $p^\mu \coloneqq (m, E, \vec{\boldsymbol{p}}) \equiv$

$(p^t, p^s, \vec{\boldsymbol{p}})$. Naturally, these components are not entirely independent. We have

$$E = m + \frac{(\vec{\boldsymbol{p}})^2}{2m} \quad \text{or} \quad p^s = p^t + \frac{(\vec{\boldsymbol{p}})^2}{2p^t}.$$

This equation will be used to define all sorts of $(d+2)$-vectors, including x^μ: $x^s = t + (\vec{\boldsymbol{x}})^2/2t$.

Now we can form a $(d+2)$-vector p^μ for the momentum and have it transform linearly, and we get a $(d+2)$-vector x^μ for the coordinates. The extra coordinate x^s transforms in exactly the way it must in order to save our scalar product (3.2). x^s is a coordinate that transforms in a non-trivial way under Galilean boosts, so that it can fix the transformation of the scalar product of Galilean spacetime. It is not really a new physical coordinate corresponding to a new physical dimension — it is merely a mathematical tool to make the boosts linear. Its transformation is

$$x^s \to x^s - \vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{x}} + \tfrac{1}{2} V^2 x^t, \tag{3.4}$$

and the Galilean boost of any vector $\vec{\boldsymbol{u}}$ is

$$\begin{cases} u^t \to u^t, \\ u^s \to u^s - \vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{u}} + \tfrac{1}{2} V^2 u^t, \\ \vec{\boldsymbol{u}} \to \vec{\boldsymbol{u}} - \vec{\boldsymbol{V}} u^t. \end{cases} \tag{3.5}$$

For the special cases of the position $(d+2)$-vector x^μ and the momentum $(d+2)$-vector p^μ this takes the form:

$$\begin{cases} t \to t, \\ x^s \to x^s - \vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{x}} + \tfrac{1}{2} V^2 t, \\ \vec{\boldsymbol{x}} \to \vec{\boldsymbol{x}} - \vec{\boldsymbol{V}} t, \end{cases} \qquad \begin{cases} m \to m, \\ E \to E - \vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{p}} + \tfrac{1}{2} V^2 m, \\ \vec{\boldsymbol{p}} \to \vec{\boldsymbol{p}} - \vec{\boldsymbol{V}} m. \end{cases}$$

3.4 Scalar product and metric

If we want the Galilean spacetime to be a Riemannian or pseudo-Riemannian manifold, the scalar product has to take the following form:

$$x \cdot y = g_{\mu\nu} x^\mu y^\nu$$

where $g_{\mu\nu}$ is the metric tensor; cf. e.g. Rindler (2006). The metric is a set of constants, independent of the vectors x and y. (Though if we are studying vector fields, both x, y and g may depend on the point on the manifold where the scalar product is evaluated.)

We want the scalar product $x \cdot y$ to reduce to the spatial scalar product $\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{y}}$ when $x^t = x^s = y^t = y^s = 0$, so we have

$$\begin{aligned} x \cdot y = {} & g_{tt} x^t y^t + g_{ts} x^t y^s + g_{ti} x^t y^i \\ & + g_{st} x^s y^t + g_{ss} x^s y^s + g_{si} x^s y^i \\ & + g_{it} x^i y^t + g_{is} x^i y^s + \underbrace{g_{ij} x^i y^j}_{=\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{y}}} . \end{aligned}$$

The g_{ij} are assumed to be known, since they simply specify what the spatial scalar product $\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{y}}$ looks like. We are using Cartesian coordinates, and measure $g_{ij} = \mathrm{diag}(1, \ldots, 1)$. The spatial metric is measurable, because we have a physical, measurable interpretation both of $\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{y}}$ on the left hand side, and of the Cartesian coordinates x^i and y^i of the right hand side $g_{ij} x^i y^j$. Once we have figured out what the other entries in the metric have to be, it is naturally our hope that the full spacetime scalar product $x \cdot y$ can have some physical interpretation allowing us to measure all the components of the metric. That is relevant because of the Principle of Special (Galilean) Relativity — if the metric is measurable, we will measure the same metric in all coordinate systems related by Galilean transformations. The metric will be form-invariant, and the Galilean transformations an isometry in the sense of e.g. Weinberg (1972).

From this we wish to determine the other components of the metric g such that $x \cdot y$ is invariant under Galilean boosts. Under (3.5) the product $x \cdot y$ transforms as

$$\begin{aligned} x \cdot y \to {} & g_{tt} x^t y^t \\ & + g_{ts} x^t (y^s - \vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{y}} + \tfrac{1}{2} V^2 y^t) \\ & + g_{ti} x^t (y^i - V^i y^t) \\ & + g_{st} (x^s - \vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{x}} + \tfrac{1}{2} V^2 x^t) y^t \\ & + g_{ss} (x^s - \vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{x}} + \tfrac{1}{2} V^2 x^t)(y^s - \vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{y}} + \tfrac{1}{2} V^2 y^t) \\ & + g_{si} (x^s - \vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{x}} + \tfrac{1}{2} V^2 x^t)(y^i - V^i y^t) \\ & + g_{it} (x^i - V^i x^t) y^t \\ & + g_{is} (x^i - V^i x^t)(y^s - \vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{y}} + \tfrac{1}{2} V^2 y^t) \\ & + g_{ij} (x^i - V^i x^t)(y^j - V^j y^t). \end{aligned}$$

Using the distributive law and collecting the first resulting terms from each term above, we get $x \cdot y$ back. The rest of the terms must be made to cancel, by choosing $g_{\mu\nu}$ for $\mu, \nu \in \{s, t\}$ intelligently. We get one term quartic in V:

$$\tfrac{1}{4} g_{ss} V^4 x^t y^t = 0.$$

If this is to hold for all v, x, y, we must have $g_{ss} = 0$. We get two terms cubic in V:

$$g_{si}(\tfrac{1}{2}V^2 x^t)(-V^i y^t) + g_{is}(-V^i x^t)(\tfrac{1}{2}V^2 y^t) = -g_{si} V^2 V^i x^t y^t = 0.$$

Here we have used the index symmetry of the metric tensor. If this is to hold for all V, x, y we must have $g_{si} = g_{is} = 0$. We get three terms quadratic in V:

$$\tfrac{1}{2}g_{ts} V^2 x^t y^t + \tfrac{1}{2}g_{st} V^2 x^t y^t + \underbrace{g_{ij} V^i V^j}_{=V^2} x^t y^t = (g_{ts} + 1) V^2 x^t y^t = 0.$$

If this is to hold for all V, x, y we get $g_{ts} = g_{st} = -1$. This is a highly interesting result, that we will come back to in a moment. We get six terms linear in V:

$$-\underbrace{g_{ts}}_{=-1} x^t \vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{y}} - g_{ti} V^i x^t y^t - \underbrace{g_{st}}_{=-1} y^t \vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{x}} - g_{it} V^i x^t y^t - \underbrace{g_{ij} V^i x^t y^j}_{=x^t \vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{y}}} - \underbrace{g_{ij} V^j x^i y^t}_{=y^t \vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{x}}} =$$

$$= -2 g_{ti} V^i x^t y^t = 0.$$

If this is to hold for all V, x, y we get $g_{ti} = 0$.

The terms that do not depend upon V sum to $x \cdot y$, so we now have $x \cdot y \to x \cdot y$, and that without any constraints on g_{tt}.

Thus the scalar product takes the form

$$x \cdot y \equiv x_\mu y^\mu = g_{tt} x^t y^t - x^t y^s - x^s y^t + \vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{y}} \tag{3.6}$$

and the metric takes the form

$$g_{\mu\nu} \hat{=} \left(\begin{array}{cc|ccccc} g_{tt} & -1 & & & \mathbf{0} & & \\ -1 & 0 & & & & & \\ \hline & & 1 & & & & \mathbf{0} \\ & & & 1 & & & \\ & \mathbf{0} & & & \ddots & & \\ & & & & & 1 & \\ & & \mathbf{0} & & & & 1 \end{array}\right) \tag{3.7}$$

or

$$\begin{cases} g_{tt} = \text{arbitrary}, \\ g_{st} = g_{ts} = -1, \\ g_{it} = g_{ti} = 0, \\ g_{ss} = g_{si} = 0, \\ g_{ij} = \delta_{ij}. \end{cases}$$

This is what we find assuming only that the scalar product is invariant under Galilean transformations and that the Galilean transformations are an isometry in the sense of e.g. Weinberg (1972). We can, however, also try to find the scalar product in an alternative way, by taking the Galilean limit of the Lorentz scalar product:

$$-E_1E_2 + \vec{\boldsymbol{p}}_1 \cdot \vec{\boldsymbol{p}}_2 \to -m_1E_2 - m_2E_1 + m_1m_2 + \vec{\boldsymbol{p}}_1 \cdot \vec{\boldsymbol{p}}_2 \tag{3.8}$$

since

$$\begin{aligned}
E_1E_2\Big|_{\text{Lorentz}} &= \sqrt{m_1^2c^4 + \vec{\boldsymbol{p}}_1^2c^2}\sqrt{m_2^2c^4 + \vec{\boldsymbol{p}}_2^2c^2} = \\
&= m_1m_2c^4\left(1 + \frac{\vec{\boldsymbol{p}}_1^2}{m_1c^2} + \frac{\vec{\boldsymbol{p}}_2^2}{m_2c^2}\right)^{\frac{1}{2}} = \\
&= m_1m_2c^4\left(1 + \frac{1}{2}\left(\frac{\vec{\boldsymbol{p}}_1^2}{m_1^2c^2} + \frac{\vec{\boldsymbol{p}}_2^2}{m_2^2c^2}\right) + \mathcal{O}\left(\frac{1}{c^4}\right)\right) \simeq \\
&\simeq m_2c^2\cdot\frac{\vec{\boldsymbol{p}}_1^2}{2m_1^2} + m_1c^2\cdot\frac{\vec{\boldsymbol{p}}_2^2}{2m_2^2} + m_1m_2c^4 = \\
&\overset{(3.3)}{=} m_2c^2\left(E_1 - m_1c^2\right) + m_1c^2\left(E_2 - m_2c^2\right) + m_1m_2c^4\Big|_{\text{Galilean}} = \\
&= m_1c^2E_2 + m_2c^2E_1 - m_1m_2c^4\Big|_{\text{Galilean}}
\end{aligned}$$

The subscripts "Lorentz" and "Galilean" refer to the definition of energy used — the Galilean energy is defined by (3.3) and is the Lorentz energy in the Galilean limit (including the mass term). Taking $c = 1$ above, we get (3.8). (We always keep c in our expressions when taking the Galilean limit.)

The right hand side of (3.8) can be interpreted as the scalar product of the $(d+2)$-momenta $p_1^\mu \mathrel{\hat{=}} (m_1, E_1, \vec{\boldsymbol{p}}_1)$ and $p_2^\mu \mathrel{\hat{=}} (m_2, E_2, \vec{\boldsymbol{p}}_2)$,

$$g_{\mu\nu}p_1^\mu p_2^\nu = -m_1E_2 - m_2E_1 + m_1m_2 + \vec{\boldsymbol{p}}_1 \cdot \vec{\boldsymbol{p}}_2,$$

which gives us the metric (3.7) with $g_{tt} = +1$. Thus $g_{tt} = +1$, while not strictly necessary to make an invariant scalar product, can be considered more natural than other choices.

For an overview of Galilean tensor calculus, see Pinski (1968).

3.5 The Galilean spacetime is Lorentzian

If we take $g_{tt} = 0$ in (3.7), we obtain the metric of Minkowski space in so called *light-cone coordinates*. If we take a vector $x = x^\mu\partial_\mu = x^0\partial_0 + x^1\partial_1 + \cdots + x^d\partial_d$

in Minkowski space, we can define

$$\begin{cases} x^+ = \frac{1}{\sqrt{2}}\left(x^0 + x^1\right), \\ x^- = \frac{1}{\sqrt{2}}\left(x^0 - x^1\right), \end{cases} \qquad \Leftrightarrow \qquad \begin{cases} x^0 = \frac{1}{\sqrt{2}}\left(x^+ + x^-\right), \\ x^1 = \frac{1}{\sqrt{2}}\left(x^+ - x^-\right). \end{cases} \tag{3.9}$$

This is the sign convention used by e.g. Zwiebach (2009), but beware that some authors prefer the opposite sign convention in the definition of x^-, so that $x^- = \frac{1}{\sqrt{2}}\left(x^1 - x^0\right)$ instead.

We can write the vector x in light-cone coordinates: $x = x^+\partial_+ + x^-\partial_- + x^2\partial_2 + \cdots + x^d\partial_d$. The metric becomes

$$\mathrm{d}s^2 = -(\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2 + \cdots (\mathrm{d}x^d)^2 = -2\,\mathrm{d}x^+\mathrm{d}x^- + (\mathrm{d}x^2)^2 + \cdots (\mathrm{d}x^d)^2$$

or

$$g_{\mu\nu} \doteq \left(\begin{array}{cc|ccccc} 0 & -1 & & & \mathbf{0} & & \\ -1 & 0 & & & & & \\ \hline & & 1 & & & & \mathbf{0} \\ & & & 1 & & & \\ & \mathbf{0} & & & \ddots & & \\ & & & & & 1 & \\ & & \mathbf{0} & & & & 1 \end{array}\right)$$

So, the Galilean spacetime can be thought of as a Lorentzian spacetime in light-cone coordinates (provided that we take $g_{tt} = 0$). They differ in dimensionality, in that a D dimensional spacetime with the above metric corresponds to either a Galilean space with spatial dimension $d = D - 2$, or a Lorentzian spacetime with spatial dimension $d = D - 1$. These two pictures are actually mathematically equivalent. When we wish to think Galilean, we take x^+ to be the Galilean time coordinate x^t, and x^- to be the strange, extra Galilean coordinate x^s, and think of the rest as the spatial coordinates. When we wish to think of the same spacetime in Lorentzian terms, we take $x^0 = \frac{1}{\sqrt{2}}\left(x^+ + x^-\right)$ to be the Lorentzian time coordinate, and we get an extra spatial coordinate $x^1 = \frac{1}{\sqrt{2}}\left(x^+ - x^-\right)$. When we wish to think Galilean, we take p^+ to be the mass m, and p^- to be the Galilean energy, and think of the rest of p^μ as the Galilean spatial momentum. When we wish to think of the same spacetime in Lorentzian terms, we take p^0 to be the Lorentzian energy, and get an extra Lorentzian component p^1 of the spatial momentum.

The result of this is that we have managed to embed a system with Galilean symmetry in a mathematical framework with Lorentzian symmetry. The Galilean spacetime is Lorentzian. This enables us to use the entire apparatus of relativistic models on systems with Galilean relativity, and raises the hope that we might be able to use the AdS/CFT correspondence to study e.g. superconductors at so called “non-relativistic” temperatures.

We get the alternative sign convention by letting $x^- \to -x^-$ in the definition (3.9). In this sign convention, the metric is

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + \cdots (dx^d)^2 = +2\, dx^+ dx^- + (dx^2)^2 + \cdots (dx^d)^2$$

or

$$g_{\mu\nu} \doteq \left(\begin{array}{cc|ccccc} 0 & +1 & & & \mathbf{0} & & \\ +1 & 0 & & & & & \\ \hline & & 1 & & & & \mathbf{0} \\ & & & 1 & & & \\ & \mathbf{0} & & & \ddots & & \\ & & & & & 1 & \\ & & \mathbf{0} & & & & 1 \end{array}\right). \tag{3.10}$$

Having only positive numbers in the metric tensor makes many calculations somewhat easier, particularly when we need to raise and lower indices a lot. Having a non-diagonal metric tensor is bad enough as it is, so unless otherwise noted we will use this positive-metric convention for the light-cone coordinates from now on. Letting $x^- \to -x^-$ does have the minor disadvantage, however, that p^- will no longer be the energy E. We will have $p^- = -E$ in the positive-metric convention.

Chapter 4

Clifford Algebra and Spinors

4.1 Clifford algebra

This section builds heavily on Lounesto (2001).

4.1.1 Introducing the Clifford algebra

In superstring theory, we work in 10 dimensions. Rather than treating the entire spacetime manifold, we concentrate on the tangent space. Let $V = \mathbb{R}^{(9,1)}$ be this space. (That's Minkowski space with nine spatial dimensions, with positive eigenvalues in the metric, and one time dimension, with negative eigenvalue in the metric.) We introduce an orthonormal basis $\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_9$, where $\boldsymbol{e}_0$ is along the time direction. $V = \operatorname{span}_{\mathbb{R}}\{\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_9\}$.

We want to define a product of vectors. The first thing that comes to mind is the scalar product, but we want to be a bit more general, while still retaining some of the key properties of the scalar product. We want to define the *Clifford product*, sometimes called the *geometric product*. We want it to obey the usual laws of multiplication: we want associativity, distributivity, multiplication with a scalar, and that the square of a vector is the squared length of that vector. Consider the following:

$$\boldsymbol{v}^2 = (\alpha \boldsymbol{e}_1 + \beta \boldsymbol{e}_2)^2 = \alpha^2 \boldsymbol{e}_1^2 + \alpha\beta \boldsymbol{e}_1 \boldsymbol{e}_2 + \alpha\beta \boldsymbol{e}_2 \boldsymbol{e}_1 + \beta^2 \boldsymbol{e}_2^2$$

If we are using an orthonormal basis, we have $\boldsymbol{e}_1^2 = \boldsymbol{e}_2^2 = 1$, and we want $\boldsymbol{v}^2 = \alpha^2 + \beta^2$. Using the expression above, we must have $\boldsymbol{e}_1\boldsymbol{e}_2 = -\boldsymbol{e}_2\boldsymbol{e}_1$. If we insist that our product be commutative, then $\boldsymbol{e}_1\boldsymbol{e}_2 = 0$ and we get the ordinary scalar product. If we do not insist on commutativity, we get the Clifford product. Now $\boldsymbol{e}_1\boldsymbol{e}_2 = \boldsymbol{e}_{12}$, an object we call a *bivector*. In three dimensions, a bivector formed of two vectors is dual to the ordinary cross product of the two vectors. You can think of a bivector as an oriented area.

In general, for vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ we have the Clifford product

$$\boldsymbol{uv} = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \wedge \boldsymbol{v}, \tag{4.1}$$

where $\boldsymbol{u} \cdot \boldsymbol{v}$ is the ordinary scalar product, and $\boldsymbol{u} \wedge \boldsymbol{v}$ is the *exterior product* or *wedge product*. If we want to emphasise that the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ can be multiplied using (4.1), we call them *Clifford vectors*. Equation (4.1) works even when we include $\boldsymbol{e}_0$, which squares to -1. We have $\boldsymbol{e}_{12} = \boldsymbol{e}_1\boldsymbol{e}_2 = \boldsymbol{e}_1 \wedge \boldsymbol{e}_2$. It is possible to define even higher order *multivectors* in the same manner, e.g. $\boldsymbol{e}_{125} = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_5 = \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_5$. The wedge product is antisymmetric for Clifford vectors. It is symmetric for even multivectors, i.e. multivectors composed of an even number of Clifford vectors, since e.g. $\boldsymbol{e}_{12} \wedge \boldsymbol{e}_5 = \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_5 = -\boldsymbol{e}_1 \wedge \boldsymbol{e}_5 \wedge \boldsymbol{e}_2 = \boldsymbol{e}_5 \wedge \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 = \boldsymbol{e}_5 \wedge \boldsymbol{e}_{12}$.

We can relate the Clifford product $\boldsymbol{uv}$ to the scalar product, by using the fact that the scalar product is the symmetric part of the Clifford product (4.1):

$$\boldsymbol{uv} + \boldsymbol{vu} = 2\boldsymbol{u} \cdot \boldsymbol{v}. \tag{4.2}$$

Written in terms of the base elements $\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_9$, where $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = g_{ij}$, we get

$$\boldsymbol{e}_i\boldsymbol{e}_j + \boldsymbol{e}_j\boldsymbol{e}_i = 2g_{ij}. \tag{4.3}$$

This relation defines the Clifford algebra, more or less. To be precise, "the Clifford algebra is by definition the associative algebra that is freely generated by a unit element $\mathbf{1}$ and the basis elements $\boldsymbol{e}_i$, $i = 1, 2, \ldots, n$, modulo the anti-commutation relations $\boldsymbol{e}_i\boldsymbol{e}_j + \boldsymbol{e}_j\boldsymbol{e}_i = 2g(\boldsymbol{e}_i, \boldsymbol{e}_j)\mathbf{1}$." (Fuchs and Schweigert, 2003, section 20.2). Freely generated simply means that the Clifford algebra is spanned by all products of all the basis vectors, with real coefficients (when we treat it as a real algebra). Taking it modulo the anti-commutation relations means that we take the freely generated algebra and use the relation (4.3) to identify $\boldsymbol{e}_2\boldsymbol{e}_1$ with $-\boldsymbol{e}_1\boldsymbol{e}_2$, and to identify $\boldsymbol{e}_1\boldsymbol{e}_1$ with the unit element of the algebra, which we denote $\mathbf{1}$, and so on. We use the symbol $\mathcal{C}\ell_{p,q}$ to denote the Clifford algebra corresponding to a vector space with p spatial dimensions and q time dimensions; we are primarily interested in $p = 9$ spatial dimensions and $q = 1$ time dimensions, but $p = 3$, $q = 1$ is also of interest. (If $q = 0$ we normally omit it: $\mathcal{C}\ell_p \coloneqq \mathcal{C}\ell_{p,0}$.) Also, we normally consider both the vectors and the scalars to be a part of the Clifford algebra ($\mathbb{R}^{(p,q)} \subset \mathcal{C}\ell_{p,q}$ and $\mathbb{R} \subset \mathcal{C}\ell_{p,q}$, respectively), so that there is no need to distinguish between $1 \in \mathbb{R}$ and $\mathbf{1} \in \mathcal{C}\ell_{p,q}$. $\{1, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_{12}\}$ forms a basis for $\mathcal{C}\ell_2$, and $\{1, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_{12}, \boldsymbol{e}_{13}, \boldsymbol{e}_{23}, \boldsymbol{e}_{123}\}$ forms a basis for $\mathcal{C}\ell_3$. In general, we choose a basis consisting of products of zero or more basis vectors $\boldsymbol{e}_i$, sorted in ascending order using (4.3), which also ensures that each basis vector appears at most once in the basis elements of $\mathcal{C}\ell_{p,q}$. If the space is n dimensional, this gives us 2^n basis elements, so the dimension of $\mathcal{C}\ell_{p,q}$ is 2^{p+q}.

We will later discuss explicit representations of the Clifford algebra in terms of matrices or operators (section 4.2), but before we do that it is instructive to discuss some of the geometrical uses of the Clifford product and get a feel for the geometrical significance of the Clifford algebra. But first, I'd like to introduce some mathematical concepts that we will be using.

The degree of a multivector. The degree of a basis element (such as $\boldsymbol{e}_{234}$) is simply the number of basis vectors it contains: $\boldsymbol{e}_{234} = \boldsymbol{e}_2\boldsymbol{e}_3\boldsymbol{e}_4$ has degree three (it is a trivector). An element x of the Clifford algebra is called pure of degree k, if it can be written as a linear combination of basis elements of degree k. x is called *even* if it can be written as a linear combination of basis elements of even degree, and we write $x \in \mathcal{C}\ell^+_{p,q}$. Similarly $\mathcal{C}\ell^-_{p,q}$ are the elements with odd degree. A general element x of the Clifford algebra can of course be a sum of both even and odd parts.

The grade involution takes an element x and changes the sign of the odd terms: $x \to \hat{x} = -x$ if $x \in \mathcal{C}\ell^-$ and $x \to \hat{x} = x$ if $x \in \mathcal{C}\ell^+$, and $x \to \hat{x} = x_{\text{even}} - x_{\text{odd}}$ if $x = x_{\text{even}} + x_{\text{odd}}$ to begin with ($x_{\text{even}} \in \mathcal{C}\ell^+$ and $x_{\text{odd}} \in \mathcal{C}\ell^-$). The word *involution* simply means that the operation is its own inverse; if we apply the grade involution twice to the same element, we get back the original element.

The reversion $x \to x^t$ simply reverses the order of the basis vectors: $\boldsymbol{e}^t_{12} = \boldsymbol{e}_{21}$, $\boldsymbol{e}^t_{123} = \boldsymbol{e}_{321}$ and so on. This is useful when defining the norm $|x|$ of an element of the Clifford algebra.

$$x^t = (-1)^{k(k-1)/2}x, \quad \text{where } x \text{ is pure with degree } k.$$

The norm $|x|$ of a Clifford vector $\boldsymbol{x}$ is defined by $|\boldsymbol{x}|^2 = \boldsymbol{x}^2$. Thus the squared norm of $\boldsymbol{x} = \alpha\boldsymbol{e}_1 + \beta\boldsymbol{e}_2$ is $\alpha^2 + \beta^2$ as expected. The norm of a vector is the length of that vector. Analogously, we define the norm of a bivector to be the area of that bivector — remember that a bivector can be thought of as an oriented area. So we want the norm of a bivector to behave the same way, e.g. $|\alpha\boldsymbol{e}_{12}|^2 = \alpha^2$, but

$$(\alpha\boldsymbol{e}_{12})^2 = \alpha^2\boldsymbol{e}_{12}\boldsymbol{e}_{12} = -\alpha^2\boldsymbol{e}_{12}\boldsymbol{e}_{21} = -\alpha^2\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_2\boldsymbol{e}_1 = -\alpha^2.$$

The problem is that we have to reverse the order of the basis vectors in the basis elements when computing x^2, and that can introduce a minus sign. The solution is to define the norm with one of the factors already reversed, using the reversal operator: $|x|^2 = xx^t$ works for all elements x, pure of degree k. For general x, xx^t need not be a pure scalar, so we define the norm to be the scalar part:

$$|x|^2 = \langle xx^t \rangle_0, \tag{4.4}$$

where $\langle\rangle_0$ denotes the part of degree 0, i.e. the scalar part.

This means that the norm of

$$x = x_0 + x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_{12}\boldsymbol{e}_{12} + x_{13}\boldsymbol{e}_{13} + x_{123}\boldsymbol{e}_{123}$$

is (the square root of)

$$|x|^2 = x_0^2 + x_1^2 + x_2^2 + x_{12}^2 + x_{13}^2 + x_{123}^2.$$

The exterior algebra is closely related to the Clifford algebra. Indeed, the Clifford algebra may be defined in terms of the exterior algebra. The exterior algebra $\bigwedge \mathbb{R}^{p,q}$ is the unital associative algebra generated by the exterior product of vectors in $\mathbb{R}^{p,q}$. (That the algebra is unital, means that $1 \in \bigwedge \mathbb{R}^{p,q}$.) We can write

$$\bigwedge \mathbb{R}^{p,q} = \underbrace{\bigwedge^0 \mathbb{R}^{p,q}}_{=\mathbb{R}} \oplus \underbrace{\bigwedge^1 \mathbb{R}^{p,q}}_{=\mathbb{R}^{p,q}=\{\boldsymbol{v}\}} \oplus \underbrace{\bigwedge^2 \mathbb{R}^{p,q}}_{=\{\boldsymbol{u}\wedge\boldsymbol{v}\}} \oplus \underbrace{\bigwedge^3 \mathbb{R}^{p,q}}_{=\{\boldsymbol{u}\wedge\boldsymbol{v}\wedge\boldsymbol{w}\}} \oplus \cdots \oplus \bigwedge^{p+q} \mathbb{R}^{p,q}.$$

You can see that we may use the same basis elements for $\bigwedge \mathbb{R}^{p,q}$ as we do for $\mathcal{C}\ell_{p,q}$. All that is needed to define $\mathcal{C}\ell_{p,q}$ in terms of $\bigwedge \mathbb{R}^{p,q}$ is a definition of the Clifford product, in terms of the exterior algebra. To do this, we introduce the interior product, or left contraction, $u \lrcorner v$ for $u, v \in \bigwedge \mathbb{R}^{p,q}$. We define $u \lrcorner v$ in the following way:

$$\boldsymbol{x} \lrcorner \boldsymbol{y} = \boldsymbol{x} \cdot \boldsymbol{y} \quad \text{for } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{p,q}, \tag{4.5}$$

$$\boldsymbol{x} \lrcorner (u \wedge v) = (\boldsymbol{x} \lrcorner u) \wedge v + \hat{u} \wedge (\boldsymbol{x} \lrcorner v), \tag{4.6}$$

$$(u \wedge v) \lrcorner w = u \lrcorner (v \lrcorner w). \tag{4.7}$$

Now we can define the Clifford product of $\boldsymbol{x} \in \mathbb{R}^{p,q}$ and $u \in \bigwedge \mathbb{R}^{p,q}$ by

$$\boldsymbol{x}u = \boldsymbol{x} \lrcorner u + \boldsymbol{x} \wedge u \tag{4.8}$$

and extend this product by linearity and associativity to all of $\bigwedge \mathbb{R}^{p,q}$ (Lounesto, 2001, section 14.2). Provided with the Clifford product, $\bigwedge \mathbb{R}^{p,q}$ becomes $\mathcal{C}\ell_{p,q}$. Using (4.8) gives the same result as (4.1) used in terms of the basis vectors, as we did previously. Thinking in terms of basis vectors $\boldsymbol{e}_i \wedge$ would be like a creation operator, adding $\boldsymbol{e}_i$ to the operand ($\boldsymbol{e}_2 \wedge \boldsymbol{e}_{13} = -\boldsymbol{e}_{123}$), and $\boldsymbol{e}_i \lrcorner$ would be like an annihilation operator, removing $\boldsymbol{e}_i$ from the operand ($\boldsymbol{e}_2 \lrcorner \boldsymbol{e}_{123} = -\boldsymbol{e}_{13}$).

4.1.2 Rotations and reflections

We have already studied the Clifford product of two vectors, in (4.1). The result is a scalar plus a bivector, where the scalar is the ordinary scalar product of the vectors, and the bivector represents an oriented area (rather like the cross product). The next step is to study the Clifford product of three vectors. Depending on the vectors involved the product can signify different things, so I want to draw your attention to a special case: the Clifford product $\boldsymbol{xyx}^{-1}$. Here $\boldsymbol{x}^{-1}$ is a Clifford vector such that $\boldsymbol{xx}^{-1} = 1$, which can be constructed as

$$\boldsymbol{x}^{-1} = \frac{\boldsymbol{x}}{\boldsymbol{x}^2}. \tag{4.9}$$

$\boldsymbol{x}^2$ is a scalar — the squared length of the vector $\boldsymbol{x}$ — so there is no problem in dividing by $\boldsymbol{x}^2$.

You should be able to see that a product of three Clifford vectors is a vector plus a trivector. As it turns out, $\boldsymbol{xyx}^{-1}$ is just a vector. Let $\boldsymbol{y} = \boldsymbol{y}_\parallel + \boldsymbol{y}_\perp$, where $\boldsymbol{y}_\parallel$ is parallel to $\boldsymbol{x}$ and $\boldsymbol{y}_\perp$ is perpendicular to $\boldsymbol{x}$:

$$\begin{aligned}\boldsymbol{xyx}^{-1} = \frac{1}{\boldsymbol{x}^2}\boldsymbol{xyx} = \frac{1}{\boldsymbol{x}^2}\boldsymbol{x}\left(\boldsymbol{y}_\parallel + \boldsymbol{y}_\perp\right)\boldsymbol{x} = \frac{1}{\boldsymbol{x}^2}\left(\boldsymbol{xy}_\parallel\boldsymbol{x} + \boldsymbol{xy}_\perp\boldsymbol{x}\right) = \\ = \frac{1}{\boldsymbol{x}^2}\left(\boldsymbol{x}^2\boldsymbol{y}_\parallel - \boldsymbol{x}^2\boldsymbol{y}_\perp\right) = \boldsymbol{y}_\parallel - \boldsymbol{y}_\perp.\end{aligned} \tag{4.10}$$

Thus $\boldsymbol{y} \to \boldsymbol{xyx}^{-1}$ is a *reflection* through a line along direction $\boldsymbol{x}$. If we instead wanted a reflection in a plane (or hyperplane) perpendicular to $\boldsymbol{x}$, we would make the transformation $\boldsymbol{y} \to -\boldsymbol{xyx}^{-1}$.

$$\text{Reflection in plane } \perp\boldsymbol{x}: \qquad y \to -\boldsymbol{xyx}^{-1}. \tag{4.11}$$

The reflection (4.11) lets us reflect a vector in any plane (or hyperplane) that we want. It is, however, not unique. $\boldsymbol{x}$ and $2\boldsymbol{x}$ give the same reflection, for example. We therefore usually impose the normalisation condition $\boldsymbol{x}^2 = 1$ (or $\boldsymbol{x}^2 = \pm 1$ in a Lorenzian spacetime). This removes most of the ambiguity, but we can't get around the fact that $\boldsymbol{x}$ and $-\boldsymbol{x}$ give rise to the same reflection.[1]

We can of course do more than one reflection as well. If we reflect in a hyperplane perpendicular to $\boldsymbol{a}$, and then in a hyperplane perpendicular to $\boldsymbol{b}$, we have the transformation $\boldsymbol{y} \to \boldsymbol{baya}^{-1}\boldsymbol{b}^{-1}$: More generally, we can

[1]Reflection through a line along a null direction is not possible. If $\boldsymbol{x}^2 = 0$, no inverse exists. Also, a vector parallel to a null vector will also be orthogonal to it, so defining $\boldsymbol{y}_\parallel$ and $\boldsymbol{y}_\perp$ gets troublesome, making it difficult to define what we mean by reflection in a null vector.

study $\boldsymbol{y} \to s\boldsymbol{y}\hat{s}^{-1}$, where s can be any Clifford element such that $s\boldsymbol{y}\hat{s}^{-1}$ exists and is a vector. We are considering elements s that have either odd or even degree, so $\hat{s} = \pm s$. This way, we respect the minus sign of (4.11). Note what happens to the length of the vector $\boldsymbol{y}$ when we apply the transformation $\boldsymbol{y} \to s\boldsymbol{y}\hat{s}^{-1} = \pm s\boldsymbol{y}s^{-1}$:

$$\boldsymbol{y}^2 \to s\boldsymbol{y}s^{-1}s\boldsymbol{y}s^{-1} = s\boldsymbol{y}^2s^{-1} = \begin{bmatrix} \boldsymbol{y}^2 \in \mathbb{R}, \\ \text{commutes} \end{bmatrix} = \boldsymbol{y}^2ss^{-1} = \boldsymbol{y}^2.$$

The transformation $\boldsymbol{y} \to s\boldsymbol{y}\hat{s}^{-1}$ leaves the length of vectors intact. This is, in other words, an orthogonal transformation. Given the normalisation requirement $ss^t = \pm 1$, which is the natural generalisation to our $\boldsymbol{x}^2 = \pm 1$ that we had when we only did reflections in one hyperplane, these transformations form the group $\mathrm{Pin}(p, q)$:

$$\mathrm{Pin}(p, q) = \{s \in \mathcal{C}\ell^+_{p,q} \cup \mathcal{C}\ell^-_{p,q} | ss^t = \pm 1 \text{ and } \forall \boldsymbol{x} \in \mathbb{R}^{p,q}, s\boldsymbol{x}\hat{s}^{-1} \in \mathbb{R}^{p,q}\}.$$

(The $s \in \mathcal{C}\ell^+_{p,q} \cup \mathcal{C}\ell^-_{p,q}$ means that we study either an even number of reflections, or an odd number of reflections, not some strange combination of both.)

So we have a group $\mathrm{Pin}(p, q)$ of orthogonal transformations. Is it *the* group of orthogonal transformations, $\mathrm{O}(p, q)$? It turns out it is not. As you will remember, $s \in \mathrm{Pin}(p, q)$ and $-s \in \mathrm{Pin}(p, q)$ both correspond to the same transformation in $\mathrm{O}(p, q)$. $\mathrm{Pin}(p, q)$ is a double cover of $\mathrm{O}(p, q)$.

Rotations are also orthogonal transformations, and may be the first ones that come to mind. Rotations can actually be constructed in terms of reflections: two reflections becomes a rotation. This is perhaps easiest to see in two dimensions, explicitly in terms of the rotation and reflection matrices. The same argument applies in more dimensions, since the two reflection vectors (perpendicular to the hyperplanes we reflect in) will always span a plane where the two-dimensional arguments apply. Two reflections means $s \in \mathcal{C}\ell^+$, and we get the rotation group $\mathrm{Spin}(p, q)$:

$$\mathrm{Spin}(p, q) = \{s \in \mathcal{C}\ell^+_{p,q} | ss^t = \pm 1 \text{ and } \forall \boldsymbol{x} \in \mathbb{R}^{p,q}, s\boldsymbol{x}s^{-1} \in \mathbb{R}^{p,q}\}. \qquad (4.12)$$

This is a double cover of the special orthogonal group $\mathrm{SO}(p, q)$ (s and $-s$ correspond to the same rotation in $\mathrm{SO}(p, q)$). For $p = 3$ and $q = 0$ we get the group $\mathrm{Spin}(3)$, which is isomorphic to $\mathrm{SU}(2)$, the special unitary group, which you may know from quantum mechanics. This is known as an *accidental isomorphism*, which means that you can't count on the Spin group always being a special unitary group in higher dimensions (cf. Fuchs and Schweigert, 2003, section 20.10).

Relativity complicates matters. Two reflections does not necessarily become a rotation if we are working in Minkowski-space. In Minkowski-space, a reflection might actually be a time reversal. A time reversal followed by a spatial reflection does not equal a rotation. The Lorentz group SO(p,q) breaks up into four disconnected subsets, as discussed in e.g. Peskin and Schroeder (1995, section 3.6): they can be proper (no spatial reflection) or improper (spatial reflection), orthochronous (no time reversal) or nonorthochronous (time reversal). Going from $\mathrm{Pin}(p,q)$ to $\mathrm{Spin}(p,q)$ removes the improper orthochronous transformations and the proper nonorthochronous transformations (i.e. reflections in one axis), but we are left both with the proper, orthochronous transformations (rotations and boosts), as well as the improper, nonorthochronous transformations. The proper, orthochronous transformations are more physical[2], so we define the group $\mathrm{Spin}_+(p,q)$ which removes the improper, nonorthochronous ones:

$$\mathrm{Spin}_+(p,q) = \{s \in \mathcal{C}\ell^+_{p,q} | ss^t = 1 \text{ and } \forall \boldsymbol{x} \in \mathbb{R}^{p,q}, s\boldsymbol{x}s^{-1} \in \mathbb{R}^{p,q}\}.$$

The difference is that now $ss^t = +1$.

4.2 Spinors

The game of tangloids consists of two wooden blocks, connected by three strings (top, bottom and middle).[3] When one block is rotated by 2π with respect to the other, the strings become entangled, so that nothing but a rotation can bring them back to their original state. If we rotate the block by 4π, however, it *is* possible to disentangle the strings, without resorting to rotations. (The player who can untangle the strings the fastest is the winner.) The fact that a 2π rotation does not bring us back to the original position is known as *orientation entanglement.*

A vector always returns to its original position after a rotation of 2π. Because of this, it cannot express things such as the tangloid or, for that matter, the electron wave function. Vectors don't work here, and neither does the group $\mathrm{SO}(p,q)$. We know that $s \in \mathrm{Spin}(p,q)$ and $-s \in \mathrm{Spin}(p,q)$ have the same effect on a vector, $\boldsymbol{y} \to s\boldsymbol{y}s^{-1}$, but they are in principle different. We can use this to construct objects that transform differently under s and $-s$, and this will enable us to distinguish a $+\pi$ rotation from a $-\pi$ rotation,

[2] For instance, the weak interactions have time-reversal symmetry, but are not symmetric under spatial reflections. Weak processes would not be expected to be invariant under improper, nonorthochronous Lorentz transformations — unless we do a charge conjugation at the same time. See e.g. Peskin and Schroeder (1995).

[3] The game was created by Piet Hein, and is described in Gardner (1966).

or if you wish, a 2π rotation from a 4π rotation. We call this object a spinor, and declare that a spinor transforms as $\chi \to s\chi$ for $s \in \mathrm{Spin}(p,q)$.

Component spinors are defined as follows. First, we find a matrix representation of the Clifford algebra — a set of (complex) matrices γ_μ that respect the Clifford relation (4.3). Then we define spinor space as the (complex) column vectors that the γ_μ matrices act on.

In three dimensions, we can use the Pauli sigma matrices for this. The matrix representation is given by

$$\boldsymbol{e}_1 \to \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{e}_2 \to \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \boldsymbol{e}_3 \to \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that an ordinary vector becomes a matrix when we do this:

$$\boldsymbol{x} = x^i \boldsymbol{e}_i \to x^i \sigma_i = \boldsymbol{x} \cdot \boldsymbol{\sigma}.$$

A Clifford vector is rotated an angle α about an axis $\boldsymbol{n}$ ($\boldsymbol{n}^2 = 1$) by $\boldsymbol{y} \to s\boldsymbol{y}s^{-1}$ for $s = \exp(\frac{1}{2}\alpha \boldsymbol{n}\boldsymbol{e}_{123})$ (cf. Lounesto, 2001, section 4.6). Since $(\boldsymbol{n}\boldsymbol{e}_{123})^2 = -1$ we have

$$s = \exp(\tfrac{1}{2}\alpha \boldsymbol{n}\boldsymbol{e}_{123}) = \cos\frac{\alpha}{2} + \boldsymbol{e}_{123}\boldsymbol{n} \sin\frac{\alpha}{2}.$$

This is a scalar plus a bivector, and you should be able to verify that $s \in \mathrm{Spin}(3)$ rather easily. You can also verify that $\alpha \to \alpha + 2\pi$ makes $s \to -s$. A spinor would transform as $\chi \to s\chi$. Using component spinors and the matrix representation of the Clifford algebra ($\boldsymbol{e}_i \to \sigma_i$) we get

$$\boldsymbol{e}_{123} = \sigma_1\sigma_2\sigma_3 = \mathrm{i}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and $\boldsymbol{n} \to \boldsymbol{n} \cdot \boldsymbol{\sigma}$:

$$\chi \to \exp(\tfrac{1}{2}\mathrm{i}\alpha \boldsymbol{n} \cdot \boldsymbol{\sigma})\chi$$

which is what you find in Sakurai (1994, equation (3.2.46)).[4]

In $9 + 1$ dimensions we could in principle do the same thing. We will need bigger matrices to represent $\mathcal{C}\ell_{9,1}$, though. Ten matrices with 1024 entries each ($2^5 \times 2^5$) and 32-component spinors (2^5), actually. While this is in principle doable, it does become somewhat tedious, when we need to do things in terms of spinor components. I do not feel like writing down these matrices.

[4] Actually, the sign differs, because Sakurai (1994) measures angles counterclockwise, while Lounesto (2001) measures them clockwise, as seen when regarded from the arrowhead of $\boldsymbol{n}$.

So we do something equivalent, following the spinorial gemometry methods presented in Gillard et al. (2005). We use operators Γ_μ, that can in principle be written as matrices, but can also be written in terms of the exterior product $\wedge$ and the interior product $\lrcorner$:

$$\begin{aligned} \Gamma_0\eta &= -e_5 \wedge \eta + e_5 \lrcorner \eta, & \Gamma_5\eta &= e_5 \wedge \eta + e_5 \lrcorner \eta, \\ \Gamma_i\eta &= e_i \wedge \eta + e_i \lrcorner \eta, & \Gamma_{i+5}\eta &= \mathrm{i}e_i \wedge \eta - \mathrm{i}e_i \lrcorner \eta, \quad i \in \{1,2,3,4\} \end{aligned} \tag{4.13}$$

(see e.g. Gran et al., 2005). It is straightforward to verify that these operators satisfy (4.3). These Γ_μ operators act on spinors η. If we declare that $\mathbf{1}$ is a spinor, and that the operators give spinors when acting on spinors, we find that the spinor space is $\Delta = \bigwedge(U^{\mathbb{C}})$, where $U^{\mathbb{C}} = \mathrm{span}_{\mathbb{C}}\{\boldsymbol{e}_i | i = 1, \ldots, 5\}$. We can further divide Δ into two parts: $\Delta^+ = \bigwedge^{\mathrm{even}}(U^{\mathbb{C}})$ and $\Delta^- = \bigwedge^{\mathrm{odd}}(U^{\mathbb{C}})$, of even and odd degree, respectively. These are the complex Weyl representations. As discussed in e.g. Elvin (2009) the spinors appearing in type IIB superstring theory are complex Weyl spinors of positive chirality, meaning that the most general Killing spinor of type IIB supergravity can be written as

$$\epsilon = p\mathbf{1} + q e_{1234} + u^i e_{i5} + \tfrac{1}{2} v^{ij} e_{ij} + \tfrac{1}{6} w^{ijk} e_{ijk5}.$$

Complex basis. Introduce the basis $\tilde{\boldsymbol{e}}_\mu$:

$$\begin{aligned} \tilde{\boldsymbol{e}}_\alpha &= \frac{1}{\sqrt{2}}(\boldsymbol{e}_\alpha - \mathrm{i}\boldsymbol{e}_{\alpha+5}) \quad \text{represented by} \quad \mathit{\Gamma}_\alpha = \frac{1}{\sqrt{2}}(\Gamma_\alpha - \mathrm{i}\Gamma_{\alpha+5}) = \sqrt{2}\,\boldsymbol{e}_\alpha \wedge , \\ \tilde{\boldsymbol{e}}_{\bar\alpha} &= \frac{1}{\sqrt{2}}(\boldsymbol{e}_\alpha + \mathrm{i}\boldsymbol{e}_{\alpha+5}) \quad \text{represented by} \quad \mathit{\Gamma}_{\bar\alpha} = \frac{1}{\sqrt{2}}(\Gamma_\alpha + \mathrm{i}\Gamma_{\alpha+5}) = \sqrt{2}\,\boldsymbol{e}_\alpha \lrcorner , \\ \tilde{\boldsymbol{e}}_+ &= \frac{1}{\sqrt{2}}(\boldsymbol{e}_5 + \boldsymbol{e}_0) \quad \text{represented by} \quad \mathit{\Gamma}_+ = \frac{1}{\sqrt{2}}(\Gamma_5 + \Gamma_0) = \sqrt{2}\,\boldsymbol{e}_5 \lrcorner , \\ \tilde{\boldsymbol{e}}_- &= \frac{1}{\sqrt{2}}(\boldsymbol{e}_5 - \boldsymbol{e}_0) \quad \text{represented by} \quad \mathit{\Gamma}_- = \frac{1}{\sqrt{2}}(\Gamma_5 - \Gamma_0) = \sqrt{2}\,\boldsymbol{e}_5 \wedge , \\ &\text{where } \alpha \in \{1,2,3,4\}. \end{aligned} \tag{4.14}$$

The reason for these particular choices is of course the simple form of the operator representation. Here, the $\mathit{\Gamma}_\mu$ behave like creation and annihilation operators. The $\tilde{\boldsymbol{e}}_\mu$ (and $\mathit{\Gamma}_\mu$) obey a variant of (4.3):

$$\tilde{\boldsymbol{e}}_\mu \tilde{\boldsymbol{e}}_\nu + \tilde{\boldsymbol{e}}_\nu \tilde{\boldsymbol{e}}_\mu = 2\tilde{g}_{\mu\nu}, \tag{4.15}$$

where $\tilde{g}_{+-} = \tilde{g}_{-+} = 1$ and $\tilde{g}_{\alpha\bar\alpha} = \tilde{g}_{\bar\alpha\alpha} = \delta_{\alpha\bar\alpha}$ are the nonvanishing components of $\tilde{g}_{\mu\nu}$.

When we use the complex basis $\tilde{\boldsymbol{e}}_\mu$, we need to use complex coordinates in order to be able to express the same Clifford elements as in the real case, e.g.

the element $\alpha \boldsymbol{e}_6$ in the real basis equals $\frac{1}{\sqrt{2}}\mathrm{i}\alpha(\tilde{\boldsymbol{e}}_1 - \tilde{\boldsymbol{e}}_{\bar{1}})$ in the complex basis. This naturally leads us to the complex Clifford algebra rather than the real one that we have used so far. But it is the real Clifford algebra $\mathcal{C}\ell_{p,q}$ that we use when we define the Spin groups and do geometry. Those elements of the complex algebra that are not expressible in the real basis are unwanted. We want to take the *real* Clifford algebra, and express it in the complex basis. To do this, we take the complex algebra generated by the $\tilde{\boldsymbol{e}}_\mu$, and add the reality condition $s^* = s$, where $*$ means complex conjugation. (Sometimes we consider a complex conjugation operator $*$ that acts on everything to the right of it, $*st = s^*t^**$, in which case the reality condition is $*s = s*$.) The conjugation in s^* acts on the complex coefficients in the ordinary way, and on the basis vectors (making up the basis elements) by $*\tilde{\boldsymbol{e}}_\alpha = \tilde{\boldsymbol{e}}_{\bar{\alpha}}*$ and $*\tilde{\boldsymbol{e}}_{\bar{\alpha}} = \tilde{\boldsymbol{e}}_\alpha*$, as can be seen in (4.14) — $\tilde{\boldsymbol{e}}_+$ and $\tilde{\boldsymbol{e}}_-$ are real.

The reality condition $s^* = s$ does not translate directly to $\mathcal{O}^* = \mathcal{O}$, when when we go to the operator representation, $s \to \mathcal{O}$ with $\mathcal{O}$ expressed in terms of the $\mathit{\Gamma}_\mu$ operators. This is because we already used the imaginary unit i when defining the operators Γ_μ corresponding to real Clifford vectors $\boldsymbol{e}_\mu$, cf. (4.13). In (4.14) we see that $*\mathit{\Gamma}_\alpha = \mathit{\Gamma}_\alpha*$, even though $*\tilde{\boldsymbol{e}}_\alpha = \tilde{\boldsymbol{e}}_{\bar{\alpha}}*$. Complex conjugation on the Clifford algebra side, corresponds to a slightly more complicated operator than complex conjugation in the representation. That operator should still include complex conjugation, to deal with the coefficients properly, but the complex conjugation operator does not give the right result on the operator representation of the basis elements. Thus we declare that $*$ corresponds to $C*$, or equivalently, that s^* corresponds to $C\mathcal{O}^*$, for some operator C whose job it is to correct the complex conjugation of the basis elements, so that $C*\mathit{\Gamma}_\alpha = \mathit{\Gamma}_{\bar{\alpha}}C*$ just as $*\tilde{\boldsymbol{e}}_\alpha = \tilde{\boldsymbol{e}}_{\bar{\alpha}}*$ on the Clifford algebra side. Our next task is to find that operator C.

Since $*\boldsymbol{e}_\mu = \boldsymbol{e}_\mu*$, we want $C*\Gamma_\mu = \Gamma_\mu C*$. We have

$$C*\Gamma_\mu = \begin{cases} C\Gamma_\mu* & \text{if } \mu \in \{0,1,2,3,4,5\}, \\ -C\Gamma_\mu* & \text{if } \mu \in \{6,7,8,9\}. \end{cases}$$

Thus we need to find a C that commutes with Γ_0 through Γ_5, and anticommutes with Γ_6 through Γ_9. Now, according to (4.3), each Γ_μ operator will anticommute with all *other* Γ_ν operators (and obviously commute with itself). A $\Gamma_\mu\Gamma_\nu$ pair, will commute with all Γ_ρ operators if $\rho \neq \mu, \rho \neq \nu$, and anticommute if $\rho = \mu \neq \nu$ or $\rho = \nu \neq \mu$. Thus e.g. $\Gamma_6\Gamma_7$ commutes with Γ_0 through Γ_5, anticommute with Γ_6 and Γ_7, and then commute with Γ_8 and Γ_9. This is close to what we want, but we want an operator that anticommutes with Γ_8 and Γ_9 too. But we can obviously do that by using the pair $\Gamma_8\Gamma_9$ in addition to $\Gamma_6\Gamma_7$:

$$C = \Gamma_6\Gamma_7\Gamma_8\Gamma_9 \tag{4.16}$$

does the job. Using (4.13) we can write it explicitly in terms of exterior and interior products as

$$C = \prod_{i=1}^{4} \big((\boldsymbol{e}_i\wedge) - (\boldsymbol{e}_i \lrcorner)\big) \tag{4.17}$$

where it is understood that the factors are sorted in the order of ascending i ($i = 1$ to the left). Using the C of (4.16) and (4.17) the complex conjugation works out as it should:

$$\begin{array}{lcl} *\boldsymbol{e}_\mu = \boldsymbol{e}_\mu * & \rightarrow & C*\Gamma_\mu = \Gamma_\mu C* \\ *\tilde{\boldsymbol{e}}_\alpha = \tilde{\boldsymbol{e}}_{\bar{\alpha}} * & \rightarrow & C*\mathit{\Gamma}_\alpha = \mathit{\Gamma}_{\bar{\alpha}} C* \\ *\tilde{\boldsymbol{e}}_{\bar{\alpha}} = \tilde{\boldsymbol{e}}_\alpha * & \rightarrow & C*\mathit{\Gamma}_{\bar{\alpha}} = \mathit{\Gamma}_\alpha C* \\ *\tilde{\boldsymbol{e}}_\pm = \tilde{\boldsymbol{e}}_\pm * & \rightarrow & C*\mathit{\Gamma}_\pm = \mathit{\Gamma}_\pm C* \end{array} \tag{4.18}$$

Now, the reality condition $*s = s*$ that we need to impose on elements s of the complex Clifford algebra when we construct the Spin group and do geometry, takes the form $C*\mathcal{O} = \mathcal{O}C*$ in the operator representation. The condition for a spinor η to be real is simply $C*\eta = \eta$, though spinors will in general be complex, even in the representation of the real Clifford algebra.

Chapter 5

Finding the Killing spinors with Galilean symmetry

5.1 Gauge fixing of the general Killing spinor

Type IIB string theory has a $\mathrm{Spin}(9,1)$ symmetry, which we break up into a $\mathrm{Spin}(3,1)$ symmetry that corresponds to the macroscopic dimensions, and the rest, corresponding to an internal Calabi–Yau space. We use local symmetry transformations to transform a general spinor into a simpler representative — this is the gauge fixing of the general Killing spinor.

5.1.1 The Galilean subgroup of $\mathrm{Spin}(3,1)$

The four-dimensional space with $\mathrm{Spin}(3,1)$ symmetry will be our Galilean spacetime. $\tilde{\boldsymbol{e}}_+$ will be the Galilean time direction, $\tilde{\boldsymbol{e}}_-$ will be the direction corresponding to the Galilean extra coordinate, which we called x^s in chapter 3 (or $-x^s$, really, since we are using the positive-metric convention (3.10) for the light-cone coordinates). There are now only two dimensions left for the Galilean space, but we are quite content with studying two-dimensional systems — in fact, many interesting superconductors have superconducting planes that one would hope could be described with the AdS/CFT correspondence. The Galilean space will be spanned by $\boldsymbol{e}_1$ and $\boldsymbol{e}_6$ (with real coefficients), or equivalently, in the complex basis, by $\tilde{\boldsymbol{e}}_1$ and $\tilde{\boldsymbol{e}}_{\bar{1}}$ (with complex coefficients, and a reality condition on the resulting vectors).

The $\mathrm{Spin}(3,1)$ group is not entirely compatible with homogeneous Galilean transformations, however — all Galilean transformations must leave the mass $p^t = m$ invariant. Thus we must restrict ourselves to a subgroup of $\mathrm{Spin}(3,1)$ that leaves the component of vectors that points in the $\tilde{\boldsymbol{e}}_+$ direction invariant. The Galilean energy-momentum tensor is

$$\boldsymbol{p} = m\tilde{\boldsymbol{e}}_+ - E\tilde{\boldsymbol{e}}_- + p^1\boldsymbol{e}_1 + p^6\boldsymbol{e}_6$$

where E is the energy and p^I are the components of the Galilean spatial momentum. After a Galilean transformation we have

$$\boldsymbol{p} \to \boldsymbol{p}' = m\tilde{\boldsymbol{e}}_+ - E'\tilde{\boldsymbol{e}}_- + p'^1\boldsymbol{e}_1 + p'^6\boldsymbol{e}_6.$$

The time component of vectors, in this case the mass m, is left invariant by Galilean transformations. Note that

$$\boldsymbol{p} \cdot \tilde{\boldsymbol{e}}_- = m \tag{5.1}$$

and that

$$\boldsymbol{p}' \cdot \tilde{\boldsymbol{e}}_- = m. \tag{5.2}$$

Doing a Galilean transformation on (5.1) we get $\boldsymbol{p}' \cdot \tilde{\boldsymbol{e}}'_- = m$, which together with (5.2) yields $\boldsymbol{p}' \cdot (\tilde{\boldsymbol{e}}'_- - \tilde{\boldsymbol{e}}_-) = 0$ for all $\boldsymbol{p}$, so that $\tilde{\boldsymbol{e}}'_- = \tilde{\boldsymbol{e}}_-$.

The conclusion is that in order to leave the Galilean time component of vectors invariant (x^+ and p^+ and so on), the basis vector $\tilde{\boldsymbol{e}}_-$ must be left invariant by the transformation.

The Spin(3, 1) group forms a subset of $\mathcal{C}\ell^+_{3,1}$, according to (4.12). If the basis vectors generating the $\mathcal{C}\ell_{3,1}$ lie in directions $\boldsymbol{e}_0$, $\boldsymbol{e}_1$, $\boldsymbol{e}_5$ and $\boldsymbol{e}_6$, the most general element of $\mathcal{C}\ell^+_{3,1}$ can be written as

$$\begin{aligned} s = \alpha_{0156}\boldsymbol{e}_0\boldsymbol{e}_1\boldsymbol{e}_5\boldsymbol{e}_6 + \alpha_{01}\boldsymbol{e}_0\boldsymbol{e}_1 + \alpha_{05}\boldsymbol{e}_0\boldsymbol{e}_5 + \alpha_{06}\boldsymbol{e}_0\boldsymbol{e}_6 + \\ + \alpha_{15}\boldsymbol{e}_1\boldsymbol{e}_5 + \alpha_{16}\boldsymbol{e}_1\boldsymbol{e}_6 + \alpha_{56}\boldsymbol{e}_5\boldsymbol{e}_6 + \beta \end{aligned} \tag{5.3}$$

for $\alpha_{\dots} \in \mathbb{R}$ and $\beta \in \mathbb{R}$. If it is an element of the Spin(3, 1) group, s acts on a vector $\boldsymbol{v}$ as $s\boldsymbol{v}s^{-1}$. If we want it to leave $\tilde{\boldsymbol{e}}_-$ invariant, this means that $s\tilde{\boldsymbol{e}}_-s^{-1} = \tilde{\boldsymbol{e}}_-$, or (multiplying by s on the right)

$$s\tilde{\boldsymbol{e}}_- = \tilde{\boldsymbol{e}}_-s. \tag{5.4}$$

Inserting (5.3) into (5.4) we get the most general $s \in \mathcal{C}\ell^+_{3,1}$ that leaves $\tilde{\boldsymbol{e}}_-$ invariant:

$$s = \underbrace{A\boldsymbol{e}_0\boldsymbol{e}_1 + A\boldsymbol{e}_1\boldsymbol{e}_5}_{s_A} + \underbrace{B\boldsymbol{e}_0\boldsymbol{e}_6 - B\boldsymbol{e}_5\boldsymbol{e}_6}_{s_B} + \underbrace{C\boldsymbol{e}_1\boldsymbol{e}_6}_{s_C} + \beta \tag{5.5}$$

$$\begin{cases} s_A = A(\boldsymbol{e}_0 - \boldsymbol{e}_5)\boldsymbol{e}_1 = -A\tilde{\boldsymbol{e}}_-(\tilde{\boldsymbol{e}}_1 + \tilde{\boldsymbol{e}}_{\bar{1}}), \\ s_B = B(\boldsymbol{e}_0 - \boldsymbol{e}_5)\boldsymbol{e}_6 = -B\,\mathrm{i}\tilde{\boldsymbol{e}}_-(\tilde{\boldsymbol{e}}_1 - \tilde{\boldsymbol{e}}_{\bar{1}}), \\ s_C = C\boldsymbol{e}_1\boldsymbol{e}_6 = -C\,\mathrm{i}\,\frac{1}{2}\left(\tilde{\boldsymbol{e}}_1\tilde{\boldsymbol{e}}_{\bar{1}} - \tilde{\boldsymbol{e}}_{\bar{1}}\tilde{\boldsymbol{e}}_1\right) \end{cases}$$

If we want s to be an element in Spin(3, 1) we need $ss^t = 1$. Calculating ss^t in (5.5) we get

$$ss^t = C^2 + \beta^2 \overset{!}{=} 1. \tag{5.6}$$

Thus $s_A, s_B \notin \mathrm{Spin}(3,1)$ by themselves — at least one of C and β has to be nonzero. We can fulfill (5.6) by putting $C = \sin\frac{\theta}{2}$ and $\beta = \cos\frac{\theta}{2}$ — then $s = s_C + \beta$ would be a rotation in the $\boldsymbol{e}_1\boldsymbol{e}_6$ plane by an angle θ.

So we associate $s_C + \beta$ with the rotations in the $\boldsymbol{e}_1\boldsymbol{e}_6$ plane. Similarly, we associate $s_A + 1$ (and $s_B + 1$) with the Galilean boosts that leave $\boldsymbol{e}_6$ (and $\boldsymbol{e}_1$, respectively) invariant. Do note, however, that a Galilean boost is not simply a pure Lorentz boost in light-cone coordinates. It is a Lorentz transformation all right, but not a pure boost. To take an example, consider how $s = s_A + 1$ acts on a vector $\boldsymbol{v}$ written in the basis $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_5, \boldsymbol{e}_6\}$:

$$(s_A + 1)\boldsymbol{v}(s_A + 1)^t = \begin{pmatrix} 1 & 2\,A & 0 & 2\,A \\ -2\,A & 1 - 2\,A^2 & 0 & -2\,A^2 \\ 0 & 0 & 1 & 0 \\ 2\,A & 2\,A^2 & 0 & 2\,A^2 + 1 \end{pmatrix} \boldsymbol{v}.$$

A pure Lorentz boost would give a symmetric transformation matrix. The $s_A + 1$ transformation is still a Lorentz transformation, and it would not be entirely wrong to call it a Lorentz boost, but it is not a *pure* one where the coordinate system is simply boosted by some velocity $\boldsymbol{V}$. Such a boost should involve both $\tilde{\boldsymbol{e}}_-$ and $\tilde{\boldsymbol{e}}_+$, whereas s_A and s_B are defined solely in terms of $\tilde{\boldsymbol{e}}_-$.

Going over to the spinor representation, we can write (5.5) in terms of

$$\Gamma_{1\bar{1}} = \frac{1}{2}\left(\Gamma_1\Gamma_{\bar{1}} - \Gamma_{\bar{1}}\Gamma_1\right) \quad = (\mathsf{e}_1 \wedge \mathsf{e}_1 \lrcorner) - (\mathsf{e}_1 \lrcorner\, \mathsf{e}_1 \wedge), \tag{5.7}$$

$$\Gamma_{-1} = \frac{1}{2}\left(\Gamma_-\Gamma_1 - \Gamma_1\Gamma_-\right) \quad = (\mathsf{e}_5 \wedge \mathsf{e}_1 \wedge) - (\mathsf{e}_1 \wedge \mathsf{e}_5 \wedge) = 2\left(\mathsf{e}_5 \wedge \mathsf{e}_1 \wedge\right), \tag{5.8}$$

$$\Gamma_{-\bar{1}} = \frac{1}{2}\left(\Gamma_-\Gamma_{\bar{1}} - \Gamma_{\bar{1}}\Gamma_-\right) \quad = (\mathsf{e}_5 \wedge \mathsf{e}_1 \lrcorner) - (\mathsf{e}_1 \lrcorner\, \mathsf{e}_5 \wedge) = 2\left(\mathsf{e}_5 \wedge \mathsf{e}_1 \lrcorner\right). \tag{5.9}$$

The interesting generators of (homogeneous) Galilean transformations, can thus be represented in spinor space as

$$\begin{aligned} s_A &\to -A\left(\Gamma_{+1} + \Gamma_{+\bar{1}}\right) \equiv -a\,\frac{1}{2}\left(\Gamma_{+1} + \Gamma_{+\bar{1}}\right), \\ s_B &\to -B\,\mathrm{i}\left(\Gamma_{+1} - \Gamma_{+\bar{1}}\right) \equiv -b\,\frac{1}{2}\,\mathrm{i}\left(\Gamma_{+1} - \Gamma_{+\bar{1}}\right), \\ s_C &\to -C\,\mathrm{i}\Gamma_{1\bar{1}}, \end{aligned}$$

where we have rescaled A and B (to a and b) in order to introduce a factor of $\frac{1}{2}$ that will later turn out to be expedient. A general element of the Galilean subgroup of Spin(3, 1) is then given by

$$G(a,b,\theta) \coloneqq -a\,\frac{1}{2}\,(\mathit{\Gamma}_{-1} + \mathit{\Gamma}_{-\bar{1}}) - b\,\frac{1}{2}\,\mathrm{i}\,(\mathit{\Gamma}_{-1} - \mathit{\Gamma}_{-\bar{1}}) - \sin\frac{\theta}{2}\,\mathrm{i}\mathit{\Gamma}_{1\bar{1}} + \cos\frac{\theta}{2} \tag{5.10}$$

in the spinor representation.

5.1.2 Spin(6) and the Calabi-Yau space

Spin(9, 1) is constructed from the Clifford algebra generated by $\boldsymbol{e}_0$ to $\boldsymbol{e}_9$, and the corresponding spinor space can be represented by the exterior algebra of e_1 to e_5. We single out $\boldsymbol{e}_0$, $\boldsymbol{e}_1$, $\boldsymbol{e}_5$ and $\boldsymbol{e}_6$ to make Spin(3, 1), containing the homogeneous Galilean transformations — in spinor space this corresponds to e_1 and e_5. Left, we have $\boldsymbol{e}_2$ to $\boldsymbol{e}_4$, and $\boldsymbol{e}_7$ to $\boldsymbol{e}_9$, which together generate Spin(6). In the spinor representation these correspond to e_2, e_3 and e_4.

A general Spin(9, 1) spinor can be written as a linear combination of terms of the form $\eta_{3,1} \wedge \eta_6$, where $\eta_{3,1}$ is a Spin(3, 1) spinor (associated with the exterior algebra of e_1 and e_5) and η_6 is a Spin(6) spinor (associated with the exterior algebra of e_2, e_3 and e_4).

What we want in the end are spinors with Galilean symmetry, and if that was all there is to it, we might ignore how things transform under Spin(6) entirely. However, compactified string theory places some stringent requirements on the compactified space — see e.g. Greene (1997). There are more general ways of doing it, but one popular way is to say that the compactified space is a generalised Calabi–Yau manifold. In our case, the generalised Calabi–Yau manifold will encompass the real directions $\boldsymbol{e}_2$ to $\boldsymbol{e}_4$ and $\boldsymbol{e}_7$ to $\boldsymbol{e}_9$, corresponding to the complex directions $\tilde{\boldsymbol{e}}_2$, $\tilde{\boldsymbol{e}}_3$, $\tilde{\boldsymbol{e}}_4$. With three complex dimensions, we call the space a (generalised) Calabi–Yau three-fold, and the interesting property of our generalised Calabi–Yau three-fold, for our purposes, is that it is a space with SU(3) structure.[1] SU(3) structure means that we want our spinors to be invariant under SU(3).

Our next task, then, is to figure out how SU(3) and Spin(6) are related. A Spin(6) spinor is in

$$\begin{aligned}\operatorname{span}\{1, e_2, e_3, e_4, e_{34}, e_{24}, e_{23}, e_{234}\} &= \\ &= \underbrace{\operatorname{span}\{1, e_{34}, e_{24}, e_{23}\}}_{=4_{\text{even}}} \oplus \underbrace{\operatorname{span}\{e_2, e_3, e_4, e_{234}\}}_{=4_{\text{odd}}}.\end{aligned}$$

[1] An ordinary Calabi–Yau three-fold has SU(3) holonomy, rather than SU(3) structure.

It is interesting to note that a spinor in $\mathsf{4}_{\text{even}}$ remains in $\mathsf{4}_{\text{even}}$ when Spin(6) acts on it; $\mathsf{4}_{\text{odd}}$ is likewise closed under Spin(6). (An element of Spin(6) is represented by linear combinations of even Clifford elements, so in the complex basis, where the Γ operators are creation and annihilation operators acting on spinor space, we directly see that even (or odd) spinors go to even (or odd) spinors, by an even number of creation/annihilation operations.)

It is known that Spin(6) is isomorphic to SU(4) (see e.g. Fuchs and Schweigert, 2003; Lounesto, 2001). Thus, somehow, the action of Spin(6) on $\mathsf{4}_{\text{odd}}$ and $\mathsf{4}_{\text{even}}$ is equivalent to the action of an SU(4) matrix. We can of course construct the group SU(3) by taking a subset of SU(4) matrices — say block diagonal SU(4) matrices with an SU(3) matrix in the first block and identity in the second. Thus a four-component object transforming under SU(4) transforms as a three-component object and a one-component object under SU(3): $\mathsf{4}_{\mathrm{SU}(4)} = (\mathsf{3}+\mathsf{1})_{\mathrm{SU}(3)}$. Looking at $\mathsf{4}_{\text{odd}}$ and $\mathsf{4}_{\text{even}}$, there seems to be a natural choice for a division into $\mathsf{3}+\mathsf{1}$: we want $\mathrm{span}\{\mathsf{e}_2, \mathsf{e}_3, \mathsf{e}_4\}$ and $\mathrm{span}\{\mathsf{e}_{34}, \mathsf{e}_{24}, \mathsf{e}_{23}\}$ to be closed under SU(3). We need to implement SU(3) in terms of Spin(6) in a way that accomplishes this.

This can be done by using $\Gamma_{\alpha\bar\beta} = \frac{1}{2}\left(\Gamma_\alpha \Gamma_{\bar\beta} - \Gamma_{\bar\beta}\Gamma_\alpha\right)$ — these operators contain one creation and one annihilation operator, so that $\mathrm{span}\{\mathsf{e}_2, \mathsf{e}_3, \mathsf{e}_4\}$ is closed under their operation. There are nine such operators, but most of them are not elements of Spin(6) as they stand — we need to impose the reality condition $C{*}\mathcal{O} = \mathcal{O}C{*}$ on each operator $\mathcal{O}$. The operators with $\alpha = \beta$ are simple in this respect — we only need to multiply by i: $\mathrm{i}\Gamma_{2\bar2}$, $\mathrm{i}\Gamma_{3\bar3}$ and $\mathrm{i}\Gamma_{4\bar4}$. The others can be combined into pairs yielding the two operators $\Gamma_{\alpha\bar\beta} + \Gamma_{\bar\alpha\beta}$ and $\mathrm{i}\Gamma_{\alpha\bar\beta} - \mathrm{i}\Gamma_{\bar\beta\alpha}$. Now, having taken the reality condition into account, we have nine operators that are in Spin(6) and map $\mathrm{span}\{\mathsf{e}_2, \mathsf{e}_3, \mathsf{e}_4\}$ to $\mathrm{span}\{\mathsf{e}_2, \mathsf{e}_3, \mathsf{e}_4\}$, and $\mathrm{span}\{\mathsf{e}_{34}, \mathsf{e}_{24}, \mathsf{e}_{23}\}$ to $\mathrm{span}\{\mathsf{e}_{34}, \mathsf{e}_{24}, \mathsf{e}_{23}\}$. That is what we mean by the $\mathsf{3}_{\mathrm{SU}(3)}$ in our $\mathsf{4}_{\mathrm{SU}(4)} = (\mathsf{3}+\mathsf{1})_{\mathrm{SU}(3)}$ subdivision; the $\mathsf{1}_{\mathrm{SU}(3)}$ means that e_{234} and $\mathsf{1}$ are left invariant. Of the nine Spin(6) operators that respect the $\mathsf{3}_{\mathrm{SU}(3)}$, three do not respect the $\mathsf{1}_{\mathrm{SU}(3)}$ as they stand: $\mathrm{i}\Gamma_{2\bar2}$, $\mathrm{i}\Gamma_{3\bar3}$ and $\mathrm{i}\Gamma_{4\bar4}$ do not annihilate $\mathsf{1}$ and e_{234} (which the Lie algebra generators should do if the corresponding Lie group leaves them invariant). Taking linear combinations of these three, we get two operators that do annihilate $\mathsf{1}$ and e_{234}. The hope, then, is that the resulting eight generators can be identified with the eight generators of SU(3) — then $\mathsf{1}$ and e_{234} would be the spinors left invariant by SU(3). That the eight Spin(6) generators we have singled out can indeed be identified with the SU(3) generators, can be seen explicitly by seeing how the generators act on a spinor in $\mathrm{span}\{\mathsf{e}_2, \mathsf{e}_3, \mathsf{e}_4\}$ or a spinor in $\mathrm{span}\{\mathsf{e}_{34}, \mathsf{e}_{24}, \mathsf{e}_{23}\}$. Thus we would use $\mathsf{1}$ and e_{234} as the η_6 part of the Spin(9, 1) Killing spinors.

5.1.3 The interesting spinors

As noted above, a general Spin$(9,1)$ spinor is a linear combination of terms of the form $\eta_{3,1} \wedge \eta_6$, where $\eta_{3,1}$ is a Spin$(3,1)$ spinor (associated with the exterior algebra of e_1 and e_5) and η_6 is a Spin(6) spinor (associated with the exterior algebra of e_2, e_3 and e_4). The coefficients of such a linear combination will be functions of spacetime. We have seen that if the SU(3) holonomy is to be respected, we should use $\mathbf{1}$ and e_{234} as the η_6 part of Killing vectors. The spinors appearing in type IIB are complex Weyl spinors of positive chirality — that means that we want Spin$(9,1)$ spinors of even degree: a (complex) linear combination of $\mathbf{1}, e_{15}, e_{1234}$ and e_{2345}. The $\eta_{3,1}$ part is built from e_1 and e_5, and the η_6 part is built from $\mathbf{1}$ and e_{234}.

It will be expedient to express a general Spin$(9,1)$ spinor in terms of a basis of purely real spinors, in the sense that $C{*}\eta = \eta$, and purely imaginary spinors, in the sense that $C{*}\eta = -\eta$:

$$
\begin{aligned}
\eta_1 &= 1 + e_{1234}, & \eta_5 &= \mathrm{i}\eta_1, \\
\eta_2 &= \mathrm{i}\left(1 - e_{1234}\right), & \eta_6 &= \mathrm{i}\eta_2, \\
\eta_3 &= e_{15} + e_{2345}, & \eta_7 &= \mathrm{i}\eta_3, \\
\eta_4 &= \mathrm{i}\left(e_{15} - e_{2345}\right), & \eta_8 &= \mathrm{i}\eta_4,
\end{aligned}
$$

where η_1 through η_4 are real ($C{*}\eta_i = \eta_i$) and η_5 through η_8 are imaginary ($C{*}\eta_i = -\eta_i$).

The spinors written in a basis of η_1 through η_8 will automatically be invariant under the SU(3) transformations associated with the generalised Calabi–Yau space, and, given appropriate conditions on the coefficients, they can be taken to be the Killing spinors corresponding to Galilean symmetry. Finding the exact conditions on the coefficients, involves solving the Killing spinor equation, but there are some things we can do before we do that.

For one thing, we can bring them to a canonical form using local Galilean transformations. We also know that the space of the Killing spinors should be closed under (global) Galilean transformations — taking a Galilean transformation of a Killing spinor should give us a Killing spinor. We next turn our attention to the canonical form of the Killing spinors.

5.1.4 The action of the Galilean group

Let a real spinor $\eta = \alpha\eta_1 + \beta\eta_2 + \gamma\eta_3 + \delta\eta_4$ be represented by the column vector

$$\eta \doteq \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}.$$

Next, we study the action of a general homogeneous Galilean transformation (5.10) on this spinor:

$$G(A, B, \theta) \coloneqq A\,\frac{1}{2}\,(\mathit{\Gamma}_{+1} + \mathit{\Gamma}_{+\bar{1}}) + B\,\frac{1}{2}\,\mathrm{i}\,(\mathit{\Gamma}_{+1} - \mathit{\Gamma}_{+\bar{1}}) + \sin\frac{\theta}{2}\,\mathrm{i}\mathit{\Gamma}_{1\bar{1}} + \cos\frac{\theta}{2} \quad (5.10)$$

The first step is to find out how $\mathit{\Gamma}_{+1}, \mathit{\Gamma}_{+\bar{1}}$ and $\mathit{\Gamma}_{1\bar{1}}$ act on η using (5.7) to (5.9):

	η_1	η_2	η_3	η_4
$\mathit{\Gamma}_{1\bar{1}}$	$-1 + e_{1234}$	$\mathrm{i}\,(-1 - e_{1234})$	$e_{15} - e_{2345}$	$\mathrm{i}\,(e_{15} + e_{2345})$
$\mathit{\Gamma}_{-1}$	$0 - 2e_{15}$	$\mathrm{i}\,(0 - 2e_{15})$	0	0
$\mathit{\Gamma}_{-\bar{1}}$	$0 - 2e_{2345}$	$\mathrm{i}\,(0 + 2e_{2345})$	0	0

Then we go to the linear combinations that appear in (5.10), which are real on the Clifford algebra side (though not real, strictly speaking, in the representation):

	η_1	η_2	η_3	η_4
$-\mathrm{i}\mathit{\Gamma}_{1\bar{1}}$	$\mathrm{i}\,(1 - e_{1234})$	$-\,(1 + e_{1234})$	$-\mathrm{i}\,(e_{15} - e_{2345})$	$(e_{15} + e_{2345})$
$-\frac{1}{2}\left(\mathit{\Gamma}_{-1} + \mathit{\Gamma}_{-\bar{1}}\right)$	$e_{15} + e_{2345}$	$\mathrm{i}\,(e_{15} - e_{2345})$	0	0
$-\frac{1}{2}\,\mathrm{i}\left(\mathit{\Gamma}_{-1} - \mathit{\Gamma}_{-\bar{1}}\right)$	$\mathrm{i}\,(e_{15} + e_{2345})$	$-\,(e_{15} + e_{2345})$	0	0

which simplifies to

	η_1	η_2	η_3	η_4
$-\mathrm{i}\mathit{\Gamma}_{1\bar{1}}$	η_2	$-\eta_1$	$-\eta_4$	η_3
$-\frac{1}{2}\,(\mathit{\Gamma}_{+1} + \mathit{\Gamma}_{+\bar{1}})$	η_3	η_4	0	0
$-\frac{1}{2}\,\mathrm{i}\,(\mathit{\Gamma}_{+1} - \mathit{\Gamma}_{+\bar{1}})$	η_4	$-\eta_3$	0	0

(5.11)

Now, knowing how these operators act on $\eta_1, \eta_2, \eta_3, \eta_4$, we can express them as matrices multiplying the column vector $(\alpha, \beta, \gamma, \delta)^T$ representing $\eta = \alpha\eta_1+$

$\beta\eta_2 + \gamma\eta_3 + \delta\eta_4$. Looking at (5.11) we see that

$$\mathrm{i}\Gamma_{1\bar{1}}\eta = \alpha\eta_2 - \beta\eta_1 - \gamma\eta_4 + \delta\eta_3 = -\beta\eta_1 + \alpha\eta_2 + \delta\eta_3 - \gamma\eta_4 \doteq \begin{pmatrix} -\beta \\ \alpha \\ \delta \\ -\gamma \end{pmatrix}.$$

Thus

$$\mathrm{i}\Gamma_{1\bar{1}} \doteq \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Similarly:

$$\frac{1}{2}\left(\Gamma_{+1} + \Gamma_{+\bar{1}}\right) \doteq \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \text{and}$$

$$\frac{1}{2}\mathrm{i}\left(\Gamma_{+1} - \Gamma_{+\bar{1}}\right) \doteq \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Inserting everything we know into (5.10) we get a matrix representation of how homogeneous Galilean transformations act on real spinors.

$$G(a,b,\theta) \doteq \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} & 0 & 0 \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} & 0 & 0 \\ a & -b & \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ b & a & -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \tag{5.12}$$

5.1.5 The orbits of the real spinors

Of course, a general spinor would be a linear combination of the real spinors $\{\eta_i\}_{i=1}^4$ and the purely imaginary spinors $\{\eta_i\}_{i=5}^8$, but either set of spinors transform among themselves, in exactly the same way. We study the real spinor $\eta = \alpha\eta_1 + \beta\eta_2 + \gamma\eta_3 + \delta\eta_4$.

In group theory, the orbit of a point x in some set X is the set of points in X that can be related to x by some element of the group. Thus, the orbit of a spinor η is the set $\{G(a,b,\theta)\eta : a,b,\theta \in \mathbb{R}\}$. Since the homogeneous Galilean transformations are linear transformations, one orbit will be the trivial $\{0\}$ — if $\eta = 0$, then $G\eta = 0$, for all a, b and θ.

Thinking about (5.12) in terms of 2×2 blocks, we see two rotation matrices on the diagonal, leading us to consider the quantities $I_1 := \sqrt{\alpha^2 + \beta^2}$ and $I_2 := \sqrt{\gamma^2 + \delta^2}$. Doing the analysis, we find infinitely many orbits, that we can describe by I_1 and I_2 as follows:

The $I_1\eta_1$ orbit: $\{\eta = \alpha\eta_1 + \beta\eta_2 + \gamma\eta_3 + \delta\eta_4 :$
$I_1 := \sqrt{\alpha^2 + \beta^2} \neq 0$ and $I_2 := \sqrt{\gamma^2 + \delta^2}$ arbitrary.$\}$

The $I_2\eta_3$ orbit: $\{\eta = \alpha\eta_1 + \beta\eta_2 + \gamma\eta_3 + \delta\eta_4 :$
$I_1 := \sqrt{\alpha^2 + \beta^2} = 0$ and $I_2 := \sqrt{\gamma^2 + \delta^2} \neq 0.\}$

All spinors in the $I_1\eta_1$ orbit, can be brought to $I_1\eta_1$ by Galilean transformations — this is perhaps easiest to see if we use the group property to do it in two steps: First use $G(0, 0, \theta)$ to rotate away the coefficient of η_2, then use $G(a, b, 0)$ to get rid of the coefficients of η_3 and η_4. Since a and b would be multiplying the η_1 coefficient in this second step, it only works if the η_1 coefficient is nonzero after the first step — if it is not, $I_1 = 0$. In that case we are in a $I_2\eta_3$ orbit, where the coefficients of η_1 and η_2 are already zero, and the coefficient of η_4 may be rotated away by $G(0, 0, \theta)$.

One $I_1\eta_1$ orbit for each positive I_1, and one $I_2\eta_3$ orbit for each positive I_2: infinitely many orbits. (And one trivial orbit, $\eta = 0$.) It looks, however, as though we have two types of orbits (three if you include the trivial one). To distinguish them, we define the orbit-type as the orbits that have the same stabiliser subgroup, which is the subgroup that leaves the spinor invariant. In the trivial orbit, the stabiliser subgroup is the entire group of homogeneous Galilean transformations: any $G(a, b, \theta)$ leaves the spinor invariant. In the $I_2\eta_3$ orbit-type, any $G(a, b, 0)$ leaves the spinor invariant; but in the $I_1\eta_1$ orbit-type, only the trivial subgroup consisting of the identity transformation $G(0, 0, 0)$ will leave the spinors invariant.

There are thus three orbit-types:

Representative	**Orbit**
$\eta = 0$	$\{0\}$
$\eta = I_1\eta_1$	$\{\alpha\eta_1 + \beta\eta_2 + \gamma\eta_3 + \delta\eta_4 : I_1 \equiv \sqrt{\alpha^2 + \beta^2} \neq 0\}$
$\eta = I_2\eta_3$	$\{\gamma\eta_3 + \delta\eta_4 : I_2 \equiv \sqrt{\gamma^2 + \delta^2} \neq 0\}$

5.1.6 The orbits of the complex spinors

Now we study the full spinors

$$\eta = \alpha\eta_1 + \beta\eta_2 + \gamma\eta_3 + \delta\eta_4 + \tilde{\alpha}\eta_5 + \tilde{\beta}\eta_6 + \tilde{\gamma}\eta_7 + \tilde{\delta}\eta_8$$

where $\eta_{i+4} = \mathrm{i}\eta_i$ for $i = 1, \ldots, 4$. One might represent η with an eight-component vector, and represent G as an 8×8 matrix, but that would not really help our intuition for things. Rather, we let G remain what it was, a 4×4 matrix, and represent η as a 4×2 matrix. The subgroup of $\mathrm{Spin}(3,1)$ acts as $G(a, b, \theta)$ multiplied from the left as usual. Now, apart from the $\mathrm{Spin}(9,1)$ group, there is also a U(1) symmetry group in the underlying type IIB theory that we may use to find a canonical form for the spinors. We may treat this group as a separate matrix. Given how we represent η, this matrix is actually multiplied by η from the right. The full group action is:

$$\eta \doteq \begin{pmatrix} \alpha & \tilde{\alpha} \\ \beta & \tilde{\beta} \\ \gamma & \tilde{\gamma} \\ \delta & \tilde{\delta} \end{pmatrix} \mapsto G(A, B, \theta) \begin{pmatrix} \alpha & \tilde{\alpha} \\ \beta & \tilde{\beta} \\ \gamma & \tilde{\gamma} \\ \delta & \tilde{\delta} \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}. \tag{5.13}$$

We start by considering the simpler case when $a = b = 0$ and $G(a, b, \theta)$ is block diagonal — then we can do the analysis in terms of a 2×2 matrix M, signifying either $\begin{pmatrix} \alpha & \tilde{\alpha} \\ \beta & \tilde{\beta} \end{pmatrix}$ or $\begin{pmatrix} \gamma & \tilde{\gamma} \\ \delta & \tilde{\delta} \end{pmatrix}$, being multiplied by orthogonal matrices on either side: $M \mapsto O_1 M O_2$. Now, we don't expect $I_1 \coloneqq \sqrt{\alpha^2 + \beta^2}$, or $I_2 \coloneqq \sqrt{\gamma^2 + \delta^2}$ to be invariant anymore, because the ϕ matrix on the right rotates α into $\tilde{\alpha}$ and β into $\tilde{\beta}$, γ into $\tilde{\gamma}$ and δ into $\tilde{\delta}$. However, we would expect

$$\mathscr{I}_1 \coloneqq \sqrt{\alpha^2 + \beta^2 + \tilde{\alpha}^2 + \tilde{\beta}^2} \quad \text{and} \quad \mathscr{I}_2 \coloneqq \sqrt{\gamma^2 + \delta^2 + \tilde{\gamma}^2 + \tilde{\delta}^2} \tag{5.14}$$

to be invariant. This may be guessed from the rotation matrices, but we can also see it directly, since $\mathscr{I}_1$ and $\mathscr{I}_2$ are simply the square root of $\mathrm{tr}(MM^T)$ for the respective M:

$$\begin{aligned} \mathrm{tr}(MM^T) \mapsto \mathrm{tr}(O_1 M O_2 (O_1 M O_2)^T) = \mathrm{tr}(O_1 M O_2 O_2^T M^T O_1^T) = \\ = \mathrm{tr}(O_1 M M^T O_1^T) = \mathrm{tr}(M M^T O_1^T O_1) = \mathrm{tr}(MM^T). \end{aligned}$$

We also get an invariant from the determinant:

$$\det M \mapsto \det(O_1 M O_2) = (\det O_1)(\det M)(\det O_2) = \det M$$

This gives us

$$\mathscr{I}_3 \coloneqq \alpha\tilde{\beta} - \beta\tilde{\alpha} \quad \text{and} \quad \mathscr{I}_4 \coloneqq \gamma\tilde{\delta} - \delta\tilde{\gamma}. \tag{5.15}$$

We want to find the orbits of the complex spinors, and a canonical representative in each orbit. We are starting with the simpler problem of doing the same for a 2×2 matrix multiplied by (potentially different) rotation matrices on either side. We have, I claim, one orbit for each value of $\mathscr{I}_1$ and $\mathscr{I}_3$ (or $\mathscr{I}_2$ and $\mathscr{I}_4$ if we think of M as the lower 2×2 matrix). To see that this is so, we can find a canonical representative — show that for an arbitrary 2×2 matrix M, the transformation can bring it to diagonal form expressed in terms of $\mathscr{I}_1$ and $\mathscr{I}_3$.

Perform the matrix multiplication and call the resulting matrix M'. We want $M'_{12} = M'_{21} = 0$, to place the nonzero entries on the diagonal. $M'_{12} = 0$ and $M'_{21} = 0$ are not very nice equations by themselves, so we study $M'_{12} + M'_{21} = 0$ and $M'_{12} - M'_{21} = 0$ instead:

$$\begin{cases} (\alpha - \tilde{\beta})\sin\left(\phi + \frac{\theta}{2}\right) + (\tilde{\alpha} + \beta)\cos\left(\phi + \frac{\theta}{2}\right) = 0 & \text{(sum)} \\ (\alpha + \tilde{\beta})\sin\left(\phi - \frac{\theta}{2}\right) + (\tilde{\alpha} - \beta)\cos\left(\phi - \frac{\theta}{2}\right) = 0 & \text{(difference).} \end{cases} \tag{5.16}$$

The solution is

$$\begin{cases} \phi + \frac{\theta}{2} = -\tan^{-1}\left(\frac{\tilde{\alpha}+\beta}{\alpha-\tilde{\beta}}\right) + n\pi, & n \in \mathbb{Z}, \\ \phi - \frac{\theta}{2} = -\tan^{-1}\left(\frac{\tilde{\alpha}-\beta}{\alpha+\tilde{\beta}}\right) + m\pi, & m \in \mathbb{Z}. \end{cases} \tag{5.17}$$

This can safely be interpreted in a sloppy way, so that e.g. $\tan^{-1}(1/0) = \tan^{-1}(\pm\infty) = \pm\frac{\pi}{2}$. When the above generates an indeterminate angle sum or angle difference, that means that the corresponding angle is truly arbitrary — in certain cases only one angle is needed to rotate the 2×2 matrix to diagonal form. The important thing is that we can always make (5.16) work, by choosing the angles as in (5.17). So we can always do

$$\begin{pmatrix} \alpha & \tilde{\alpha} \\ \beta & \tilde{\beta} \end{pmatrix} \rightarrow \begin{pmatrix} \alpha' & 0 \\ 0 & \tilde{\beta}' \end{pmatrix}.$$

The diagonal form is, however, not unique. We can do

$$\begin{pmatrix} \alpha' & 0 \\ 0 & \tilde{\beta}' \end{pmatrix} \rightarrow \begin{pmatrix} -\alpha' & 0 \\ 0 & -\tilde{\beta}' \end{pmatrix}$$

using $\theta = 0$, $\phi = \pi$, and we can do

$$\begin{pmatrix} \alpha' & 0 \\ 0 & \tilde{\beta}' \end{pmatrix} \rightarrow \begin{pmatrix} \tilde{\beta}' & 0 \\ 0 & \alpha' \end{pmatrix}$$

using $\frac{\theta}{2} = \phi = \frac{\pi}{2}$. If we want a unique representative of the orbit, we can use the transformations just mentioned to make sure that $|\alpha'| \geq |\tilde{\beta}'|$ and

$\alpha' \geq 0$. Then, we can even give an explicit form for α' and $\tilde{\beta}'$ in terms of the invariants $\mathscr{I}_1$ and $\mathscr{I}_3$:

$$\begin{cases} \alpha' = \sqrt{\frac{\mathscr{I}_1^2}{2} + \sqrt{\frac{\mathscr{I}_1^4}{4} - \mathscr{I}_3^2}}, \\ \tilde{\beta}' = \operatorname{sgn}(\mathscr{I}_3)\sqrt{\frac{\mathscr{I}_1^2}{2} - \sqrt{\frac{\mathscr{I}_1^4}{4} - \mathscr{I}_3^2}}. \end{cases} \tag{5.18}$$

Analogously we have

$$\begin{cases} \gamma' = \sqrt{\frac{\mathscr{I}_2^2}{2} + \sqrt{\frac{\mathscr{I}_2^4}{4} - \mathscr{I}_4^2}}, \\ \tilde{\delta}' = \operatorname{sgn}(\mathscr{I}_4)\sqrt{\frac{\mathscr{I}_2^2}{2} - \sqrt{\frac{\mathscr{I}_2^4}{4} - \mathscr{I}_4^2}}. \end{cases} \tag{5.19}$$

The general case with 4×2 matrices representing our spinors, is given by the above analysis not only when $a = b = 0$ in $G(a, b, \theta)$, but also when $\mathscr{I}_1 = 0$, which implies that the spinor is of the form

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \gamma & \tilde{\gamma} \\ \delta & \tilde{\delta} \end{pmatrix}.$$

For such spinors, a and b do not matter — only θ in $G(a, b, \theta)$ has the power to change the spinor. The conclusion is that all spinors with $\mathscr{I}_1 = 0$ are in the same orbit as the representative

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \gamma' & 0 \\ 0 & \tilde{\delta}' \end{pmatrix},$$

where γ' and $\tilde{\delta}'$ are given by (5.19) in terms of $\mathscr{I}_2$ and $\mathscr{I}_4$, which in turn are defined by (5.14) and (5.15).

If $\mathscr{I}_1 > 0$, we may use θ and ϕ to transform the spinor to something of the form

$$\begin{pmatrix} \alpha' & 0 \\ 0 & \tilde{\beta}' \\ \gamma & \tilde{\gamma} \\ \delta & \tilde{\delta} \end{pmatrix}$$

where α' and $\tilde{\beta}'$ are given by (5.18) in terms of $\mathscr{I}_1$ and $\mathscr{I}_3$, which in turn are defined by (5.14) and (5.15). One might argue that I should put a prime on

$\gamma, \tilde{\gamma}, \delta$ and $\tilde{\delta}$ too, since they will in general also change when we transform the upper part to diagonal form, but I have left them without prime so as not to confuse them with the γ' and $\tilde{\delta}'$ in (5.19) — here $\gamma, \tilde{\gamma}, \delta$ and $\tilde{\delta}$ are transformed, but still pretty arbitrary. This transformed spinor we want to transform again to make it even simpler, but this time using the a and b of $G(a, b, \theta)$ and not using θ and ϕ at all — that would mess up the diagonal form of the upper part of the spinor. Thus, we study $G(a, b, 0)$ on the spinor:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & -b & 1 & 0 \\ b & a & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha' & 0 \\ 0 & \tilde{\beta}' \\ \gamma & \tilde{\gamma} \\ \delta & \tilde{\delta} \end{pmatrix} = \begin{pmatrix} \alpha' & 0 \\ 0 & \tilde{\beta}' \\ a\alpha' + \gamma & -b\tilde{\beta}' + \tilde{\gamma} \\ b\alpha' + \delta & a\tilde{\beta}' + \tilde{\delta} \end{pmatrix} =$$

(Choose $a = -\gamma/\alpha'$ and $b = -\delta/\alpha'$. Since $\mathscr{I}_1 > 0$, we know that α' must be positive too and we can divide by it. $\tilde{\beta}'$ on the other hand, may be zero — will be zero if $\mathscr{I}_3 = 0$.)

$$= \begin{pmatrix} \alpha' & 0 \\ 0 & \tilde{\beta}' \\ 0 & \frac{\delta}{\alpha'}\tilde{\beta}' + \tilde{\gamma} \\ 0 & -\frac{\gamma}{\alpha'}\tilde{\beta}' + \tilde{\delta} \end{pmatrix}.$$

We are left with one guaranteed nonzero component, and three potentially non-zero components. Of those $\tilde{\beta}'$ can be expressed as $\mathscr{I}_3/\alpha'$. Could the other two possibly be expressed as $\mathscr{I}_5/\alpha'$ and $\mathscr{I}_6/\alpha'$ for invariants $\mathscr{I}_5$ and $\mathscr{I}_6$? What would $\mathscr{I}_5$ and $\mathscr{I}_6$ be, in general? In this particular case, when we started with a spinor whose upper 2×2 component was already diagonal, we would have $\mathscr{I}_5 = \delta\tilde{\beta} + \tilde{\gamma}\alpha$ and $\mathscr{I}_6 = -\gamma\tilde{\beta} + \tilde{\delta}\alpha$ (now dropping the primes, as we try to find general expressions for completely arbitrary spinors). These definitions would not, however, be invariant, as can be seen from the transformation

$$\begin{pmatrix} \alpha & \tilde{\alpha} \\ \beta & \tilde{\beta} \\ \gamma & \tilde{\gamma} \\ \delta & \tilde{\delta} \end{pmatrix} \xrightarrow{\phi = \frac{\pi}{2}} \begin{pmatrix} -\tilde{\alpha} & \alpha \\ -\tilde{\beta} & \beta \\ -\tilde{\gamma} & \gamma \\ -\tilde{\delta} & \delta \end{pmatrix}$$

since $\delta\tilde{\beta} + \tilde{\gamma}\alpha \to -\tilde{\delta}\beta - \gamma\tilde{\alpha}$ and $-\gamma\tilde{\beta} + \tilde{\delta}\alpha \to \tilde{\gamma}\beta - \delta\tilde{\alpha}$. This would lead us to guess that perhaps $\mathscr{I}_5 = \alpha\tilde{\gamma} - \gamma\tilde{\alpha} + \delta\tilde{\beta} - \beta\tilde{\delta}$ and $\mathscr{I}_6 = \beta\tilde{\gamma} - \gamma\tilde{\beta} + \alpha\tilde{\delta} - \delta\tilde{\alpha}$ would be invariant — in the case $\beta = \tilde{\alpha} = 0$ they yield the same expression as above, and they are invariant under a transformation with $\phi = \frac{\pi}{2}$. To see that $\mathscr{I}_5$ and $\mathscr{I}_6$ are invariant under a general transformation, involving both $G(a, b, \theta)$ and ϕ, it is perhaps easiest to turn to a computer algebra system. They are indeed invariants.

So, collecting the expressions for invariants that allow us to take an arbitrary spinor and directly write down a canonical representative, we get:

$$\begin{cases} \mathscr{I}_1 := \sqrt{\alpha^2+\beta^2+\tilde{\alpha}^2+\tilde{\beta}^2}, & \\ \mathscr{I}_2 := \sqrt{\gamma^2+\delta^2+\tilde{\gamma}^2+\tilde{\delta}^2} & \text{(invariant if } \mathscr{I}_1 = 0), \\ \mathscr{I}_3 := \alpha\tilde{\beta} - \beta\tilde{\alpha}, & \\ \mathscr{I}_4 := \gamma\tilde{\delta} - \delta\tilde{\gamma} & \text{(invariant if } \mathscr{I}_1 = 0), \\ \mathscr{I}_5 := \alpha\tilde{\gamma} - \gamma\tilde{\alpha} + \delta\tilde{\beta} - \beta\tilde{\delta}, & \\ \mathscr{I}_6 := \beta\tilde{\gamma} - \gamma\tilde{\beta} + \alpha\tilde{\delta} - \delta\tilde{\alpha}, & \end{cases}$$

$$\begin{cases} \alpha' = \sqrt{\frac{\mathscr{I}_1^2}{2} + \sqrt{\frac{\mathscr{I}_1^4}{4} - \mathscr{I}_3^2}}, \\ \gamma' = \sqrt{\frac{\mathscr{I}_2^2}{2} + \sqrt{\frac{\mathscr{I}_2^4}{4} - \mathscr{I}_4^2}}. \end{cases}$$

In terms of these, we can write down the orbits directly as:

The 0 orbit: $\{\eta = 0 : \quad \mathscr{I}_1 = 0 \text{ and } \mathscr{I}_2 = 0\}$

The $\gamma'\eta_3 + \dfrac{\mathscr{I}_4}{\gamma'}\eta_8$ orbit:

$$\{\eta = \gamma\eta_3 + \delta\eta_4 + \tilde{\gamma}\eta_7 + \tilde{\delta}\eta_8 : \quad \mathscr{I}_1 = 0 \text{ and } \mathscr{I}_2 > 0\}$$

The $\alpha'\eta_1 + \dfrac{\mathscr{I}_3}{\alpha'}\eta_6 + \dfrac{\mathscr{I}_5}{\alpha'}\eta_7 + \dfrac{\mathscr{I}_6}{\alpha'}\eta_8$ orbit:

$$\{\eta = \alpha\eta_1 + \beta\eta_2 + \gamma\eta_3 + \delta\eta_4 + \tilde{\alpha}\eta_5 + \tilde{\beta}\eta_6 + \tilde{\gamma}\eta_7 + \tilde{\delta}\eta_8 : \quad \mathscr{I}_1 > 0 \text{ and } \mathscr{I}_2 \text{ irrelevant}\}.$$

So this is how we do it if we get a completely general spinor and wish to find the canonical representative spinor of the corresponding orbit — we just calculate some $\mathscr{I}_i$'s. That is not, however, what we want to do. We don't want to start with a completely general spinor; we don't much care about the general spinors. We care about the orbits of spinors, and what we need is one spinor that can represent each orbit. Ignoring $\eta = 0$, we wish to study the spinors $\eta = \gamma\eta_3 + \tilde{\delta}\eta_8$ and $\eta = \alpha\eta_1 + \tilde{\beta}\eta_6 + \tilde{\gamma}\eta_7 + \tilde{\delta}\eta_8$. If we know what happens to them, we know everything we wish to know about the completely general spinor, since it can be obtained by transforming these representatives. We thus study two cases: one with two functions and one with four.

5.2 Local and global

We have used local symmetries (Galilean and U(1)) to get a general spinor into one of the forms $\eta = \alpha\eta_1 + \tilde{\beta}\eta_6 + \tilde{\gamma}\eta_7 + \tilde{\delta}\eta_8$ or $\eta = \gamma\eta_3 + \tilde{\delta}\eta_8$, where the Greek coefficients are functions of spacetime. This means that we in general have four (or two) functions.

The local symmetries we used are the symmetries of the underlying theory that are compatible with choosing the +, 1, $\bar{1}$ directions to be the Galilean directions. Other symmetries of the underlying theory would mix the directions. We have not yet, however, made sure that the solution has Galilean symmetry.

We want the solution to have a global Galilean symmetry. So we take our two representative spinors, and act on them with a global Galilean symmetry transformation — global meaning that this time around, the a, b and θ are not functions, but some real values. Doing Galilean transformations on the set of Killing spinors should give us back the set of Killing spinors, to make the Galilean transformation a symmetry. Thus all spinors resulting from the Galilean transformations are Killing spinors. The Galilean transformation of the spinors is given in (5.12).

$$\begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} & 0 & 0 \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} & 0 & 0 \\ a & -b & \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ b & a & -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \tilde{\beta} \\ 0 & \tilde{\gamma} \\ 0 & \tilde{\delta} \end{pmatrix} =$$

$$= \cos\frac{\theta}{2}\begin{pmatrix} \alpha & 0 \\ 0 & \tilde{\beta} \\ 0 & \tilde{\gamma} \\ 0 & \tilde{\delta} \end{pmatrix} + \sin\frac{\theta}{2}\begin{pmatrix} 0 & -\tilde{\beta} \\ \alpha & 0 \\ 0 & \tilde{\delta} \\ 0 & -\tilde{\gamma} \end{pmatrix} + a\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \alpha & 0 \\ 0 & \tilde{\beta} \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -\tilde{\beta} \\ \alpha & 0 \end{pmatrix}.$$

Thus acting with a global Galilean transformation on $\eta = \alpha\eta_1 + \tilde{\beta}\eta_6 + \tilde{\gamma}\eta_7 + \tilde{\delta}\eta_8$ gives us a linear combination of

- $\alpha\eta_1 + \tilde{\beta}\eta_6 + \tilde{\gamma}\eta_7 + \tilde{\delta}\eta_8$,
- $\alpha\eta_2 - \tilde{\beta}\eta_5 + \tilde{\delta}\eta_7 - \tilde{\gamma}\eta_8$,
- $\alpha\eta_3 + \tilde{\beta}\eta_8$, and
- $\alpha\eta_4 - \tilde{\beta}\eta_7$.

In this case, we have $N = 4$ Killing spinors.

The orbits described by $\eta = \gamma\eta_3 + \tilde{\delta}\eta_8$ under global Galilean transformations:

$$\begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} & 0 & 0 \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} & 0 & 0 \\ a & -b & \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ b & a & -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \gamma & 0 \\ 0 & \tilde{\delta} \end{pmatrix} =$$

$$= \cos\frac{\theta}{2} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \gamma & 0 \\ 0 & \tilde{\delta} \end{pmatrix} + \sin\frac{\theta}{2} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \tilde{\delta} \\ -\gamma & 0 \end{pmatrix}.$$

Thus acting with a global Galilean transformation on $\eta = \gamma\eta_3 + \tilde{\delta}\eta_8$ gives us a linear combination of

- $\gamma\eta_3 + \tilde{\delta}\eta_8$, and
- $-\gamma\eta_4 + \tilde{\delta}\eta_7$.

In this case we have $N = 2$ Killing spinors.

5.3 Killing spinors and Killing vectors

The Killing vectors are the vectors associated with $(\kappa_+^{IJ})_B e^B$. Elvin (2009, eq. 4.23) gives the expression for the Killing vector coordinates as:

$$\left(\kappa_+^{IJ}\right)_B = B(\eta^I, C{*}\Gamma_B\eta^J) + B(C{*}\eta^I, \Gamma_B\eta^J). \tag{5.20}$$

Here, $B(\cdot,\cdot)$ is a Spin(9, 1) invariant Majorana inner product, given by Elvin (2009, eq. 4.4):

$$B(\eta,\theta) = \langle \Gamma_0 C{*}\eta, \theta\rangle. \tag{5.21}$$

Combining (5.20) and (5.21) we get

$$\begin{aligned}(\kappa_+^{IJ})_B &= \langle \Gamma_0 C{*}\eta^I, C{*}\Gamma_B\eta^J\rangle + \langle C{*}\Gamma_0 C{*}\eta^I, \Gamma_B\eta^J\rangle = \\ &= \langle \Gamma_0 C{*}\eta^I, C{*}\Gamma_B\eta^J\rangle + \langle \Gamma_0\eta^I, \Gamma_B\eta^J\rangle. \end{aligned} \tag{5.22}$$

Note that these coordinates are real, so we should be using the real basis for the Killing vectors. Going over to the complex basis using (4.14), we would get complex coordinates, if we want to express the same Killing vectors, but $(\kappa_+^{IJ})_B$ is entirely real.

For the first set of orbits we have

$$\begin{aligned}
\eta^1_{\mathrm{A}} &\coloneqq \alpha\,\eta_1 + \tilde\beta\,\eta_6 + \tilde\gamma\,\eta_7 + \tilde\delta\,\eta_8, & C{*}\eta^1_{\mathrm{A}} &= \alpha\,\eta_1 - \tilde\beta\,\eta_6 - \tilde\gamma\,\eta_7 - \tilde\delta\,\eta_8,\\
\eta^2_{\mathrm{A}} &\coloneqq \alpha\,\eta_2 - \tilde\beta\,\eta_5 + \tilde\delta\,\eta_7 - \tilde\gamma\,\eta_8, & C{*}\eta^2_{\mathrm{A}} &= \alpha\,\eta_2 + \tilde\beta\,\eta_5 - \tilde\delta\,\eta_7 + \tilde\gamma\,\eta_8,\\
\eta^3_{\mathrm{A}} &\coloneqq \alpha\,\eta_3 + \tilde\beta\,\eta_8, & C{*}\eta^3_{\mathrm{A}} &= \alpha\,\eta_3 - \tilde\beta\,\eta_8,\\
\eta^4_{\mathrm{A}} &\coloneqq \alpha\,\eta_4 - \tilde\beta\,\eta_7, & C{*}\eta^4_{\mathrm{A}} &= \alpha\,\eta_4 + \tilde\beta\,\eta_7,
\end{aligned}$$

$$\begin{aligned}
\eta^1_{\mathrm{A}} &= \left(\alpha+\tilde\beta\right)\,\mathsf{e}_{1234} + \left(-\tilde\delta + \mathrm{i}\,\tilde\gamma\right)\,\mathsf{e}_{15} + \left(\tilde\delta + \mathrm{i}\,\tilde\gamma\right)\,\mathsf{e}_{2345} - \left(-\alpha+\tilde\beta\right)\,\mathsf{1},\\
\eta^2_{\mathrm{A}} &= -\mathrm{i}\,\left(\alpha+\tilde\beta\right)\,\mathsf{e}_{1234} + \left(\mathrm{i}\,\tilde\delta + \tilde\gamma\right)\,\mathsf{e}_{15} - \left(-\mathrm{i}\,\tilde\delta + \tilde\gamma\right)\,\mathsf{e}_{2345} - \mathrm{i}\,\left(-\alpha+\tilde\beta\right)\,\mathsf{1},\\
\eta^3_{\mathrm{A}} &= -\left(-\alpha+\tilde\beta\right)\,\mathsf{e}_{15} + \left(\alpha+\tilde\beta\right)\,\mathsf{e}_{2345},\\
\eta^4_{\mathrm{A}} &= -\mathrm{i}\,\left(-\alpha+\tilde\beta\right)\,\mathsf{e}_{15} - \mathrm{i}\,\left(\alpha+\tilde\beta\right)\,\mathsf{e}_{2345},
\end{aligned}$$

$$\begin{aligned}
C{*}\eta^1_{\mathrm{A}} &= -\left(-\alpha+\tilde\beta\right)\,\mathsf{e}_{1234} - \left(-\tilde\delta + \mathrm{i}\,\tilde\gamma\right)\,\mathsf{e}_{15} - \left(\tilde\delta + \mathrm{i}\,\tilde\gamma\right)\,\mathsf{e}_{2345} + \left(\alpha+\tilde\beta\right)\,\mathsf{1},\\
C{*}\eta^2_{\mathrm{A}} &= \mathrm{i}\,\left(-\alpha+\tilde\beta\right)\,\mathsf{e}_{1234} - \left(\mathrm{i}\,\tilde\delta + \tilde\gamma\right)\,\mathsf{e}_{15} + \left(-\mathrm{i}\,\tilde\delta + \tilde\gamma\right)\,\mathsf{e}_{2345} + \mathrm{i}\,\left(\alpha+\tilde\beta\right)\,\mathsf{1},\\
C{*}\eta^3_{\mathrm{A}} &= \left(\alpha+\tilde\beta\right)\,\mathsf{e}_{15} - \left(-\alpha+\tilde\beta\right)\,\mathsf{e}_{2345},\\
C{*}\eta^4_{\mathrm{A}} &= \mathrm{i}\,\left(\alpha+\tilde\beta\right)\,\mathsf{e}_{15} + \mathrm{i}\,\left(-\alpha+\tilde\beta\right)\,\mathsf{e}_{2345}.
\end{aligned}$$

Equation (5.22) can be reduced to coefficients times $\langle \boldsymbol{e}_1, \Gamma_B \boldsymbol{e}_{15}\rangle$, $\langle \boldsymbol{e}_1, \Gamma_B \boldsymbol{e}_{2345}\rangle$, $\langle \boldsymbol{e}_{234}, \Gamma_B \boldsymbol{e}_{15}\rangle$ and $\langle \boldsymbol{e}_{234}, \Gamma_B \boldsymbol{e}_{2345}\rangle$, and we get

$$\begin{aligned}
\kappa^{11}_+ &= 4(\alpha^2+\tilde\beta^2+\tilde\gamma^2+\tilde\delta^2)\boldsymbol{e}^0 - 8\tilde\beta\tilde\delta\boldsymbol{e}^1 + 4(-\alpha^2-\tilde\beta^2+\tilde\gamma^2+\tilde\delta^2)\boldsymbol{e}^5 + 8\tilde\beta\tilde\gamma\boldsymbol{e}^6\\
\kappa^{12}_+ &= 8\tilde\beta\tilde\gamma\boldsymbol{e}^1 + 8\tilde\beta\tilde\delta\boldsymbol{e}^6\\
\kappa^{13}_+ &= 4\tilde\beta\tilde\delta\boldsymbol{e}^0 + (-4\alpha^2-4\tilde\beta^2)\boldsymbol{e}^1 + 4\tilde\beta\tilde\delta\boldsymbol{e}^5\\
\kappa^{14}_+ &= -4\tilde\beta\tilde\gamma\boldsymbol{e}^0 - 4\tilde\beta\tilde\gamma\boldsymbol{e}^5 + (-4\alpha^2-4\tilde\beta^2)\boldsymbol{e}^6\\
\kappa^{22}_+ &= 4(\alpha^2+\tilde\beta^2+\tilde\gamma^2+\tilde\delta^2)\boldsymbol{e}^0 + 8\tilde\beta\tilde\delta\boldsymbol{e}^1 + 4(-\alpha^2-\tilde\beta^2+\tilde\gamma^2+\tilde\delta^2)\boldsymbol{e}^5 - 8\tilde\beta\tilde\gamma\boldsymbol{e}^6\\
\kappa^{23}_+ &= -4\tilde\beta\tilde\gamma\boldsymbol{e}^0 - 4\tilde\beta\tilde\gamma\boldsymbol{e}^5 + (4\alpha^2+4\tilde\beta^2)\boldsymbol{e}^6\\
\kappa^{24}_+ &= -4\tilde\beta\tilde\delta\boldsymbol{e}^0 + (-4\alpha^2-4\tilde\beta^2)\boldsymbol{e}^1 - 4\tilde\beta\tilde\delta\boldsymbol{e}^5\\
\kappa^{33}_+ &= (4\alpha^2+4\tilde\beta^2)\boldsymbol{e}^0 + (4\alpha^2+4\tilde\beta^2)\boldsymbol{e}^5\\
\kappa^{34}_+ &= 0\\
\kappa^{44}_+ &= (4\alpha^2+4\tilde\beta^2)\boldsymbol{e}^0 + (4\alpha^2+4\tilde\beta^2)\boldsymbol{e}^5\\
\kappa^{IJ}_+ &= \kappa^{JI}_+
\end{aligned}$$

For the second set of orbits we have

$$\eta_{\mathrm{B}}^1 \coloneqq \gamma\,\eta_3 + \tilde{\delta}\,\eta_8, \qquad C{*}\eta_{\mathrm{B}}^1 = \gamma\,\eta_3 - \tilde{\delta}\,\eta_8,$$
$$\eta_{\mathrm{B}}^2 \coloneqq \gamma\,\eta_4 - \tilde{\delta}\,\eta_7, \qquad C{*}\eta_{\mathrm{B}}^2 = \gamma\,\eta_4 + \tilde{\delta}\,\eta_7,$$

$$\eta_{\mathrm{B}}^1 = \left(-\tilde{\delta} + \gamma\right)\, e_{1,5} + \left(\tilde{\delta} + \gamma\right)\, e_{2,3,4,5},$$
$$\eta_{\mathrm{B}}^2 = \mathrm{i}\,\left(-\tilde{\delta} + \gamma\right)\, e_{1,5} - \mathrm{i}\,\left(\tilde{\delta} + \gamma\right)\, e_{2,3,4,5},$$

$$C{*}\eta_{\mathrm{B}}^1 = \left(\tilde{\delta} + \gamma\right)\, e_{1,5} + \left(-\tilde{\delta} + \gamma\right)\, e_{2,3,4,5},$$
$$C{*}\eta_{\mathrm{B}}^2 = \mathrm{i}\,\left(\tilde{\delta} + \gamma\right)\, e_{1,5} - \mathrm{i}\,\left(-\tilde{\delta} + \gamma\right)\, e_{2,3,4,5}.$$

Equation (5.22) can be reduced to coefficients times $\langle \boldsymbol{e}_1, \Gamma_B \boldsymbol{e}_{15}\rangle$, $\langle \boldsymbol{e}_1, \Gamma_B \boldsymbol{e}_{2345}\rangle$, $\langle \boldsymbol{e}_{234}, \Gamma_B \boldsymbol{e}_{15}\rangle$ and $\langle \boldsymbol{e}_{234}, \Gamma_B \boldsymbol{e}_{2345}\rangle$, and we get

$$\kappa_+^{11} = \left(4\,\tilde{\delta}^2 + 4\,\gamma^2\right)\,\boldsymbol{e}^0 + \left(4\,\tilde{\delta}^2 + 4\,\gamma^2\right)\,\boldsymbol{e}^5$$
$$\kappa_+^{12} = 0$$
$$\kappa_+^{22} = \left(4\,\tilde{\delta}^2 + 4\,\gamma^2\right)\,\boldsymbol{e}^0 + \left(4\,\tilde{\delta}^2 + 4\,\gamma^2\right)\,\boldsymbol{e}^5$$
$$\kappa_+^{IJ} = \kappa_+^{JI}$$

We have many more Killing vectors in the $N = 4$ case compared to $N = 2$ here. More Killing vectors means more symmetry and makes it easier to solve the equations; the $N = 2$ case with only two Killing vectors is correspondingly much harder.

Chapter 6

Outlook and future work

The next step is to insert the spinors found in the previous chapter into the Killing spinor equation. Remember that the coefficients of the Killing spinors found there are functions of spacetime, but if they are to be true Killing spinors those functions cannot be truly arbitrary. The Killing spinor equation is a first order differential equation given by the supersymmetry variation of the gravitino — I won't go in to the details here; you can read about it in e.g. Elvin (2009). The upshot of it all, is that you get an equation of the form $\mathcal{D}_M \epsilon = 0$, where $\mathcal{D}$ is a supercovariant derivative. Linear differential equations are in principle easy to solve, though you would want to write a computer program to do this explicitly.

That the Killing spinors respect Galilean symmetry is ensured by the ansatz in chapter 5, but the supercovariant derivative also involve some other fields, such as the five-form field strength F and the three-form field strength G, which are also a part of the solution. We may therefore need to impose Galilean symmetry on those as well.

When we have found the functions in the spinor, we insert them into the corresponding equation for the Killing vectors. The expected final result is a list of all possible ansätze for geometries in this theory, that respect Galilean symmetry. Some of them will describe superconductors. Some of them might not have been examined before.

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